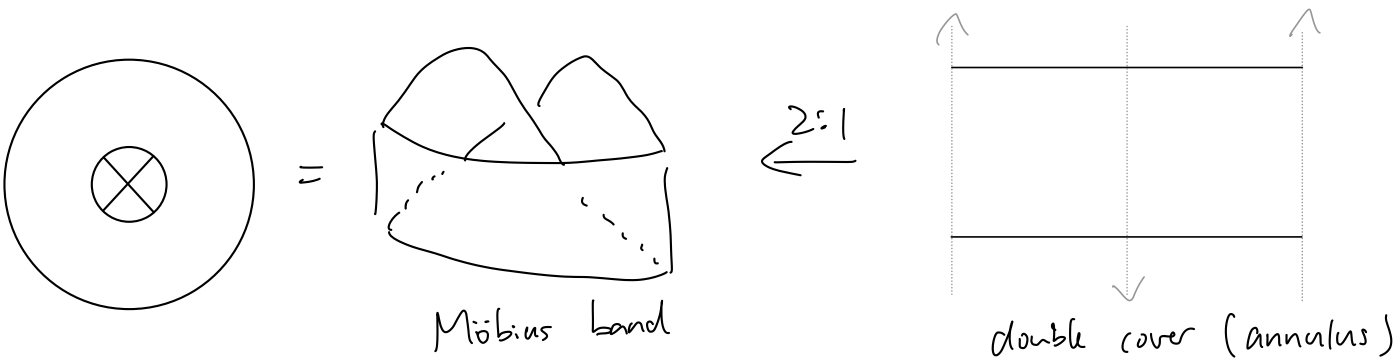
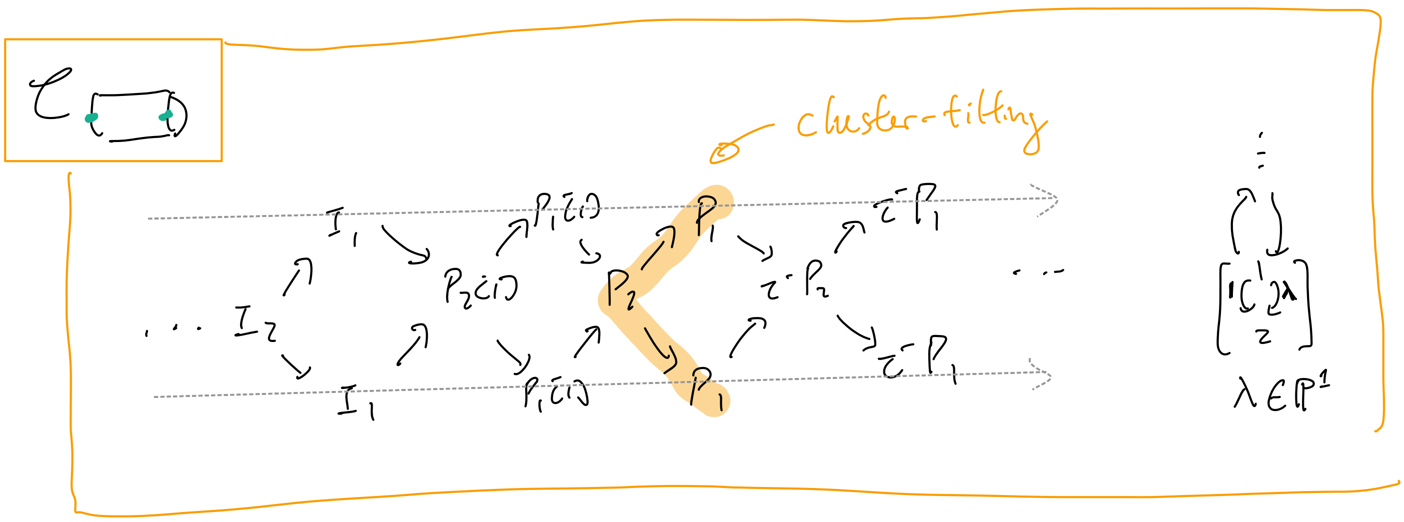
Categorification of unpunctured non-orientable marked surfaces

work in progress, jt. with V. Bazier-Matte, K. Wright





★ Green texts = extra info

Summary

(Known):

Topology		Category
Orientable surface w/ triangulation	$\overset{\mathrm{I}}{\xleftrightarrow{1:1}}$	gentle Jacobian algebra
		$\rightsquigarrow$ cluster category
		$\rightsquigarrow$
$\rightsquigarrow$ curves (+ ...)	$\overset{\mathrm{II}}{\xleftrightarrow{1:1}}$	indec. (families of) obj's
triangulations	$\overset{\mathrm{III}}{\xleftrightarrow{1:1}}$	cluster-tilting obj.'s

$\rightsquigarrow$ IV

Categorification of mutation (= flip) and associated cluster algebra [Fomin-Shapiro-Thurston, Fock-Goncharov]

Question : Non-orientable surfaces (NoS) ?

Idea : $\exists$ orientable double cover $\tilde{S}$ for an NoS S.

i.e., $S = \tilde{S}/\sigma$ $\quad$ σ : orientation-reversing auto.

$\sigma^2 = 1$

Today : Goal I, II, "III"

§1 Surface topology

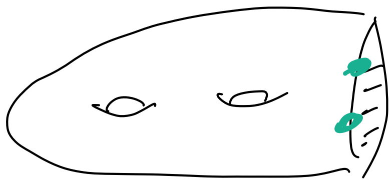
Setup

$\mathbb{S}$: Surface = compact 2-dim^{l}/$\mathbb{R}$ w/ non-empty boundary $\partial\mathbb{S} \neq \phi$

$\mathbb{M}$: finite set of marked points in $\partial\mathbb{S}$ ($\therefore$ unpunctured)

s.t. each boundary component contains ≥ 1 marked pt.

