GENERALISED CLUSTER CATEGORIES FROM n-CY TRIPLES

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1. CLUSTER THEORY

- Fomin & Zelevinsky (2002): cluster algebras

$\rightsquigarrow$ categorifications:

- Buan-Marsh-Reineke-Reiten-Todorov (2006):

  $K = \bar{K}$ field, Q: finite quiver with no loops/cycles
  cluster category of Q: the orbit category

  $$\mathcal{C}_Q := D^b(\operatorname{mod} KQ) / (\tau^{-1} \circ \Sigma)^{\mathbb{Z}}$$

- Amiot (2009) & Guo (2011):

  Chain dg algebra

  $$A = \cdots \longrightarrow A_2 \longrightarrow A_1 \longrightarrow A_0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

  with some properties (A is 3-CY bimodule)
  generalised cluster category:

  Verdier quotient $\mathcal{C}_A := \operatorname{per} A / D^b(A)$

Main properties: (1) $\mathcal{C}_A$ triangulated, Hom-finite, 2-Calabi-Yau

(2) $\mathcal{C}_A$ has 2-cluster tilting object A

(3) $\mathrm{Hom}_{\mathcal{C}_A}(A,A) \cong \mathrm{Hom}_{\mathrm{per}A}(A,A)$

- Guo: higher case (n instead of 2 above)

2. IYAMA-YANG GENERALISATION (2018)

Recall

$\mathcal{T}$ triangulated

→ torsion pair on $\mathcal{T}$: pair $(\mathcal{X},\mathcal{Y})$ of full subcategories of $\mathcal{T}$ s.t.

$$\mathcal{X} = {}^{\perp_{\mathcal{T}}}\mathcal{Y}, \quad \mathcal{Y} = \mathcal{X}^{\perp_{\mathcal{T}}}, \quad \mathcal{X} * \mathcal{Y} = \mathcal{T}$$

→ t-structure on $\mathcal{T}$: pair $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ of full subcategories of $\mathcal{T}$ s.t.

- $\mathcal{T}^{\geq 1} := \Sigma^{-1}\mathcal{T}^{\geq 0} \subseteq \mathcal{T}^{\geq 0}$
- $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 1})$ torsion pair

NB: $\forall m \in \mathbb{Z}$, $(\mathcal{T}^{\leq m} := \Sigma^{-m}\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq m} := \Sigma^{-m}\mathcal{T}^{\geq 0})$ is a t-structure.

→ let $n \geq 3$, k field, $(\mathcal{T}, \mathcal{T}^{fd}, \mathcal{M})$ an n-Calabi-Yau triple:

- $\mathcal{T}$: k-linear, Hom-finite, Krull-Schmidt Δed cat
- $\mathcal{T}^{fd} \subseteq \mathcal{T}$: Δed subcategory s.t. $(\mathcal{T}, \mathcal{T}^{fd})$ is relative n-Calabi-Yau:

$$D\mathcal{T}(X, Y) \simeq \mathcal{T}(Y, \Sigma^n X)$$

  bifunctorial isomorphism $\forall X \in \mathcal{T}^{fd}, Y \in \mathcal{T}$.

- $\mathcal{M} = \operatorname{add} M \subseteq \mathcal{T}$ silting subcategory:

$$\mathcal{T}(\mathcal{M}, \Sigma^{>0}\mathcal{M}) = 0 \quad \& \quad \mathcal{T} = \operatorname{thick}(\mathcal{M})$$

  with right adjacent t-structure

$$(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}) = ((\Sigma^{<0}\mathcal{M})^{\perp_{\mathcal{T}}}, (\Sigma^{>0}\mathcal{M})^{\perp_{\mathcal{T}}})$$

Then I-Y generalised cluster category:

the Verdier quotient $\mathcal{T}/\mathcal{T}^{fd}$

Main properties : (1) $\mathcal{T}/\mathcal{T}^{fd}$ triangulated, Hom-finite, $(n-1)$-CY

(2) $\mathcal{T}/\mathcal{T}^{fd}$ has $(n-1)$-ct object M

(3) $\mathrm{Hom}_{\mathcal{T}/\mathcal{T}^{fd}}(M,M) \cong \mathrm{Hom}_{\mathcal{T}}(M,M)$

Goal : Give deeper understanding of $\mathcal{T}/\mathcal{T}^{fd}$ using more classic homological tools

3. Hom-SPACES :

$\forall p > 0$, $X \in \mathcal{T}$, $\exists !$ (up to unique iso) truncation triangle

$$\underbrace{X^{\leq -p}}_{\in \mathcal{T}^{\leq -p}} \xrightarrow{f^{-p}} X \longrightarrow \underbrace{X^{\geq -p+1}}_{\in \mathcal{T}^{\geq -p+1}} \longrightarrow \Sigma X^{\leq -p}$$

