Globally rigid braced triangulations

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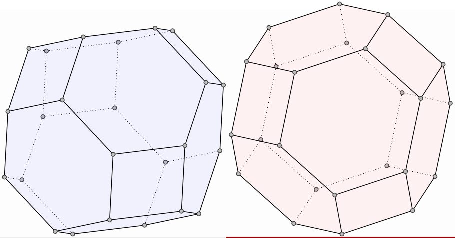
Cauchy's theorem

Consider a convex polyhedron P in $\mathbb{R}^3$. In the *graph $G(P)$ of the polyhedron* the vertices are the vertices of P, with two vertices adjacent if they form the endpoints of an edge of P.

Theorem (Cauchy, 1813)

Let P_1 and P_2 be convex polyhedra in $\mathbb{R}^3$ whose graphs are isomorphic and for which corresponding faces are pairwise congruent. Then P_1 and P_2 are congruent.



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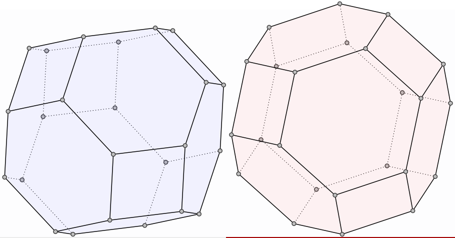

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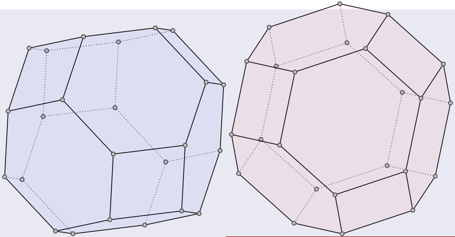
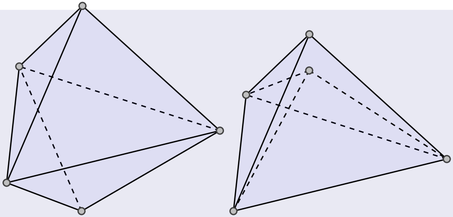


A non-convex example



The graphs of convex polyhedra

Theorem (Steinitz)

A graph G is the graph of some convex polyhedron P in $\mathbb{R}^3$ if and only if G is 3-connected and planar.

Theorem (Steinitz, 1906)

A graph G is the graph of some convex polyhedron P in $\mathbb{R}^3$ with triangular faces if and only if G is a maximal planar graph.

We shall simply call a maximal planar graph (or planar triangulation) a *triangulation*.

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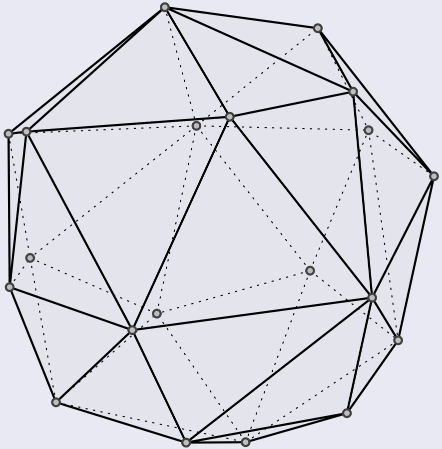
Convex polyhedra with triangular faces



A convex polyhedron with triangular faces

Bar-and-joint frameworks

A d-dimensional (bar-and-joint) *framework* is a pair (G, p), where $G = (V, E)$ is a graph and p is a map from V to $\mathbb{R}^d$. We consider the framework to be a straight line *realization* of G in $\mathbb{R}^d$.

Two realizations (G, p) and (G, q) of G are *equivalent* if $||p(u) - p(v)|| = ||q(u) - q(v)||$ holds for all pairs u, v with $uv \in E$, where $||.||$ denotes the Euclidean norm in $\mathbb{R}^d$. Frameworks (G, p), (G, q) are *congruent* if $||p(u) - p(v)|| = ||q(u) - q(v)||$ holds for all pairs u, v with $u, v \in V$.

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Bar-and-joint frameworks II.

The framework (G, p) is *rigid* if there exists an $\epsilon > 0$ such that, if (G, q) is equivalent to (G, p) and $||p(u) - q(u)|| < \epsilon$ for all $v \in V$, then (G, q) is congruent to (G, p).

Equivalently, the framework is rigid if every continuous deformation that preserves the edge lengths results in a congruent framework.

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Triangulated convex polyhedra

Corollary

Let P be a convex polyhedron with triangular faces and let $(G(P), p)$ be the corresponding bar-and-joint realization of its graph in three-space. Then $(G(P), p)$ is rigid.

Proof sketch

Consider a continuous motion of the vertices of $(G(P), p)$ which preserves the edge lengths. Then it must also preserve the faces as well as the convexity in a small enough neighbourhood. Thus it results in a congruent realization by Cauchy's theorem.

