Cluster structures for Grassmannians

jt work w.
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let $1 \leq k < n$, $k \leq n/2$

(1) Background

k=2 — arbitrary k

Theorem (Fomin-Zelevinsky '03, Scott '05)

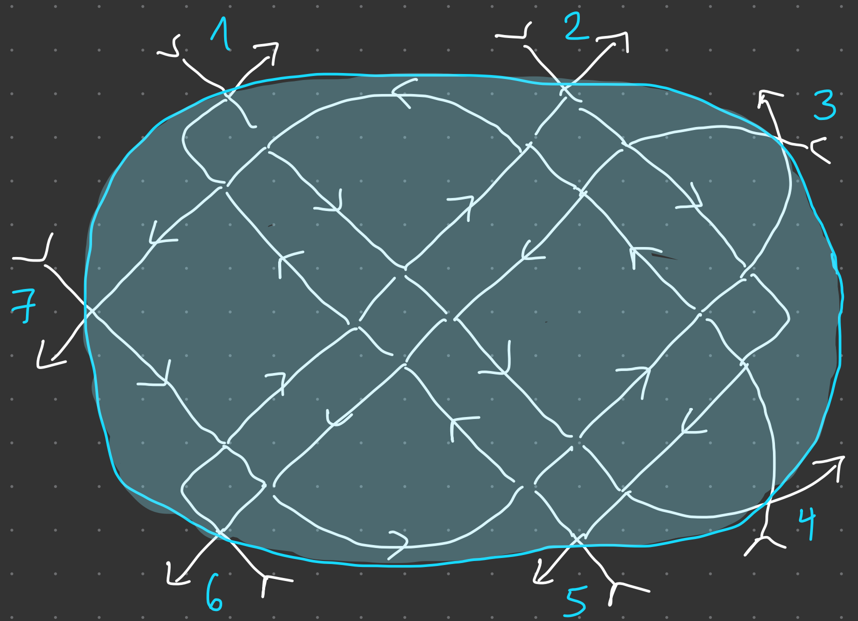
The ring $\mathbb{C}[\widehat{Gr(k,n)}]$ has a cluster algebra structure
The Plücker coordinates are cluster variables. Mutation arises from short Plücker relations.

$\boxed{k=2}$ All cluster variables are Plücker coordinates: Variables:
$\{p_{ab}$, (a,b) a diagonal in $P_n\} \cup \{p_{i,i+1} \mid 1 \leq i \leq n\}$. Clusters $\Leftrightarrow$ triangul. of P_n, mutation $\longleftrightarrow$ quadrilateral flip in triangul.

$\boxed{k \text{ arbitrary}}$ Plücker coord's $\subset$ {cluster var's}. $\exists$ clusters of Plücker coord's.

From Postnikov diagrams on P_n: coll. of oriented curves in P_n $i \rightsquigarrow j = i+k$ alt. crossings