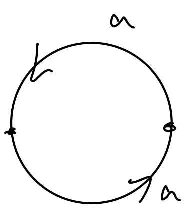
and $(\mathbb{S}, \mathbb{M}) \neq$ 1, 2, 3-gon



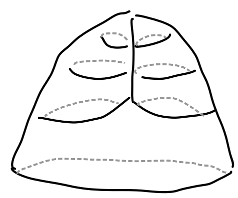
$|\mathbb{M}| = 2.$

§1.1 Working with non-orientable surfaces (NoS)

- $\mathbb{RP}^2$ =



- crosscap := $\mathbb{RP}^2 \setminus$ disc

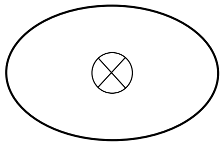
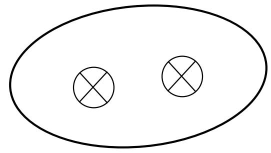


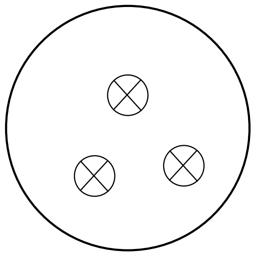
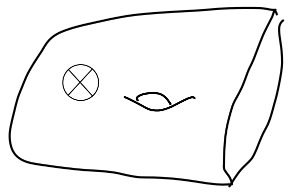
$\overset{\text{homeo}}{\simeq}$ Möbius strip,

In practice, represent by the symbol $\otimes$ (or , , etc.)

Rmk: Some literature call $\mathbb{RP}^2$ the crosscap instead.

E.g.

 = Möbius strip ,  = Klein bottle \ disc ,

 =  = Dyck's surface \ disc .

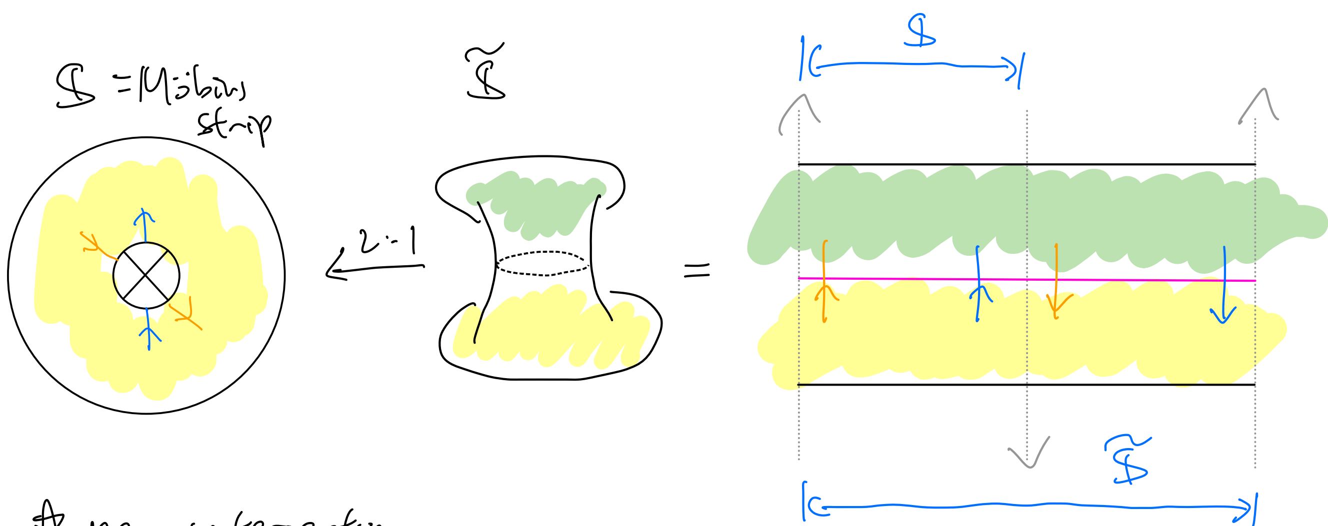
§1.2 Double cover

$$\boxed{\text{(N)OS } (\mathbb{S},\mathbb{M}) \longleftrightarrow (\tilde{\mathbb{S}},\mathbb{M})/\sigma}$$

where

$(\tilde{\mathbb{S}},\mathbb{M})$:= Orientable double cover of a marked nos. $(\mathbb{S},\mathbb{M})$

σ : orientation-reversing auto. of order 2.



✡ no intersection between the 2 curves shown!

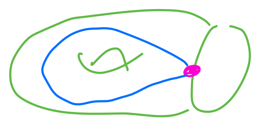
§1.2 Objects of interest

γ: **Curve** on $(\mathcal{S}, \mathbb{M})$ $:\Longleftrightarrow$ either

- **closed** $\gamma \simeq S^1$ s.t. $\gamma \cap \mathbb{M} = \phi$, non-contractible

or

- **non-closed** $\gamma : [0,1] \to \mathcal{S}$, $\gamma(0), \gamma(1) \in \mathbb{M}$, $\gamma(0,1) \subset \mathcal{S} \setminus \partial \mathcal{S}$



$\in$

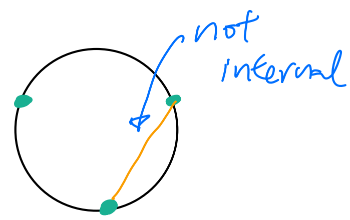
* Always considered up to isotopies that fix $\partial \mathcal{S}$ pointwise

Non-crossing (set of) curves $:\Longleftrightarrow$ no intersection except possibly at endpoints.