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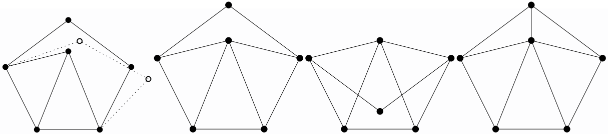

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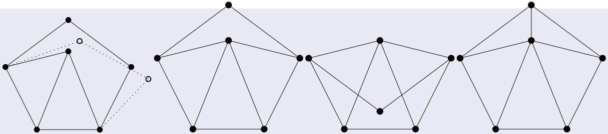

Bar-and-joint frameworks: generic realizations

Testing rigidity is NP-hard for $d \geq 2$ (T.G. Abbot, 2008). Testing global rigidity is NP-hard for $d \geq 1$ (J.B. Saxe, 1979).

The framework is *generic* if there are no algebraic dependencies between the coordinates of the vertices.

The rigidity (resp. global rigidity) of frameworks in $\mathbb{R}^d$ is a generic property, that is, the rigidity (resp. global rigidity) of (G, p) depends only on the graph G and not the particular realization p, if (G, p) is generic. We say that the graph G is *rigid* (*globally rigid*) in $\mathbb{R}^d$ if every (or equivalently, if some) generic realization of G in $\mathbb{R}^d$ is rigid.

For example, triangulations are (minimally) rigid in $\mathbb{R}^3$.

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Globally rigid graphs - necessary conditions

We say that G is *redundantly rigid in* $\mathbb{R}^d$ if removing any edge of G results in a rigid graph.

Theorem (B. Hendrickson, 1992)

Let G be a globally rigid graph in $\mathbb{R}^d$. Then either G is a complete graph on at most $d+1$ vertices, or G is
(i) $(d+1)$-connected, and
(ii) redundantly rigid in $\mathbb{R}^d$.

Global rigidity on the line and in the plane

Lemma

Graph G is globally rigid in $\mathbb{R}^1$ if and only if G is a complete graph on at most two vertices or G is 2-connected.

Theorem (B. Jackson, T. J., 2005)

Let G be a 3-connected and redundantly rigid graph in $\mathbb{R}^2$ on at least four vertices. Then G can be obtained from K_4 by extensions and edge-additions.

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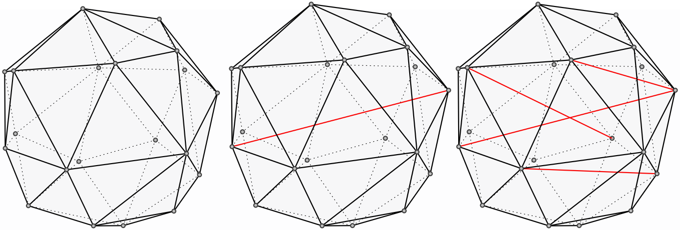
Braced triangulations

In what follows we shall call a graph $H = (V, E + B)$ a *braced triangulation* if it is obtained from a triangulation $G = (V, E)$ by adding a set B of new edges (called *bracing edges*). In the special case when $|B| = 1$ we say that H is a *uni-braced* triangulation.



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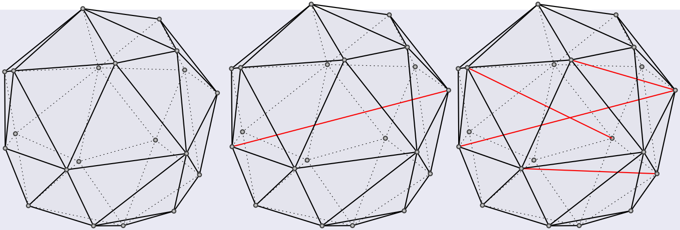

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Braced triangulations II.

Theorem (Whiteley, 1988)

Every 4-connected uni-braced triangulation is redundantly rigid in $\mathbb{R}^3$.

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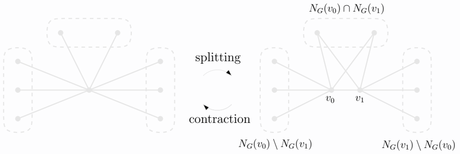
Vertex splitting

Let $H = (V, E)$ be a graph. For a vertex $v \in V$ we use $N_H(v)$ to denote the set of neighbours of v in H. Given a vertex $v_1 \in V$ and a partition $\{U_{12}, U_1, U_2\}$ of $N_H(v)$ with $|U_{12}| = k$, the *k-vertex splitting* operation at v_1 with respect to $\{U_{12}, U_1, U_2\}$ removes the edges connecting v_1 to U_2 and inserts a new vertex v_2 as well as new edges between v_2 and $v_1 \cup U_{12} \cup U_2$. The operation is *nontrivial* if U_1 and U_2 are both non-emtpy.