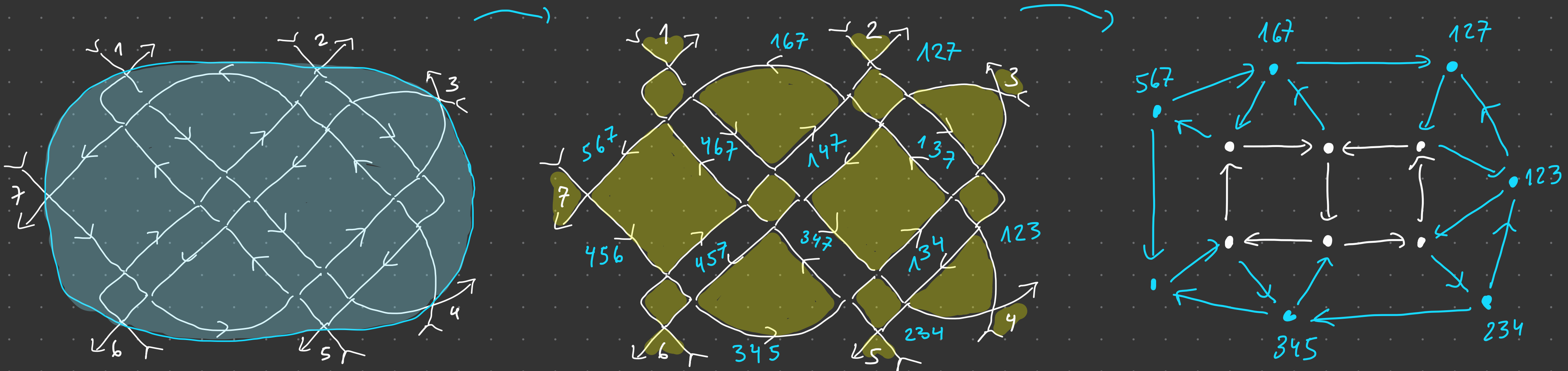
Example $k=3, n=7$



* subdivides P_n into oriented / alternating regions
* label altern. regions to left of curve $i \rightsquigarrow i+k$ by label i
* every k-subset appears as a label in such a diagram

$\Rightarrow \{p_I \mid I \text{ label in diagram}\}$ is a cluster.

Frozen variables: p_{I_j}, $I_j := \{j, j+1, \dots, j+k-1\}$



Keller's period. conj.

$A_{k-1} \times A_{n-k-1}$

Remark : * Cluster algebra structures on :

- (skew) Schubert varieties (Serhiyenko – Sherman-Bennett – Williams)
- open positroid varieties (Galashin – Lam)

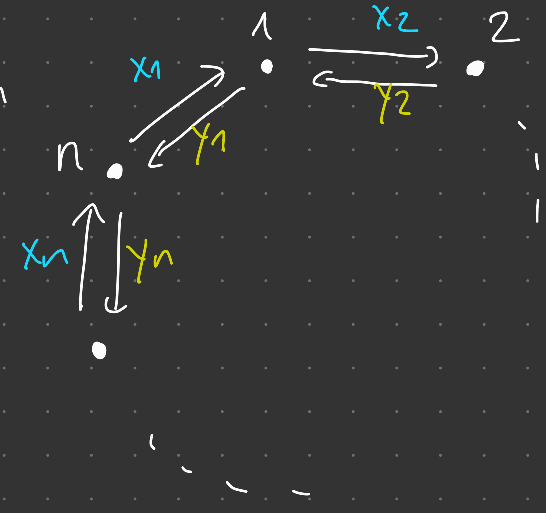
* Cluster categories associated to Grassmannian :

categorifications Geiß – Leclerc – Schröer ; $\underbrace{\text{Jensen – King – Su}}_{\text{for today}}$

Leclerc

② Cluster category $\mathcal{F}_{k,n}$

Γ_n



$$B := B_{k,n} := \widehat{\mathbb{C}\Gamma_n} \Big/ \langle\{ yx = xy,\ y^{n-k} = x^{k} \}\rangle$$

$$Z = Z(B) : \widehat{\mathbb{C}[t]} \qquad t := \sum_i x_i y_i$$

$$\mathcal{F}_{k,n} := CM(B) = \{ M \text{ B-module},\ M|_Z \text{ free} \}$$

Properties (Jensen-King-Su '16) * $\mathcal{F}_{k,n}$ is a Frobenius cat.

* $\mathcal{F}_{k,n}$ categorifies Scott's cluster algebra structure on $Gr(k,n)$
* Plücker coord's are in bijection w. rk 1-modules (later)

Note: Postnikov-diagrams [as above] give cluster-tilting objects. D such a diagr.

$$\mathbb{M}_D := \bigoplus_{I \in D} M_I \Rightarrow \mathrm{Ext}^1(\mathbb{M}_D, \mathbb{M}_D) = 0$$

Theorem (B-King-Marsh '16)

$$\mathrm{End}(\mathbb{M}_D) \cong A_D$$

$$e\,\mathrm{End}(\mathbb{M}_D)\,e \cong B_{k,n}$$

combinat. approach to $\mathcal{F}_{k,n}$

Modules in $\mathcal{F}_{k,n}$

∞-dim. (free over Z)

