The Karoubi envelope of an extriangulated category

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Outline

Idempotent complete

Definition

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Definition (Karoubi 1968)

An additive category $\mathcal{A}$ is *idempotent complete* if every idempotent morphism $e\colon A \to A$ gives a decomposition $A \cong K \oplus L$ so that $e \cong \begin{pmatrix} 0 & 0 \\ 0 & 1_L \end{pmatrix}$ with respect to this decomposition.

Idempotent completeness in literature

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Theorem (Krause 2015)

An additive category $\mathcal{A}$ is a Krull-Remak-Schmidt category if and only if it is idempotent complete and the endomorphism ring of every object is semi-perfect

Idempotent completeness in literature

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Definition (Jasso 2016)

Let n be a positive integer. An additive category $\mathcal{A}$ is n-Abelian if:

- $\mathcal{A}$ is idempotent complete
- ...

Karoubi envelope

Karoubi envelope

Definition (See for example Balmer-Schlichting 2001)

Let $\mathcal{A}$ be an additive category. The *Karoubi envelope* of $\mathcal{A}$ is denoted by $\tilde{\mathcal{A}}$ is defined as follows:

- $\mathrm{ob}(\tilde{\mathcal{A}}) := \{(A, p) \mid A \in \mathrm{ob}(\mathcal{A}), p\colon A \to A \text{ such that } p^2 = p\}$,
- A morphism in $\tilde{\mathcal{A}}$ from (A, p) to (B, q) is a morphism $\sigma\colon A \to B \in \mathcal{A}$ such that $\sigma p = q\sigma = \sigma$,
- For any object (A, p) in $\tilde{\mathcal{A}}$, the identity morphism $1_{(A,p)} = p\colon A \to A$.

Karoubi envelope

Proposition (See for example Bühler 2010)

- *The Karoubi envelope* $\tilde{\mathcal{A}}$ *is an idempotent complete.*
- *The biproduct in* $\tilde{\mathcal{A}}$ *is defined as* $(A, p) \oplus (B, q) = (A \oplus B, p \oplus q)$.
- *The inclusion* $i_{\mathcal{A}} \colon \mathcal{A} \to \tilde{\mathcal{A}}$ *where on objects* $A \mapsto i_{\mathcal{A}}(A) = (A, 1_A)$ *and on morphisms* $f \mapsto i_{\mathcal{A}}(f) = f$ *is a fully faithful additive functor.*
- *Universality. Let* $\mathcal{B}$ *be an idempotent complete category. For all additive functors* $F \colon \mathcal{A} \to \mathcal{B}$, *there exists a functor* $\tilde{F} \colon \tilde{\mathcal{A}} \to \mathcal{B}$ *and a natural isomorphism* $\alpha \colon F \Rightarrow \tilde{F} i_{\mathcal{A}}$.

Extriangulated categories

Extriangulated categories

Definition (Nakaoka-Palu 2019)

Two sequences of morphisms $A \xrightarrow{x} B \xrightarrow{y} C$, and $A \xrightarrow{x'} B' \xrightarrow{y'} C$ in $\mathcal{C}$ are *equivalent* if there exists an isomorphism $b \colon B \to B'$ such that the following diagram commutes.

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \| & & \downarrow{\scriptstyle b} & & \| \\ A & \xrightarrow{x'} & B' & \xrightarrow{y'} & C \end{array}$$

We denote the equivalence class of a sequence $A \xrightarrow{x} B \xrightarrow{y} C$, by $[A \xrightarrow{x} B \xrightarrow{y} C]$.

Extriangulated categories

Definition (Nakaoka-Palu 2019)

An *extriangulated category* is a triple $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ satisfying the following axioms.

(ET0) $\mathcal{C}$ is an additive category.

(ET1) The functor $\mathbb{E}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to Ab$ is a biadditive functor.

(ET2) The correspondence $\mathfrak{s}$ is an additive realisation of $\mathbb{E}$. A realisation is $\mathfrak{s}$ is a correspondence associating an equivalence class $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ to any $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$. We write

$$A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$$

and call this an extriangle or $\mathbb{E}$-triangle.