(Non-crossing = NC)

- **Arc** $:\Longleftrightarrow$ NC non-closed curves

- **Internal arc** $:\Longleftrightarrow$ arc $\not\simeq$ boundary interval

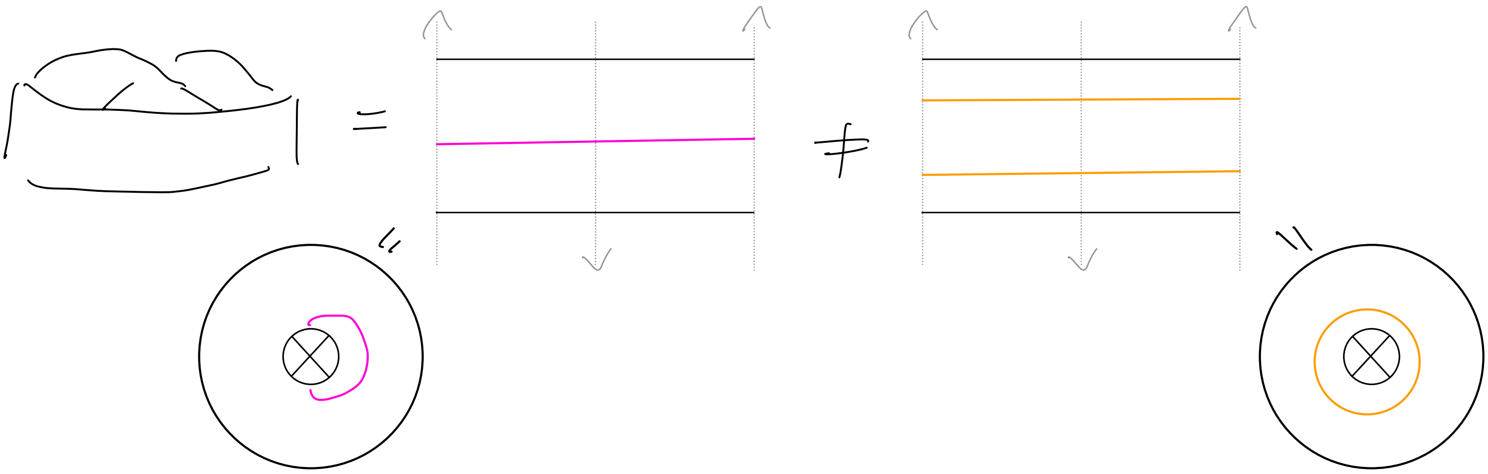


- **1-sided** closed curve $:\Longleftrightarrow$ non-orientable closed curve

  (If simple, then equiv. to $\exists$ regular nbhd $\underset{\text{homeo}}{\cong}$ Möbius strip)

- **2-sided** closed curve $:\Longleftrightarrow$ not 1-sided

  (If simple, then equiv. to $\exists$ regular nbhd $\underset{\text{homeo}}{\cong}$ annulus)



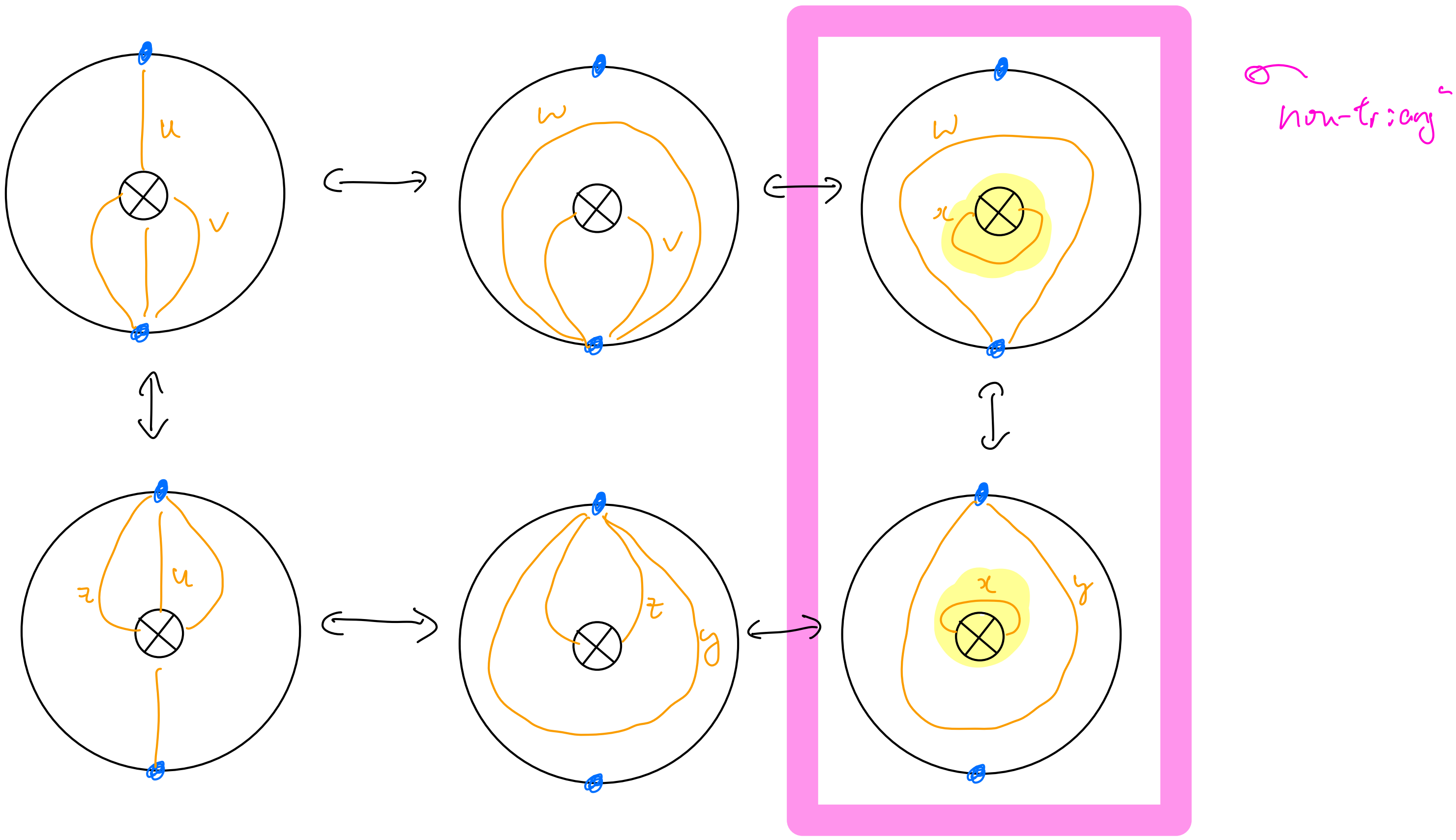
- **Quasi-arc** $:\Longleftrightarrow$ either internal arc,

  or 1-sided simple closed curve

  "no self-intersection"

- **(quasi-)triangulation** $:\Longleftrightarrow$ maximal NC set of (quasi-)arcs

E.g. M_2 (:= Möbius strip w/ 2 marked pt's) has 6 quasi-triang"'s.