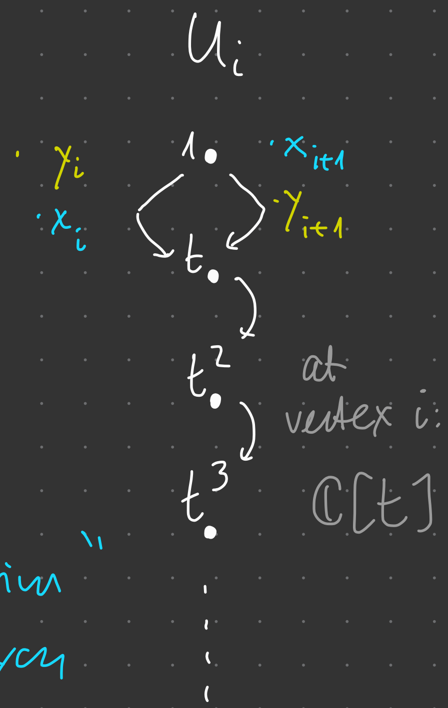
rank: # of copies at each vertex of Γ_n.

"smallest": rank 1-modules $\xleftrightarrow{1:1}$ k-subsets of $\{1,\dots,n\}$ ($\longleftrightarrow$ Plücker coord's)

For I a k-subset:

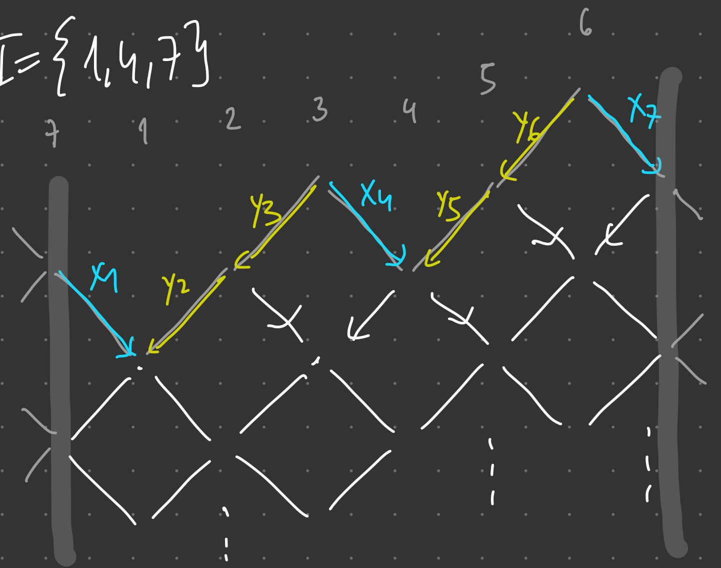
$M_I := ((U_i)_{1\le i\le n}\,;\, x_i\ y_i,\ 1\le i\le n)$ where $\left\{\begin{array}{l} x_i \\ y_i \end{array}\right.$ mult. by $\left\{\begin{array}{l} 1 \\ t \end{array}\right.$ if $i\in I$, by $\left\{\begin{array}{l} t \\ 1 \end{array}\right.$ if $i\notin I$