(ET3)

(ET3)$^{\mathrm{op}}$

(ET4) An Octahedron axiom

(ET4)$^{\mathrm{op}}$

Notation

Definition (Nakaoka-Palu 2019)

Let A, C be objects of $\mathcal{C}$. An element $\delta \in \mathbb{E}(C, A)$ is called an $\mathbb{E}$-extension, formally written (A, δ, C).
Since $\mathbb{E}$ is a bifunctor, for any $a\colon A \to A'$ and $c\colon C' \to C$, we have the following $\mathbb{E}$-extensions:

$$a_*\delta := \mathbb{E}(C, a)(\delta) \in \mathbb{E}(C, A'),$$

$$c^*\delta := \mathbb{E}(c^{\mathrm{op}}, A)(\delta) \in \mathbb{E}(C', A) \text{ and}$$

$$c^*a_*\delta = a_*c^*\delta := \mathbb{E}(c^{\mathrm{op}}, a)(\delta) \in \mathbb{E}(C', A').$$

Morphism of Extensions

Definition (Nakaoka-Palu 2019)

Let (A, δ, C) and (A', δ', C') be any pair of $\mathbb{E}$-extensions. A morphism $(a, c)\colon \delta \to \delta'$ of $\mathbb{E}$-extensions is a pair of morphisms $a\colon A \to A'$ and $c\colon C \to C'$ such that:

$$a_*\delta = c^*\delta'.$$

Relative theory of extriangulated categories

Relative theory of extriangulated categories

Definition (Herschend-Liu-Nakaoka 2021)

Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category with biadditive functor $\mathbb{E} \colon \mathcal{C}^{\text{op}} \times \mathcal{C} \to Ab$. A functor $\mathbb{F} \colon \mathcal{C}^{\text{op}} \times \mathcal{C} \to Ab$ is an additive subfunctor if:

- $\mathbb{F}(C, A)$ is a subgroup of $\mathbb{E}(C, A)$ for objects A, C,
- $\mathbb{F}(c, a) = \mathbb{E}(c, a)_{|\mathbb{F}(C,A)}$ for morphisms $a \colon A \to A'$ and $c \colon C' \to C$.

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Definition (Herschend-Liu-Nakaoka 2021)

A subfunctor $\mathbb{F}$ is closed if for any $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ the following sequences of natural transformations are exact

$$\mathbb{F}(C, -) \xRightarrow{\mathbb{F}(y,-)} \mathbb{F}(B, -) \xRightarrow{\mathbb{F}(x,-)} \mathbb{F}(A, -)$$

$$\mathbb{F}(-, A) \xRightarrow{\mathbb{F}(-,x)} \mathbb{F}(-, B) \xRightarrow{\mathbb{F}(-,y)} \mathbb{F}(-, C)$$

Relative theory of extriangulated categories

Proposition (Herschend-Liu-Nakaoka 2021)

Let $\mathbb{F} \subseteq \mathbb{E}$ be an additive subfunctor. Then $(\mathcal{C}, \mathbb{F}, \mathfrak{s}_{|\mathbb{F}})$ is extriangulated if and only if $\mathbb{F}$ is closed.

Main Theorem

Theorem

Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ *be an extriangulated category. Let* $\tilde{\mathcal{C}}$ *be the idempotent completion of* $\mathcal{C}$*. Then* $\tilde{\mathcal{C}}$ *has an extriangulated structure* $(\tilde{\mathcal{C}}, \mathbb{F}, \mathfrak{r})$*. Moreover, the embedding* $i_{\mathcal{C}} \colon \mathcal{C} \to \tilde{\mathcal{C}}$ *is an extriangulated functor.*

Biadditive functor $\mathbb{F}$

Definition

Given a pair of objects (X, p) and (Y, q) in $\tilde{\mathcal{C}}$, we define $\mathbb{F}$ on objects by setting,

$$\mathbb{F}((X, p), (Y, q)) := p^* q_* \mathbb{E}(X, Y) = \{p^* q_* \delta \mid \delta \in \mathbb{E}(X, Y)\}.$$

For the pair $(\tilde{\alpha}, \tilde{\beta})$ we define

$$\mathbb{F}(\tilde{\alpha}^{\text{op}}, \tilde{\beta})\colon \mathbb{F}((Y, q), (U, e)) \to \mathbb{F}((X, p), (V, f))$$

as follows. For $\varepsilon \in \mathbb{F}((Y, q), (U, e))$ we set

$$\mathbb{F}(\tilde{\alpha}^{\text{op}}, \tilde{\beta})(\varepsilon) := \beta_* \alpha^* \varepsilon.$$