§§2) Triangulation vs QP

2.1) Orientable case $(\tilde{S}, \tilde{M})$, $\tilde{T}$

$\rightsquigarrow$ (Q, W) : QP (Quiver with Potential)

where
$$\begin{cases} Q_0 = \{\text{internal arcs of } \tilde{T}\} \\ Q_1 = \text{CW oriented angles of triangles of } \tilde{T} \\ W = \sum\limits_{\substack{\text{internal} \\ \text{triangles}}} \end{cases}$$

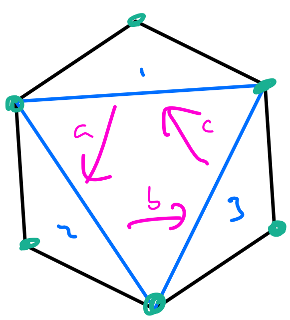
$\rightsquigarrow$ Jacobian algebra $J_{\tilde{T}} = \mathrm{Jac}(Q, W) := \Bbbk Q / (\partial_a W) = \Bbbk Q / (\text{length 2 paths in int. } \Delta\text{'s})$

N.B. $J_{\tilde{T}}$ is f.d. and a gentle algebra

Convention/Definition: QP is gentle if induced by a triang^{n}.

E.g.

$(\tilde{S}, \tilde{M})$, $\tilde{T}$:



$$Q = \begin{tikzcd} & 1 \arrow[dr, "a"] & \\ 3 \arrow[ur, "c"] & & 2 \arrow[ll, "b"] \end{tikzcd}, \quad W = abc$$

$$J_{\tilde{T}} = \begin{tikzcd} & 1 \arrow[dr] & \\ 3 \arrow[ur] & & 2 \arrow[ll] \end{tikzcd} \Big/ (ab, bc, ca)$$

Thm [Assem, Brüstle, Charbonneau-Jodoin, Plamondon]

$$\left\{ (\tilde{S}, \tilde{M}; \tilde{T}) : \begin{array}{c}\text{triangulated} \\ \text{ori. surfaces}\end{array} \right\} \overset{1:1}{\longleftrightarrow} \{ (Q, W) : \text{gentle QP} \}$$

2.2) NoS case $\quad (S, M) \xleftarrow{2:1} (\tilde{S}, \tilde{M}) \circlearrowleft \sigma$

Def ① Q : quiver

An <u>involution</u> $\sigma : Q \to Q$

In [Derksen-Weyman] language: (Q, σ) is a <u>symmetric quiver</u>

$:\Leftrightarrow \quad \sigma = (\sigma_0 : Q_0 \to Q_0,\ \sigma_1 : Q_1 \to Q_1)$

$$\text{s.t.} \begin{cases} \sigma^2 = 1 \\ \sigma(v \xrightarrow{\alpha} w) = \left(\sigma(v) \xleftarrow{\sigma(\alpha)} \sigma(w)\right) \end{cases}$$

<u>N.B.</u> Specifying involution $\sigma \Rightarrow \exists \left(\Bbbk Q \overset{\text{alg}}{\xrightarrow{\sim}} \Bbbk Q^{op} \right)$

② An <u>involution on a QP</u> (Q, W).

$:\Leftrightarrow$ an involution σ on Q s.t. $\sigma W = W$

$\sigma W = \pm W$ OK

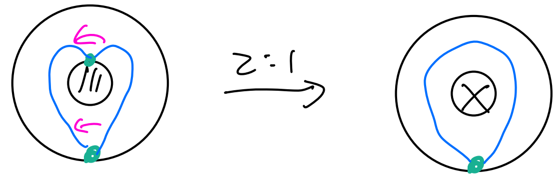
$\Leftrightarrow$ ——— '' ——— s.t. $\sigma(\partial_a W) = (\partial_a W)$

Eg.

① $1 \bullet \xrightarrow{\alpha} \bullet 2 \qquad \sigma : \begin{array}{c} 1 \longleftrightarrow 2 \\ \alpha \longleftrightarrow \alpha \end{array}$

② $1 \bullet \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \bullet 2 \qquad \sigma : \begin{array}{c} 1 \longleftrightarrow 2 \\ \alpha \longleftrightarrow \alpha \\ \beta \longleftrightarrow \beta \end{array}$

(3) $\underset{1}{\bullet} \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \underset{2}{\bullet}$ $\qquad \sigma ; \begin{array}{l} 1 \longleftrightarrow 2 \\ \alpha \longleftrightarrow \beta \end{array}$



(4) Exercise: $A_{n,n}$-quiver

$$\begin{tikzcd}[column sep=small, row sep=small]
 & 2 \arrow[r] & \cdots \arrow[r] & n \arrow[dr] & \\
1 \arrow[ur] \arrow[dr] & & & & n+1 \\
 & 2n \arrow[r] & \cdots \arrow[r] & n+2 \arrow[ur] &
\end{tikzcd}$$

has an involution

Obs: The setup $\sigma \circlearrowright (\widetilde{S, M}) \xrightarrow{2:1} (S, M)$

induces an involution σ on (Q, W)

Moreover, σ is fixed-point free (FPF)