$Z = \mathbb{C}[[t]]$

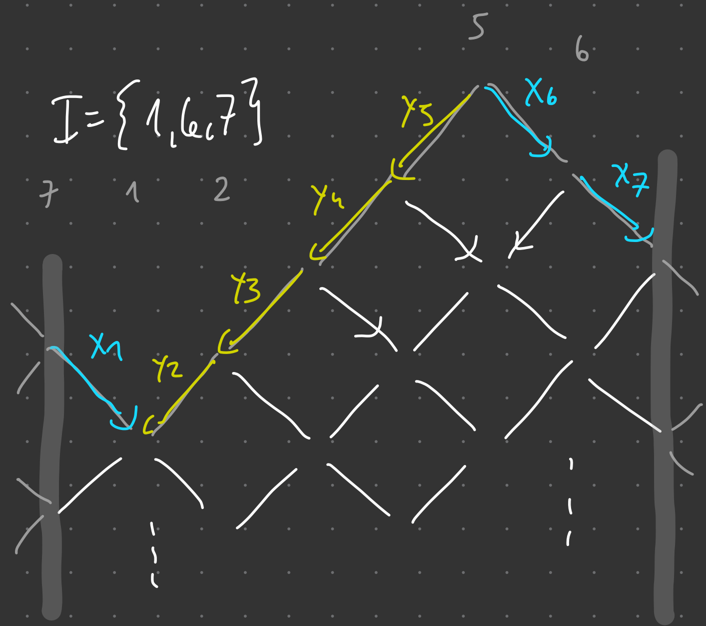


Examples ($k=3, n=7$) $\mathcal{F}_{3,7}$

$I=\{1,4,7\}$



$I=\{1,6,7\}$

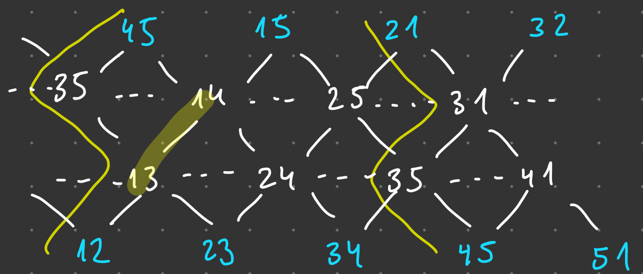


"top rim" tells you which M_I it is

Remark: $\mathcal{F}_{k,n}$: of infinite type in general

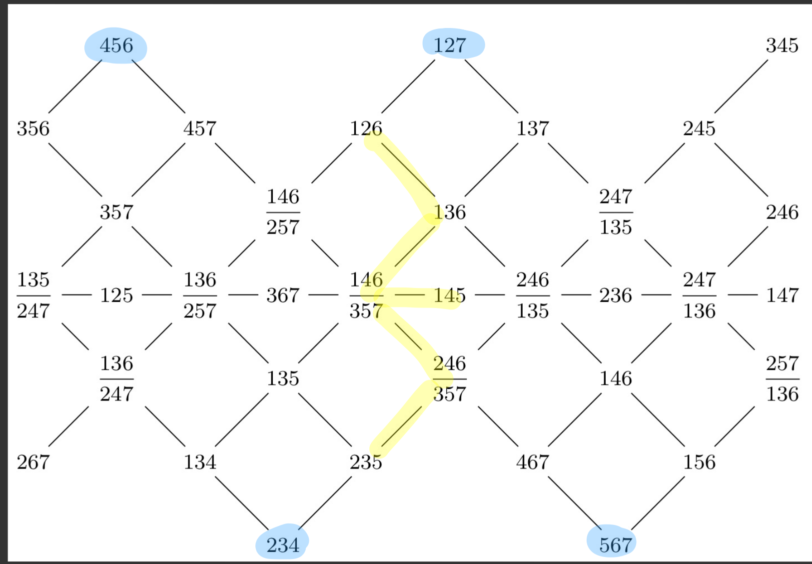
Exceptions:

$\mathcal{F}_{2,n}$ (type A_{n-3})