$\mathbb{F}$ is *like* a closed additive subfunctor of $\mathbb{E}$

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Proposition

- *$\mathbb{F}((C,p),(A,q))$ is a subgroup of $\mathbb{E}(C,A)$ for objects A, C and idempotents $p\colon C \to C, q\colon A \to A$*
- *$\mathbb{F}(c,a) = \mathbb{E}(c,a)_{|\mathbb{F}((C,p),(A,q))}$ for morphisms $a\colon (A,q) \to (A',q')$ and $c\colon (C',p') \to (C,p)$.*
- *For an $\mathbb{F}$-triangle*

$$(X,q) \xrightarrow{xq} (Y,r) \xrightarrow{py} (Z,p) \overset{\delta}{\dashrightarrow} .$$

the following sequences of natural transformations are exact.

$$\mathbb{F}(-,(X,q)) \xRightarrow{\mathbb{F}(-,xq)} \mathbb{F}(-,(Y,r)) \xRightarrow{\mathbb{F}(-,py)} \mathbb{F}(-,(Z,p))$$

$$\mathbb{F}((Z,p),-) \xRightarrow{\mathbb{F}(py,-)} \mathbb{F}((Y,r),-) \xRightarrow{\mathbb{F}(xq,-)} \mathbb{F}((X,q),-)$$

Lemma

Let $(\mathcal{A}, \mathbb{G}, \mathfrak{t})$ *be a triple satisfying (ET0), (ET1), (ET2), (ET3) and (ET3)*^{op}*. Let* δ *be an extension in* $\mathbb{G}(C, A)$ *with* $\mathfrak{t}(\delta) = [A \xrightarrow{a} B \xrightarrow{b} C]$*. Let* $(e, f)\colon \delta \to \delta$ *be a morphism of* $\mathbb{G}$*-extensions where* $e\colon A \to A$ *and* $f\colon C \to C$ *are idempotent morphisms. Then there exists an idempotent morphism* $g\colon B \to B$ *such that the diagram below commutes.*

$$\begin{array}{ccccc} A & \xrightarrow{a} & B & \xrightarrow{b} & C \\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{a} & B & \xrightarrow{b} & C \end{array}$$

Additive realisation $\mathfrak{r}$

Definition

Let $\mathfrak{r}$ be the correspondence between $\mathbb{F}$-extensions and equivalence classes of sequences of morphisms in $\tilde{\mathcal{C}}$ defined as follows. For any objects Z, X in $\mathcal{C}$ and idempotent morphisms $p\colon Z \to Z, q\colon X \to X$ in $\mathcal{C}$, let $\delta = p^* q_* \varepsilon$ be an extension in $\mathbb{F}((Z,p),(X,q))$ such that

$$\mathfrak{s}(p^* q_* \varepsilon) = [X \xrightarrow{x} Y \xrightarrow{y} Z].$$

We set

$$\mathfrak{r}(\delta) := [(X,q) \xrightarrow{xq} (Y,r) \xrightarrow{py} (Z,p)],$$

where $r\colon Y \to Y$ is an idempotent morphism such that $rx = xq$ and $yr = py$ obtained by application of the above lemma.

Triangulated case

Definition (Balmer-Schlichting 2001)

A sequence of morphisms

$$t\colon (A,q) \xrightarrow{x} (B,r) \xrightarrow{y} (C,p) \xrightarrow{\delta} (\Sigma A, \Sigma q)$$

is a distinguished triangle in $\tilde{\mathcal{C}}$ if there exists a sequence of morphisms

$$t'\colon (A',q') \xrightarrow{x'} (B',r') \xrightarrow{y'} (C',p') \xrightarrow{\delta'} (\Sigma A', \Sigma q')$$

such that $t \oplus t'$ is isomorphic to the image of a distinguished triangle in $\mathcal{C}$ under the embedding $i_{\mathcal{C}}\colon \mathcal{C} \to \tilde{\mathcal{C}}$.

Lemma

For any $\mathbb{F}$-triangle

$$(A, q) \xrightarrow{xq} (B, r) \xrightarrow{py} (C, p) \overset{\delta}{\dashrightarrow}$$

there exists an $\mathbb{F}$-triangle

$$(A', q') \xrightarrow{x'q'} (B', r') \xrightarrow{p'y'} (C', p') \overset{\delta'}{\dashrightarrow}$$

such that their direct sum is isomorphic to the image of an $\mathbb{E}$-triangle in $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ under the embedding $i_{\mathcal{C}} \colon \mathcal{C} \to \tilde{\mathcal{C}}$.