$\sigma(v) \neq v \quad \forall v \in Q_0$

$\sigma(\alpha) \neq \alpha \quad \forall v \in Q_1$

Prop [BM-C-W] (Goal I)

$$\left\{ \begin{array}{l} \text{Connected NoS} \\ \text{with triangulation} \end{array} \right\} /\text{homeo} \quad \overset{1:1}{\longleftrightarrow} \quad \left\{ \begin{array}{l} (Q, W; \sigma) \text{ s.t. } (Q, W) \text{ gentle,} \\ Q \text{ connected, } \sigma \text{ is a FPF involution} \end{array} \right\} / \text{"}\cong\text{"}$$

$$(S, M; T) \longmapsto (Q, W; \sigma)$$

Rmk "Same" argument works for similar setting

e.g. · locally gentle quivers vs dissections

· skew-gentle vs orbifold dissection

§3 Triangulation vs CTO (cluster-tilting obj)

Setup: From now on, $\boxed{\Bbbk = \text{complex numbers}}$

Orientable case

See [Brüstle-Zhang]

TOPOLOGY	CATEGORY
$(\tilde{S}, \tilde{M})$	cluster category $\mathcal{C} = \mathcal{C}_{(\tilde{S}, \tilde{M})} \circlearrowleft [1] \cong \tau$
non-closed curves	"string objects"
$\left(\text{closed curves}\right) \times \Bbbk^{\times}$	"band objects"

N.B. This is KS Hom-fin. 2-CY tri. cat.

all indec's of $\mathcal{C}_{(S,M)}$

Elements are (ω^n, λ), where ω: primitive, i.e. $\omega \neq \rho^k$ in $\pi_1(\tilde{S})$ with $k > 1$

Moreover,

TOPOLOGY	CATEGORY
Crossing between curves (+ loc. sys.)	Canonical basis (+ local sys. hom.) of $\mathrm{Ext}^1_{\mathcal{C}}(-,-) := \mathrm{Hom}_{\mathcal{C}}(-,-[1])$
arcs	rigid objects $:=$ no self-ext^n ($\mathrm{Ext}^1 = 0$)
triangulation	cluster-tilting object (CTO) $:=$ maximal rigid object. (i.e. $M \oplus N$ rigid $\Rightarrow N \in \mathrm{add}\, M$)

$\boxed{\text{Slogan: Crossings = Non-split extensions}}$

Prop [BM-C-W]

$\exists$ exact contravariant duality functors ∇ (i.e. $\nabla^2 \cong \mathrm{Id}$)

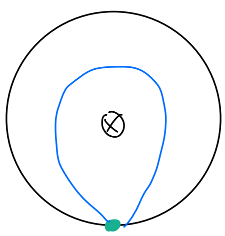
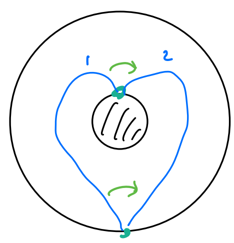
$$\begin{tikzcd} \mathcal{C} \arrow[r, "\nabla", "\sim"'] \arrow[d, "\Downarrow"] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{C} \arrow[d, "\Downarrow"] \\ \mathrm{mod}\, J_T \arrow[r, "\nabla", "\sim"'] & \mathrm{mod}\, J_T \end{tikzcd}$$

such that

① $\forall \gamma$: non-closed on $\widetilde{(S,M)}$, $\overbrace{\sigma(\gamma)}^{\text{Topology}} \longleftrightarrow \overbrace{\nabla(\gamma)}^{\text{Category}}$

② $\forall (\omega^n, \lambda) \in \left\{ \begin{matrix} \text{c.c.'s} \\ \text{on } \widetilde{(S,M)} \end{matrix} \right\} \times k^\times$, $(\sigma(\omega)^n, \lambda^{-1}) \longleftrightarrow \nabla(\omega^n, \lambda)$

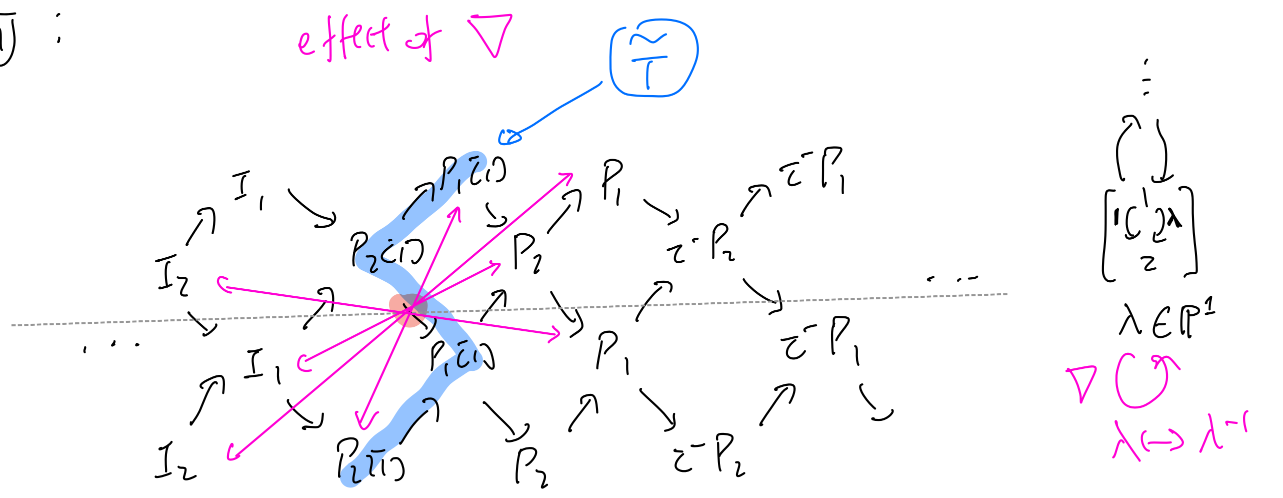
E.g.