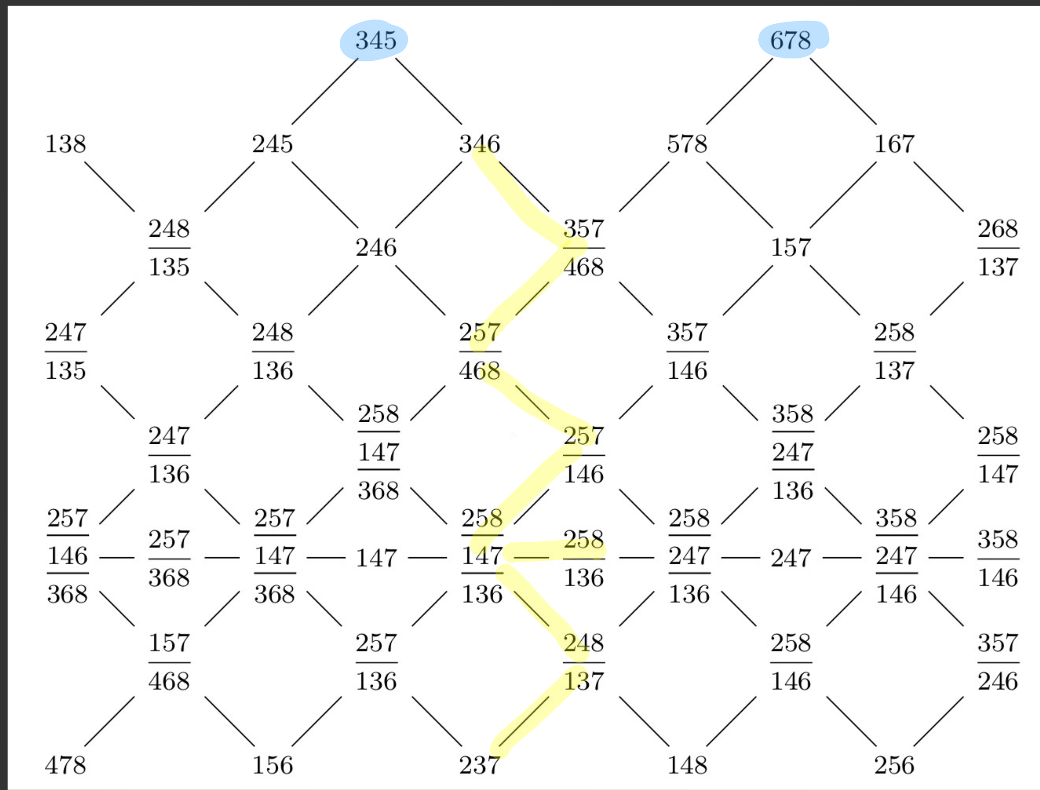


$\mathcal{F}_{3,7}$ (type E_6)

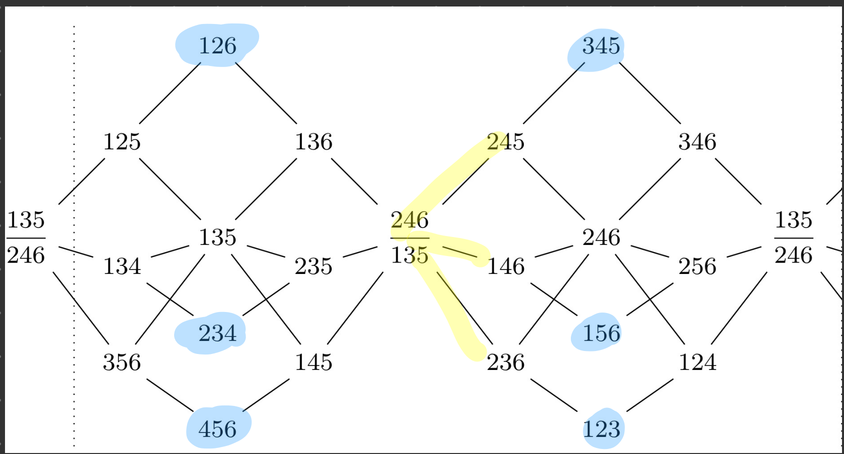
Note: "τ^{-2} adds k"



$\mathcal{F}_{3,8}$ (type E_8)



$\mathcal{F}_{3,6}$ (type D_4)



Pictures from JKS, Proc LMS

③ Infinite types

* $\mathcal{F}_{3,9}$ and $\mathcal{F}_{4,8}$: tame. The rigid indec. in tubes of ranks 2, 3, 6 $\mathcal{F}_{3,9}$, of ranks 2, 4, 4 for $\mathcal{F}_{4,8}$. } B-Bogdanic - Garcia Elsener '18

* every module in $\underline{\mathcal{F}_{k,n}}$ is τ-periodic with period a factor of $2\,\mathrm{lcm}(k,n)/k$ < work of Demonet - Luo; work of Keller

Theorem (B - B - GE - Li '21) Let C be a connected component of the Auslander-Reiten quiver of $\underline{\mathcal{F}_{k,n}}$, C of ∞ type

$\Longrightarrow$ $C \cong \mathbb{Z}A_\infty/\tau^m$ $(m>0)$

Idea: "$\underline{\mathrm{Sub}}(Q_k)$"; use a result of Liu

In $\mathcal{F}_{k,n}$: proj.-inj. belong to certain tubes w. rk 1 modules

$\longrightarrow$ In ∞ types, all tubes and all AR-sequences have ≤ 2 middle terms

④ Modules of small rank

Note: Modules in $\mathcal{F}_{k,n}$ have filtrations by rk 1 modules

↳ use rank 1-modules to "build up" higher rank modules.