Unification

Our main theorem is a unifies the exact and triangulated case.

Weakly idempotent complete

Definition (See for example Bühler 2010)

Let $\mathcal{A}$ a category. A morphism $r\colon B \to C$ is a *retraction* if there is $q\colon C \to B$ such that $rq = 1_C$. A morphism $s\colon A \to B$ is a *section* if there is $t\colon B \to A$ such that $ts = 1_A$.

Definition (See for example Selinger 2008)

Let $\mathcal{A}$ be any category and A an object in $\mathcal{A}$. An idempotent morphism $e\colon A \to A$ is said to split if it admits a retraction $r\colon A \to X$ and a section $s\colon X \to A$ such that $s \circ r = e$ and $r \circ s = 1_X$.

Weakly idempotent complete

Definition (See for example Bühler 2010)

An additive category $\mathcal{A}$ is *weakly idempotent complete* if every retraction has a kernel. Equivalently, $\mathcal{A}$ is weakly idempotent complete if every section has a cokernel.

Weakly idempotent completeness in literature

Condition (Nakaoka-Palu 2019)

Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ *be an extriangulated category. Consider the following conditions.*

1. *Let* $f: A \to B$ *and* $g: B \to C$ *be composable morphisms. If* gf *is a deflation, then* g *is also a deflation.*
2. *Let* $f: A \to B$ *and* $g: B \to C$ *be composable morphisms. If* gf *is an inflation, then* f *is also an inflation.*

Weakly idempotent completeness in literature

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2. *Let* $f: A \to B$ *and* $g: B \to C$ *be composable morphisms. If* gf *is an inflation, then* f *is also an inflation.*

Proposition

Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ *be an extriangulated category. If* $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ *satisfies one of the WIC conditions then* $\mathcal{C}$ *is weakly idempotent complete.*

Weak idempotent completion

Definition (Neeman 1990)

Let $\mathcal{A}$ be a small additive category. The *weak idempotent completion* of $\mathcal{A}$ is denoted by $\hat{\mathcal{A}}$ is defined as follows:

- $\mathrm{ob}(\hat{\mathcal{A}}) := \{(A, p) \mid A \in \mathrm{ob}(\mathcal{A}), p \colon A \to A \text{ is a split idempotent}\}$,
- A morphism in $\hat{\mathcal{A}}$ from (A, p) to (B, q) is a morphism $\sigma \colon A \to B \in \mathcal{A}$ such that $\sigma p = q\sigma = \sigma$,
- For any object (A, p) in $\hat{\mathcal{A}}$, the identity morphism $1_{(A,p)} = p \colon A \to A$.

Weak idempotent completion

Proposition (See for example Bühler 2010)

- *The weak idempotent completion* $\hat{\mathcal{A}}$ *is weakly idempotent complete.*
- *The biproduct in* $\hat{\mathcal{A}}$ *is defined as* $(A, p) \oplus (B, q) = (A \oplus B, p \oplus q)$.
- *The inclusion* $i_{\mathcal{A}} \colon \mathcal{A} \to \hat{\mathcal{A}}$ *where on objects* $A \mapsto i_{\mathcal{A}}(A) = (A, 1_A)$ *and on morphisms* $f \mapsto i_{\mathcal{A}}(f) = f$ *is a fully faithful additive functor.*
- *Universality. Let* $\mathcal{B}$ *be an idempotent complete category. For all additive functors* $F \colon \mathcal{A} \to \mathcal{B}$, *there exists a functor* $\hat{F} \colon \hat{\mathcal{A}} \to \mathcal{B}$ *and a natural isomorphism* $\alpha \colon F \Rightarrow \hat{F} i_{\mathcal{A}}$.

Theorem

Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ *be an extriangulated category such that* $\mathcal{C}$ *is small and* $(\tilde{\mathcal{C}}, \mathbb{F}, \mathfrak{r})$ *be its idempotent completion. Then the weak idempotent completion* $\hat{\mathcal{C}}$ *of* $\mathcal{C}$ *is an extension-closed subcategory of* $\tilde{\mathcal{C}}$*. Hence* $\hat{\mathcal{C}}$ *is an extriangulated category.*

Thank you!

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