$(S, M; T) =$  $\xleftarrow{2:1}$ $\widetilde{(S, M, T)} =$ 

$Q = \left(1 \underset{b}{\overset{a}{\rightrightarrows}} 2 \right)$, $W = 0$

σ : $1 \longleftrightarrow 2$, $a \longleftrightarrow b$

$\mathcal{C}_{\widetilde{(S,M)}}$:



Thm [BM-C-W] (Partial Goal II, III)

$\forall$ NoS $(\mathbb{S}, \mathbb{M})$, have

$$\{\text{non-closed curves on } (\mathbb{S},\mathbb{M})\} \xleftrightarrow{1:1} \{\nabla\gamma \oplus \gamma \mid \gamma : \text{string obj}\}$$

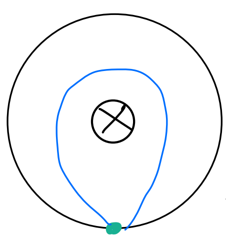
which induces

$$\left\{\begin{matrix}\text{triangulations}\\ \text{of } (\mathbb{S},\mathbb{M})\end{matrix}\right\} \xleftrightarrow{1:1} \left\{\begin{matrix}\text{basic CTO } M \in \mathcal{C}\\ \text{s.t. } \underbrace{\nabla M \cong M}_{\Leftarrow \text{ self-dual}}\end{matrix}\right\}$$

Rmk • This is mutation compatible (as long as it is possible.)

• Mutation send corresp. s.τ-tilting pair to incomparable s.τ-tilting pair.

E.g. $(\mathbb{S},\mathbb{M}) = M_1$

$\Rightarrow$ LHS $= \{$  $\}$, RHS $= \{$initial CTO$\}$

$\xleftrightarrow{1:1} \{\Lambda[1]\} \subset s\tau\text{-tilt}(\Lambda)$

What about quasi-triangulations?

and their mutations?

§§4 Symmetric representations (= ε-representations)

$\varepsilon = \{\pm 1\}$

orthogonal ($\varepsilon = +1$) [Derksen-Weyman]

symplectic ($\varepsilon = -1$)

[Boos-Cerulli Irelli]

Throughout this section,

- $(A = \Bbbk Q/I, \sigma)$: σ involution on Q fixing I.
- modules = right f.d. modules

Def

1) $\underline{\varepsilon\text{-form}}$ $:\Leftrightarrow$ $\begin{cases} \cdot \text{ symmetric bilinear form if } \varepsilon = +1, \\ \cdot \text{ skew-symm. bilinear form if } \varepsilon = -1. \end{cases}$

2) An $\underline{\varepsilon\text{-representation}}$ over (A, σ)

$:\Leftrightarrow$ · M : oridinary rep^n.

· $\langle -, - \rangle : M \times M \to \Bbbk$ non-degen. ε-form

s.t. $\begin{cases} \cdot \langle m e_i, n e_j \rangle = 0 \quad \forall j \neq \sigma(i) \text{ ; } e_i, e_j \text{ : primitive idem's.} \\ \cdot \langle m\alpha, n \rangle + \langle m, n\sigma(\alpha) \rangle = 0 \\ \quad (\text{i.e., } \sigma(\alpha) \text{ is adjoint of } \alpha) \end{cases}$

☆ M: ε-rep^n $\Rightarrow \exists\, \psi_M : M \xrightarrow{\sim} \nabla M$ as A-module s.t. $\nabla(\psi_M) = \varepsilon \psi_M$

☆ Indecomposability makes sense for ε-repⁿ's.

"Slogan"

$$\boxed{\varepsilon\text{-rep}^n \approx \begin{array}{l}\text{anti-version of}\\ \text{rep}^n \text{ over skew-group ring } \Lambda * \mathbb{Z}/2\end{array}}$$

Prop [D-W, B-CI] (Characterisation of indec ε-repⁿ's)

M : indec. ε-repⁿ $/ (A, \sigma)$

$\Rightarrow \exists\ \overline{M}$: indec. A-module

s.t. exactly one of the following holds.

a) $\nabla\overline{M} \not\cong \overline{M}$, $M = \overline{M} \oplus \nabla\overline{M}$ → call M split

b) $\nabla\overline{M} \cong \overline{M}$, $M = \overline{M} \oplus \nabla\overline{M}$ → call M ramified

c) $\nabla\overline{M} \cong \overline{M}$, $M = \overline{M}$ → call M ~~type I~~ 1-sided

Notation: ω: primitive cc, $\ell(\omega) := \# \omega \cap \tilde{\tau}$.

Thm [BM-C-W]

M: indec ε-rep^n / $\mathrm{Jac}(Q,W)$, (Q,W): gentle FPF-Symm. QP

$\Rightarrow M \cong$ exactly one of the following.

split
- $M(\gamma) \oplus M(\sigma(\gamma))$ — all strings
- $M_\lambda(\omega^n) \oplus M_{\lambda^{-1}}(\sigma(\omega)^n)$ s.t. $\begin{cases} \omega \neq \sigma(\omega) \text{ or } \lambda \notin \{\pm 1\}, \\ \text{any } n \geq 1. \end{cases}$

1-sided
- $M_\lambda(\omega)$ s.t. $\sigma(\omega) = \omega$ & $\varepsilon = (-1)^{\ell(\omega)/2} \lambda$
- $M_\lambda(\omega^2)$ s.t. $\sigma(\omega) = \omega$ & $\varepsilon = (-1)^{\frac{\ell(\omega)}{2}+1} \lambda$

ramified
- $M_\lambda(\omega^n)^{\oplus 2}$ s.t. all remaining cases

— all band

Hence

"Indec" top. obj. — Indec ε-rep^n's

$\{$1-sided c.c. ω on $(S,M)\}$ $\xleftrightarrow{1:1}$ $\{$1-sided ε-indec $M_{\varepsilon_\omega}(\omega)\}$

$\updownarrow$ 1:1

$\{$2-sided c.c. ω on $(S,M)\}$ $\xleftrightarrow{1:1}$ $\{$ramified ε-indec $M_{-\varepsilon_\omega}(\omega)^{\oplus 2}\}$

$\{$non-closed curve γ on $(S,M)\} \setminus \tilde{\tau}$ $\xleftrightarrow{1:1}$ $\{$split ε-indec "of strong type" $M(\gamma) \oplus \nabla M(\gamma)\}$

$\rightarrow$ Goal II ✓

§§5 ε-extension, ε-rigid

Go back to $\mathcal{C} = \mathcal{C}_{(\mathbb{S},\mathbb{M})}$.

