2-Dimensional Rigidity with Three Coincident Points

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Overview

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- Can we chracterise $\mathcal{R}_{uvw}(G)$ in a combinatorial way?

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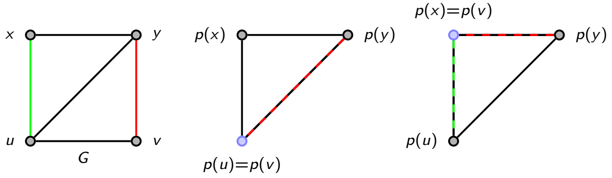

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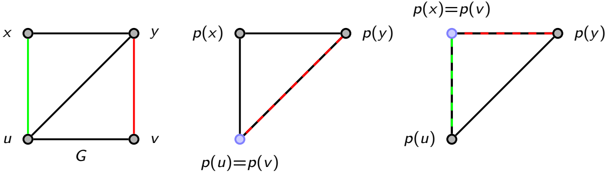

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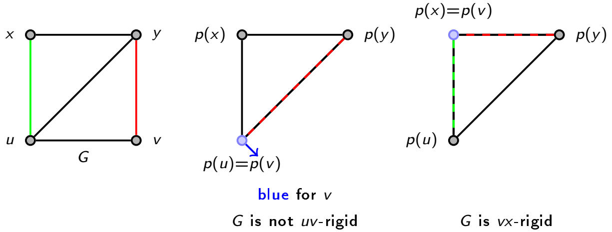

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S-Sparsity

• We say that G is *S-sparse* if, for all $H \subseteq V$ with $|H| \geq 2$, we have $i_G(H) \leq val_S(H)$, and for all S-compatible families $\mathcal{H}$, we have $i_G(\mathcal{H}) \leq val_S(\mathcal{H})$. It follows that, if G is S-sparse, then there is no edge between any distinct pair of vertices in S as $val_S(H) = 0$ for a set $H \subseteq S$.

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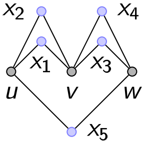

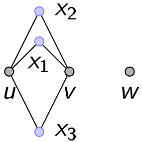

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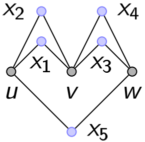

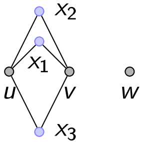

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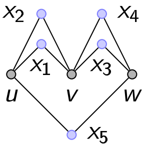

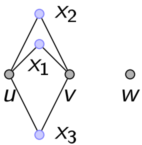

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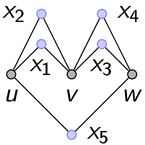

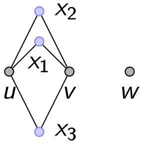

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• Let $\mathcal{H} = \{H_1, ..., H_t\}$ be an S-compatible family and let $X_1, ..., X_k$ be subsets of V of size at least two. We say that a collection $\mathcal{K} = \{X_1, ..., X_k\}$ is *thin* if
(i) $|X_i \cap X_j| \leq 1$ for all pairs $1 \leq i < j \leq k$.
The collection $\mathcal{L} = \{\mathcal{H}, X_1, ..., X_k\}$ is *thin* if (i) holds and
(ii) $H_i \cap H_j = S$ for all pairs $1 \leq i < j \leq t$, and
(iii) $|X_i \cap \bigcup_{j=1}^{t} H_j| \leq 1$ for all $1 \leq i \leq k$.

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(iii) $|X_i \cap \bigcup_{j=1}^{t} H_j| \leq 1$ for all $1 \leq i \leq k$.
• We define the value of $\mathcal{L}$ as

$$val(\mathcal{L}) = val_S(\mathcal{H}) + \sum_{i=1}^{k} 2|X_i| - 3.$$

The Count Matroid

Theorem 1 (Fekete, Jordán and Kaszanitzky ($|U| = 2$) / Jackson, G. ($|U| \geq 3$))

Let $G = (V, E)$ be a graph and $U \subseteq V$. Then the family $\mathcal{I}_G := \{F : F \subseteq E, H = (V, F) \text{ is } (U) - \text{sparse}\}$ is a family of independent sets of a matroid, $\mathcal{M}_U(G)$ on E. Moreover, the rank of a set $E' \subseteq E$ in $\mathcal{M}_U(G)$ is equal to

$$min\{val(\mathcal{L}) : \mathcal{L} \text{ is a thin cover of } E'\}.$$

Results and Examples

Theorem 2 (Fekete, Jordán and Kaszanitzky)

Let $G = (V, E)$ *be a graph and* $u, v \subseteq V$ *be distinct. Then* $\mathcal{M}_{uv}(G) = \mathcal{R}_{uv}(G)$.