There are morphisms of triangles

$$\begin{tikzcd}
X^{\leq -p} \arrow[r, "f^{-p}"] \arrow[d, "\xi^{-p}"'] & X \arrow[r] \arrow[d, equal] & X^{\geq -p+1} \arrow[r] \arrow[d, dashed] & \Sigma X^{\leq -p} \arrow[d] \\
X^{\leq -p+1} \arrow[r, "f^{-p+1}"] & X \arrow[r] & X^{\geq -p+2} \arrow[r] & \Sigma X^{\leq -p+1}
\end{tikzcd}$$

Then, we have inverse system

$$\cdots \longrightarrow X^{\leq -2} \xrightarrow{\xi^{-2}} X^{\leq -1} \xrightarrow{\xi^{-1}} X^{\leq 0} \xrightarrow{f^0} X$$

For $Y \in \mathcal{T}$, applying $\mathcal{T}(-, Y)$, we get direct system:

$$\mathcal{T}(X, Y) \longrightarrow \mathcal{T}(X^{\leq 0}, Y) \longrightarrow \mathcal{T}(X^{\leq -1}, Y) \longrightarrow \cdots \longrightarrow \varinjlim_{q} \mathcal{T}(X^{\leq -q}, Y)$$

- Over $\mathcal{T}/\mathcal{T}^{fd}$:

$\forall p \geq 0$, the truncation triangle

$$X^{\leq -p} \xrightarrow{f^{-p}} X \longrightarrow X^{\geq -p+1} \longrightarrow \Sigma X^{\leq -p}$$

$\rightsquigarrow X^{\leq -p} \cong_{\mathcal{T}/\mathcal{T}^{fd}} X$

Direct system

$$\mathcal{T}/\mathcal{T}^{fd}(X, Y) \xrightarrow{\sim} \mathcal{T}/\mathcal{T}^{fd}(X^{\leq 0}, Y) \xrightarrow{\sim} \mathcal{T}/\mathcal{T}^{fd}(X^{\leq -1}, Y) \xrightarrow{\sim} \cdots \xrightarrow{\sim} \varinjlim_{q} \mathcal{T}/\mathcal{T}^{fd}(X^{\leq -q}, Y)$$

We have commutative diagram

$$\begin{tikzcd}
\mathcal{T}(X,Y) \arrow[r] \arrow[d, "Q(-)"] & \mathcal{T}(X^{\leq 0},Y) \arrow[r] \arrow[d, "Q(-)"] & \mathcal{T}(X^{\leq -1},Y) \arrow[r] \arrow[d, "Q(-)"] & \cdots \arrow[r] & \varinjlim_q \mathcal{T}(X^{\leq -q},Y) \arrow[d, dashed, "\psi"] \\
\mathcal{T}/\mathcal{T}^{fd}(X,Y) \arrow[r, "\simeq"] & \mathcal{T}/\mathcal{T}^{fd}(X^{\leq 0},Y) \arrow[r, "\simeq"] & \mathcal{T}/\mathcal{T}^{fd}(X^{\leq -1},Y) \arrow[r, "\simeq"] & \cdots \arrow[r, "\simeq"] & \varinjlim_q \mathcal{T}/\mathcal{T}^{fd}(X^{\leq -q},Y)
\end{tikzcd}$$

- <u>Theorem</u> : Let $X, Y \in \mathcal{T}$.

(I) For $p \gg 0$, the direct system

$$\mathcal{T}(X,Y) \longrightarrow \mathcal{T}(X^{\leq 0},Y) \longrightarrow \mathcal{T}(X^{\leq -1},Y) \longrightarrow \cdots$$

stabilizes. Moreover

$$\varinjlim_q \mathcal{T}(X^{\leq -q},Y) \cong \mathcal{T}/\mathcal{T}^{fd}(X,Y)$$

(II) $\varprojlim_q \left(\varinjlim_p \mathcal{T}(\Sigma^{-1}X^{\geq -p+1}, Y^{\leq -q}) \right) \cong \mathcal{T}/\mathcal{T}^{fd}(X,Y)$

4. THE MAIN PROPERTIES:

(1) $\mathcal{T}/\mathcal{T}^{fd}$ triangulated, Hom-finite, $(n-1)$-Calabi-Yau:

- triangulated: $\mathcal{T}/\mathcal{T}^{fd}$ Verdier quotient
- Hom-finite: $\mathcal{T}$ Hom-finite & for $p \gg 0$ Theorem

$$\Rightarrow \mathcal{T}/\mathcal{T}^{fd}(X,Y) \cong \mathcal{T}(X^{\leq -p},Y)$$

- $(n-1)$-CY: Theorem + relative n-CY property

(3) $\mathrm{Hom}_{\mathcal{T}/\mathcal{T}^{fd}}(M,M) \cong \mathrm{Hom}_{\mathcal{T}}(M,M)$:

For $j \in \mathbb{Z}$, $Y = \Sigma^j M$ the direct system stabilizes at

$$p = j-n+2$$

$$\rightsquigarrow \mathcal{T}/\mathcal{T}^{fd}(X,\Sigma^j M) \cong \mathcal{T}(X^{\leq -j+n-2},\Sigma^j M)$$

$$\rightsquigarrow \mathcal{T}/\mathcal{T}^{fd}(M,M) \cong \mathcal{T}(M^{\leq n-2},M) \cong \mathcal{T}(M,M)$$