Fact: $\begin{cases} \tilde{T} : \text{triang. on } (\tilde{\mathbb{S}}, \tilde{\mathbb{M}}), \quad \Lambda := \mathrm{Jac}(Q, W) \\ \Rightarrow C := \tilde{T}[-1] \text{ is a CTO} \\ \quad \& \ \pi := \mathrm{Hom}_{\mathcal{C}}(C, -) : \mathcal{C}/_{[\tilde{T}]} \xrightarrow{\sim} \mathrm{mod}\,\Lambda \end{cases}$

Fix $\varepsilon \in \{\pm 1\}$, T : triang^n.

Def:

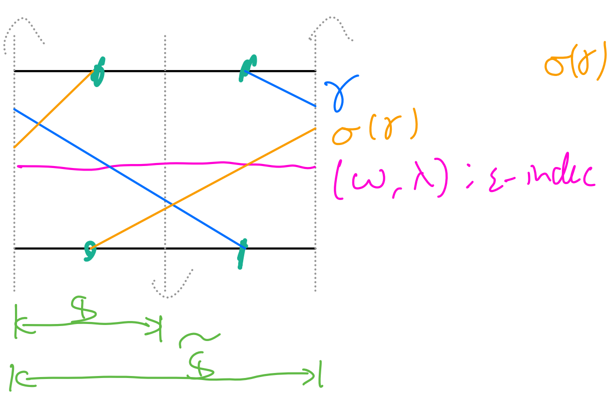
- $X \in \mathcal{C}$ is an **indec. ε-object** $:\Leftrightarrow$ • πX an indec. ε-rep^n/Λ
  or
  • $X = \alpha \oplus \underset{\nabla(\alpha)}{\sigma(\alpha)}$, some $\alpha \in \tilde{T}$

- In this case,

① $\exists\ (\nabla\bar{X} \to X \to \bar{X} \to)$: $\triangle$ in $\mathcal{C}$ with $\bar{X}$: indec

we call any such $\bar{X}$ **ε-factor** N.B. not unique !

e.g. $\mathbb{S} = M_2$:



$\sigma(\gamma) \to (\omega,\lambda) \to \gamma \to$
$\quad\quad\quad\quad\quad\quad\quad\ \ \overline{(\omega,\lambda)} = \gamma$

② Isom. $X \underset{\Psi_X}{\xleftarrow{\sim}} \nabla X$ s.t. $\nabla(\Psi_X) = \varepsilon \Psi_X$

- $X \in \mathcal{C}$ is an **ε-object** $:\Longleftrightarrow$ $X = \bigoplus$ indec ε-obj's.

- X, Y: ε-objects

  $f : X \to \overline{Y}[1]$ s.t. $\begin{cases} \overline{Y} : \varepsilon\text{-factor of } Y \\ f \circ \psi_X \circ \nabla f = 0 \end{cases}$

  $$\begin{tikzcd} \nabla\overline{Y}[-1] \arrow[d, "\nabla f"] & \\ X \arrow[r, "f"'] & \overline{Y}[1] \end{tikzcd}$$

  In this case, $\exists$ comm. dgm:

$$\begin{tikzcd} & & \nabla\overline{Y}[-1] \arrow[d, "\psi_X \nabla f"'] & \\ \overline{Y} \arrow[r] \arrow[d, equal] & C^f \arrow[r] \arrow[d] & X \arrow[r, "f"] \arrow[d] & \overline{Y}[1] \\ \overline{Y} \arrow[r] & E \arrow[r] \arrow[d] & C_{\psi_X \nabla f} \arrow[d] & \\ & \nabla\overline{Y} \arrow[r, equal] & \nabla\overline{Y} & \end{tikzcd}$$

s.t. all rows & all col^n's are Δ's of $\mathcal{C}$.

Fact: This is self-dual obj. in $\mathcal{C}$

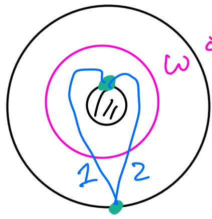
We call this E an **ε-extension of $\begin{bmatrix} X \\ Y \end{bmatrix}$**.

- An ε-extension **splits** $:\Longleftrightarrow$ $E \cong X \oplus \overline{Y} \oplus \nabla\overline{Y}$

- Write $\varepsilon\mathrm{Ext}^1(X, Y) = 0$ if all ε-ext^n of $\begin{pmatrix} X \\ Y \end{pmatrix}$ splits

- X is **indec. ε-rigid** $:\Longleftrightarrow$ $\begin{cases} X \text{ indec } \varepsilon\text{-obj} \\ \varepsilon\mathrm{Ext}^1(X, X) = 0. \end{cases}$

- X is **ε-rigid** $:\Longleftrightarrow$ $\begin{cases} \cdot\ X = \bigoplus \text{indec } \varepsilon\text{-rigid.} \\ \cdot\ \forall\ \varepsilon\text{-indec's } Y, Z \leq_{\oplus} X, \\ \quad Y \not\cong Z \Rightarrow \mathrm{Ext}^1_{\mathcal{C}}(Y, Z) = 0. \end{cases}$

Eg. $\mathcal{S} = M_1$, $\tilde{\mathcal{S}} =$  ← lift of the unique quasi-arc on $\mathcal{S}$

$\tilde{\gamma} \leftrightarrow (\alpha, \omega) = (1 \rightrightarrows 2, 0)$

$\Rightarrow \quad \pi\omega = 1 \begin{smallmatrix} 1 \\ (\\ 2 \end{smallmatrix} \lambda \qquad \varepsilon = -\lambda \in \{\pm 1\}$

$\pi\gamma = S_1$

$\pi\nabla\gamma = S_2$

$(\nabla\gamma)[-1] \to \omega \to \gamma[1]$

π: $\begin{smallmatrix} 1 & 1 \\ 2 & 2 & 2 \end{smallmatrix} \xrightarrow{\nabla f} \begin{smallmatrix} 1 \\ (\\ 2 \end{smallmatrix} \xrightarrow{f} \begin{smallmatrix} 2 & 2 & 2 \\ 1 & 1 \end{smallmatrix}$

If $f \neq 0$, this composition is <u>non-zero</u>.