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Theorem 3

Let $G = (V, E)$ be a graph and $u, v, w \subseteq V$ be distinct. Then $\mathcal{M}_{uvw}(G) = \mathcal{R}_{uvw}(G)$.

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Theorem 4 (Fekete, Jordán and Kaszanitzky)

Let $G = (V, E)$ be a graph and $u, v \in V$ be distinct vertices. Then G is uv-rigid if and only if $G - uv$ and G_{uv} are both rigid.

The graph G_{uv} is obtained from G by contracting u and v, and deleting double edges if there are any.

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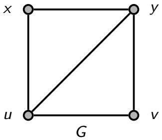

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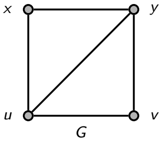

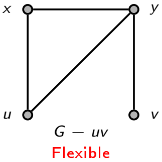

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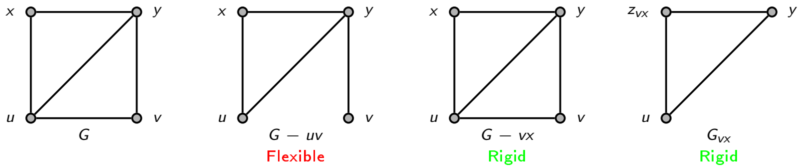

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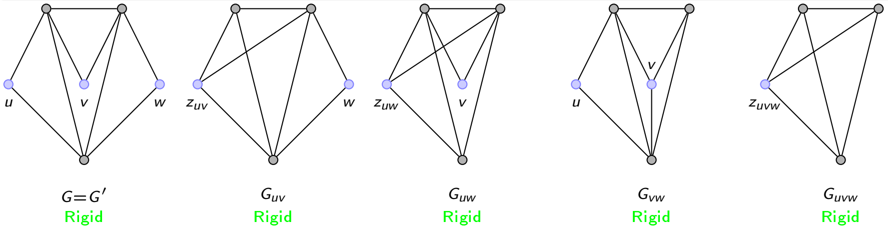

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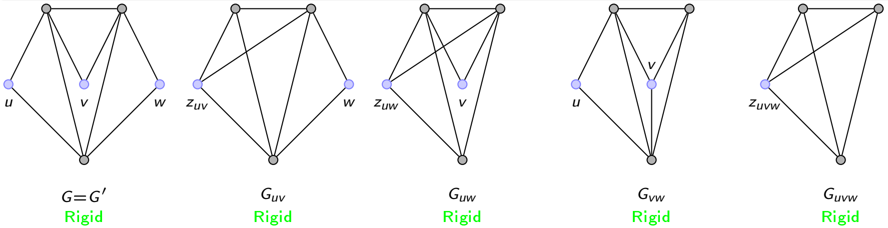

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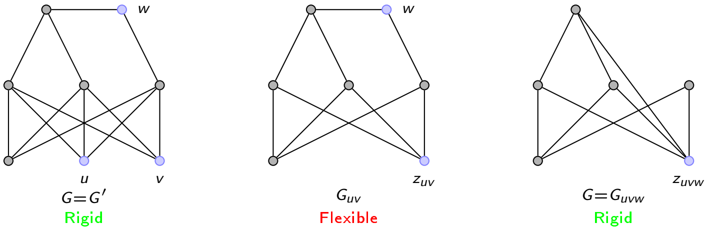

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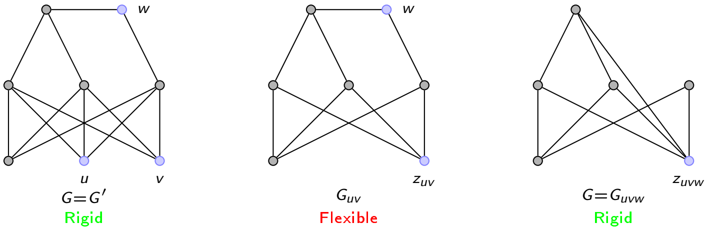

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Any Questions

Thank you!