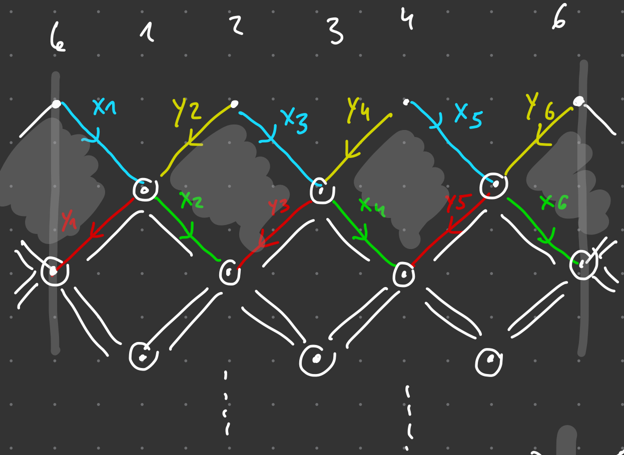
Let $M \in \mathcal{F}_{k,n}$ be rigid indec. of rk 2, with filtration $M = \mathbb{M}_I / \mathbb{M}_J$

Example (as lattice diagram):

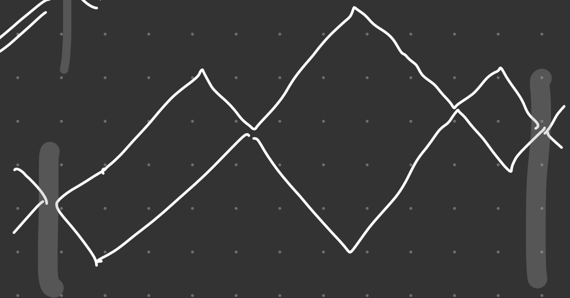
$k=3, n=6$

$$\frac{\mathbb{M}_{\{1,3,5\}}}{\mathbb{M}_{\{2,4,6\}}}$$

$\left(\frac{135}{246} \text{ in } \mathcal{F}_{3,6} \text{ picture} \right)$



3 boxes



Theorem Let $M \in \mathcal{F}_{k,n}$ have filtration $\mathbb{M}_I / \mathbb{M}_J$

(1) M is rigid indecomposable $\iff$ the rims of $\mathbb{M}_I$ & $\mathbb{M}_J$ form 3 boxes
lattice diagram

(2) The number of profiles of rigid indec. rk 2 modules is

$$N_{k,n} = \sum_{r=3}^{k} \left(\frac{2r}{3} p_1(r) + 2p_2(r) + 4p_3(r) \right) \binom{n}{2r} \binom{n-2r}{k-r}$$

$p_i(r) :=$ # of partitions $r = r_1 + r_2 + r_3$ w. $r_i \in \mathbb{Z}_{>0}$ and $|\{r_1, r_2, r_3\}| = i$

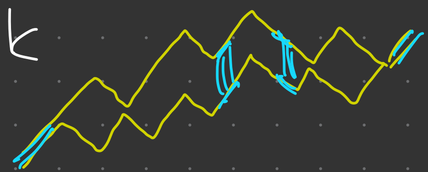
Le-Yildirim

Strategy

(A) $M = \mathbb{M}_I / \mathbb{M}_J$ indec. $\Rightarrow$ rims form ≥ 3 (quasi-) boxes. [else: $\oplus$ of two rank 1 modules]

(B) If M has ≥ 4 (quasi-) boxes or 3 (quasi-)boxes which are not all boxes $\Rightarrow$ M is not rigid: Use collapsing to reduce to $\mathcal{F}_{k,2k}$ (stretching to induce to $\mathcal{F}_{k',n'}$, $k \leq k'$, $n \leq n'$).