(2) $\underline{\mathcal{T}/\mathcal{T}^{fd}}$ <u>has $(n-1)$-cluster tilting object M</u>:

<u>ie</u> $\left[\begin{array}{l} \bullet\ \mathcal{T}/\mathcal{T}^{fd}(M, \Sigma^{1,\dots,n-2}M) = 0 \\ \bullet\ \mathcal{T}/\mathcal{T}^{fd} \cong \mathcal{M} * \Sigma\mathcal{M} * \cdots * \Sigma^{n-2}\mathcal{M} \end{array}\right.$

For $j = 1, \dots, n-2$, $\quad \mathcal{T}/\mathcal{T}^{fd}(M, \Sigma^j M) \cong \mathcal{T}(M^{\leq -j+n-2}, \Sigma^j M)$

- $-j+n-2 \geq 0 \implies \mathcal{T}^{\leq -j+n-2} \subseteq \mathcal{T}^{\leq 0}$ and so

$$\mathcal{T}/\mathcal{T}^{fd}(M, \Sigma^j M) \cong \mathcal{T}(M, \Sigma^j M)$$

- $j > 0$ and $\mathcal{M}$ silting $\implies \mathcal{T}/\mathcal{T}^{fd}(M, \Sigma^j M) = 0$.

5. NEGATIVE CLUSTER CATEGORY :

Setup so far

→ let $n \geq 3$, k field, $(\mathcal{T}, \mathcal{T}^{fd}, \mathcal{M})$ an n-Calabi-Yau triple :

- $\mathcal{T}$: k-linear, Hom-finite, Krull-Schmidt Δed cat
- $\mathcal{T}^{fd} \subseteq \mathcal{T}$: Δed subcategory s.t. $(\mathcal{T}, \mathcal{T}^{fd})$ is relative n-Calabi-Yau :

$$D\mathcal{T}(X,Y) \cong \mathcal{T}(Y, \Sigma^n X)$$

bifunctorial isomorphism $\forall X \in \mathcal{T}^{fd}, Y \in \mathcal{T}$.

- $\mathcal{M} = \operatorname{add} M \subseteq \mathcal{T}$ silting subcategory :

$$\mathcal{T}(\mathcal{M}, \Sigma^{>0}\mathcal{M}) = 0 \quad \& \quad \mathcal{T} = \operatorname{thick}(\mathcal{M})$$

with right adjacent t-structure

$$(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}) = ((\Sigma^{<0}\mathcal{M})^{\perp_\mathcal{T}}, (\Sigma^{>0}\mathcal{M})^{\perp_\mathcal{T}})$$

Setup for negative cluster category

→ let $n \geq 3$, k field, $(\mathcal{T}, \mathcal{T}^{fd}, \mathcal{M})$ a $(-n)$-Calabi-Yau triple :

- $\mathcal{T}$: k-linear, Hom-finite, Krull-Schmidt Δed cat
- $\mathcal{T}^{fd} \subseteq \mathcal{T}$: Δed subcategory s.t. $(\mathcal{T}, \mathcal{T}^{fd})$ is relative $(-n)$-Calabi-Yau :

$$D\mathcal{T}(X,Y) \cong \mathcal{T}(Y, \Sigma^{-n} X)$$

bifunctorial isomorphism $\forall X \in \mathcal{T}^{fd}, Y \in \mathcal{T}$.

- $\mathcal{M} = \operatorname{add} M \subseteq \mathcal{T}$ simple minded collection :

$$\mathcal{T}(\mathcal{M}, \Sigma^{<0}\mathcal{M}) = 0, \quad \mathcal{T} = \operatorname{thick}(\mathcal{M}), \quad \dim_k \mathcal{T}(M', M'') = \delta_{M'M''}$$

with right adjacent co-t-structure

$$(\mathcal{T}_{\geq 0}, \mathcal{T}_{\leq 0}) = ({}^{\perp_\mathcal{T}}(\Sigma^{>0}\mathcal{M}), {}^{\perp_\mathcal{T}}(\Sigma^{<0}\mathcal{M}))$$

⇝ $\mathcal{T}/\mathcal{T}^{fd}$ negative cluster category (Coelho Simões, Pauksztello, Ploog, Jin)

Properties :

- $\mathcal{T}/\mathcal{T}^{fd}$ Δed, Hom-finite, $(-n-1)$-CY
- $\mathcal{M} \subseteq \mathcal{T}/\mathcal{T}^{fd}$ $(n+1)$-simple minded system :
  - → $\dim_k \mathcal{T}/\mathcal{T}^{fd}(M', M'') = \delta_{M'M''}$, $M', M'' \in \mathcal{M}$
  - → $\mathcal{T}/\mathcal{T}^{fd}(\Sigma^{1 \dots n} M, M) = 0$
  - → $\mathcal{T}/\mathcal{T}^{fd} \cong \langle \mathcal{M} \rangle * \Sigma^{-1}\langle \mathcal{M} \rangle * \dots * \Sigma^{-n}\langle \mathcal{M} \rangle$

NB : truncation triangles in co-t-structures are not unique.