$\Rightarrow$ only split extensions everywhere.

$\Rightarrow$ ε-rigid.

Goal III

<u>Conj / "Thm"</u> [IM-C-W] ("Baby" categorification of <u>quasi</u>-triang")

$\left\{ \text{indec } \varepsilon\text{-rigid} \right\} \xleftrightarrow{1:1} \left\{ \text{quasi-arc of } (\mathcal{S}, M) \right\}$

which then induces

$\left\{ \text{maximal } \varepsilon\text{-rigid obj} \right\} \xleftrightarrow{1:1} \left\{ \text{quasi-triangulations of } (\mathcal{S}, M) \right\}$

Expectations

1) Define <u>strictly ε-rigid obj</u> X $:\Leftrightarrow$ $\varepsilon\mathrm{Ext}^1(X,X)=0$

$\Rightarrow$ (maximal ε-rigid $\Leftrightarrow$ maximal <u>strictly</u> ε-rigid)

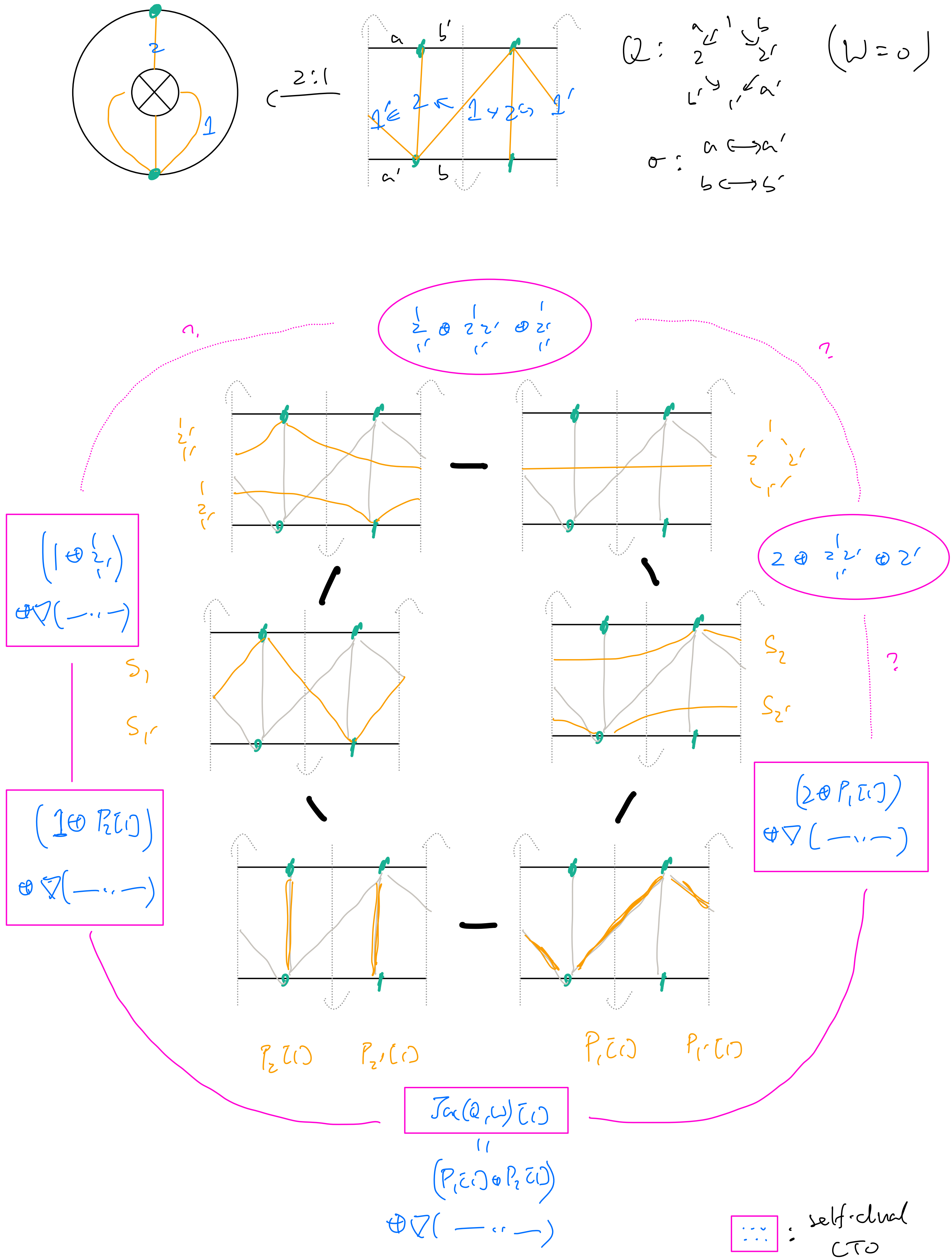
2) ε-ext^{n}'s / equiv.?? count crossings on $(\mathbb{S}, \mathbb{M})$

3) Exchange relation/Mutation of quasi-triangulation:

$$\alpha \in T \overset{\mu}{\longleftrightarrow} (T \setminus \{\alpha\}) \cup \{\alpha'\}.$$

is categorified by existence of some "special" ε-extension / equiv.??.

Part of Goal $\underline{\mathrm{IV}}$.


2:1
Q:
(W=0)
σ: a ⟷ a'
b ⟷ b'
?
S1
S1'
S2
S2'
P2[1]
P2'[1]
P1[1]
P1'[1]
Jac(Q,W)[1]
(P1[1] ⊕ P2[1]) ⊕ ∇(— ,, —)
(1 ⊕ P2[1]) ⊕ ∇(— ,, —)
(2 ⊕ P1[1]) ⊕ ∇(— ,, —)
: self-dual CTO