$n = 2k$



Idea: $\mathbb{M}_I / \mathbb{M}_J$ is rigid indec. $\Leftrightarrow$ $\mathbb{M}_{I'} / \mathbb{M}_{J'}$ is rigid indec.

for I, J k-subsets of $\{1, \dots, n\}$ for $I' = I \setminus (I \cap J \cup I^c \cap J^c)$

$J' = J \setminus (I \cap J \cup I^c \cap J^c)$

(Le - Yildirim)

4 box or 2 box, 1 quasibox: from tube in $\mathcal{F}_{k,2k}$ w. proj-inj's (below).
Or use:

<u>Theorem (B Bogdanic–Li 2021)</u>: In $\mathcal{F}_{4,8}$ there exists a 1-parameter family of indec. rk 2 modules w. profile $\{1,3,5,7\} \mid \{2,4,6,8\}$.

$\longrightarrow$ such modules are not rigid.
One can check: the syzygy of such a module has the same profile.

$$\beta \notin \{0, \pm 1, \pm 2\}$$

$\longrightarrow$ Take $V_i = Z \oplus Z$, $i = 1, \dots, 8$. M_β is defined:

$$V_1 \underset{\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}}{\overset{\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}}{\rightleftarrows}} V_2 \underset{\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}}{\overset{\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}}{\rightleftarrows}} V_3 \underset{\begin{pmatrix} t & \beta \\ 0 & 1 \end{pmatrix}}{\overset{\begin{pmatrix} 1 & -\beta \\ 0 & t \end{pmatrix}}{\rightleftarrows}} V_4 \underset{\begin{pmatrix} 1 & -\beta \\ 0 & t \end{pmatrix}}{\overset{\begin{pmatrix} t & \beta \\ 0 & 1 \end{pmatrix}}{\rightleftarrows}} V_5 \underset{\begin{pmatrix} t & 2 \\ 0 & 1 \end{pmatrix}}{\overset{\begin{pmatrix} 1 & -2 \\ 0 & t \end{pmatrix}}{\rightleftarrows}} V_6 \underset{\begin{pmatrix} 1 & -2 \\ 0 & t \end{pmatrix}}{\overset{\begin{pmatrix} t & 2 \\ 0 & 1 \end{pmatrix}}{\rightleftarrows}} V_7 \underset{\begin{pmatrix} t & 1 \\ 0 & 1 \end{pmatrix}}{\overset{\begin{pmatrix} 1 & -1 \\ 0 & t \end{pmatrix}}{\rightleftarrows}} V_8 \underset{\begin{pmatrix} 1 & -1 \\ 0 & t \end{pmatrix}}{\overset{\begin{pmatrix} t & 1 \\ 0 & 1 \end{pmatrix}}{\rightleftarrows}} V_1$$

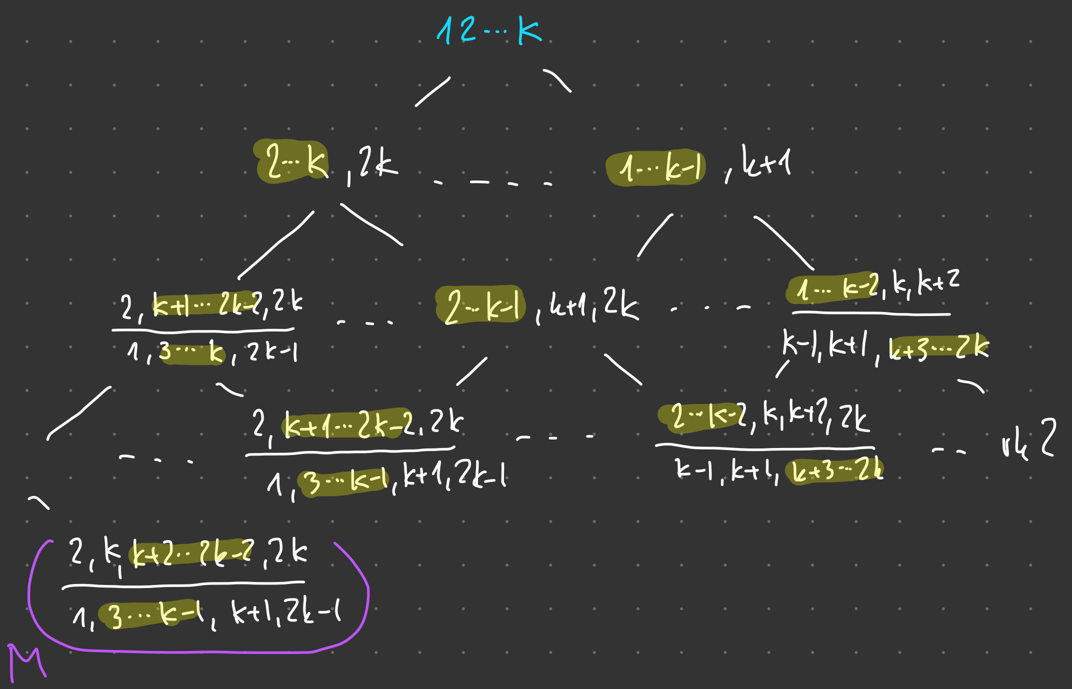
apart from $\mathcal{F}_{3,9}$ and $\mathcal{F}_{4,8}$ (and infinite): M in tube of rank m, M in rows $1,\dots,m-1$ ✱) M rigid :

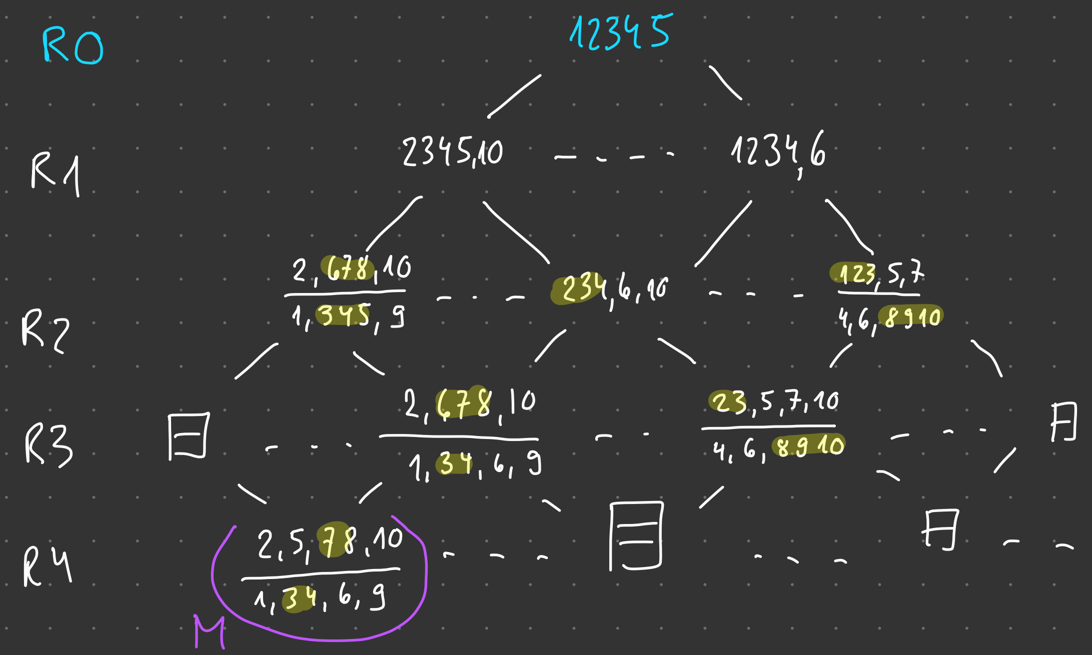
Example $\mathcal{F}_{k,2k}$ ($\mathcal{F}_{3,6}$ finite, $\mathcal{F}_{4,8}$ tame) Consider a tube w. pg.-inj.'s :

(use BBogd-GE to compute AR-sequences near mouth)

$\boxed{k=5}$

tube of rk 4





Profile of M:

: Not rigid

Can stretch, e.g. to $\mathcal{F}_{5,11}$ ⇝ tube has rank 11, M in R4, not rigid.