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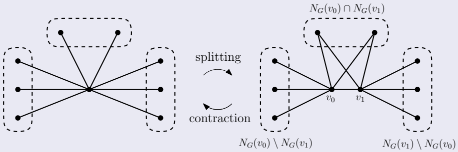

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Theorem (Steinitz, 1906)

Every triangulation can be obtained from K_4 by a sequence of 2-vertex splitting operations.

Theorem (Whiteley, 1991)

Let H be a rigid graph in $\mathbb{R}^d$ and let G be obtained from H by a $(d-1)$-vertex splitting operation. Then G is rigid in $\mathbb{R}^d$.

Theorem (Gluck, 1975)

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Vertex splitting III.

Conjecture (Whiteley, 2005)

Let H be globally rigid in $\mathbb{R}^d$ with at least $d+2$ vertices and let G be obtained from H by a nontrivial $(d-1)$-vertex-splitting operation. Then G is globally rigid in $\mathbb{R}^d$.

Theorem (T.J and S. Tanigawa, 2017)

Suppose that G can be obtained from K_{d+2} by a sequence of non-trivial $(d-1)$-vertex splitting operations. Then G is globally rigid in $\mathbb{R}^d$.

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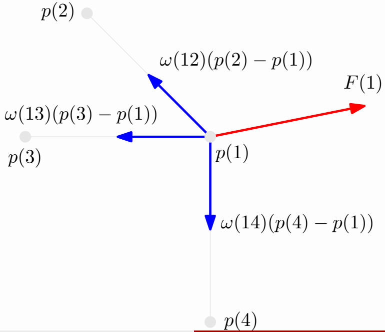
Equilibrium stress and the stress matrix

The function $\omega : e \in E \mapsto \omega(e) \in \mathbb{R}$ on framework (G, p) is *in equilibrium* with respect to $F : V \rightarrow \mathbb{R}^d$ if for each vertex $v \in V$ we have

$$\sum_{u \in N(v)} \omega(uv)(p(u) - p(v)) = -F(v). \quad (1)$$



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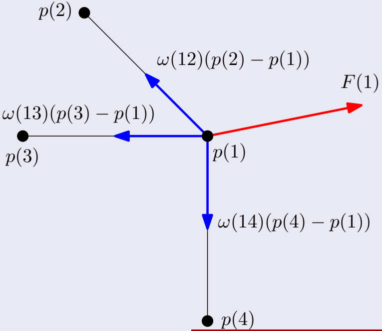

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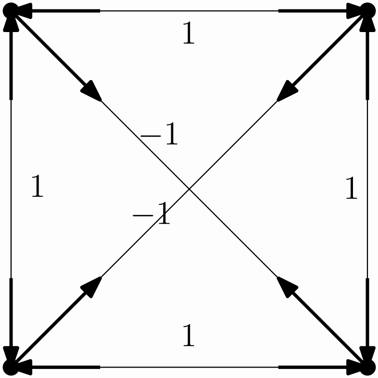

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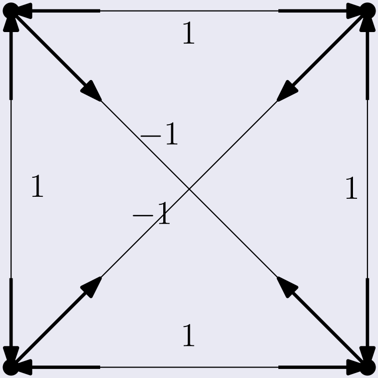

The stress matrix characterization of global rigidity

The *stress matrix* Ω of ω is a symmetric matrix of size $|V| \times |V|$ in which all row (and column) sums are zero and

$$\Omega[u, v] = -\omega(uv). \tag{2}$$

Theorem (Connelly, 2005, Gortler, Healy, Thurston, 2010)

Let (G, p) be a generic framework in $\mathbb{R}^d$ on at least $d + 2$ vertices. Then (G, p) is globally rigid in $\mathbb{R}^d$ if and only if (G, p) has an equilibrium stress ω for which the rank of the associated stress matrix Ω is $|V| - d - 1$.

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Non-degenerate stresses

Let (G, p) be a d-dimensional framework and let ω be a stress on (G, p). For a given vertex v of G and a given non-empty subset $X \subseteq N_G(v)$ we define $\omega \circ p(X) \in \mathbb{R}^d$ by

$$\omega \circ p(X) := \sum_{u \in X} \omega(uv)(p(u) - p(v)).$$

We say that ω is *degenerate* (resp. *non-degenerate*) with respect to a d-subpartition $\{X_1, \ldots, X_d\}$ of $N_G(v)$ if the set of vectors $\{\omega \circ p(X_i) : 1 \leq i \leq d\}$ is linearly dependent (linearly independent, respectively). Due to the equilibrium condition, ω is always degenerate with respect to a d-partition of $N_G(v)$. We say that ω is *non-degenerate* if it is non-degenerate with respect to every vertex v and every proper d-subpartition of the neighborhood of v.

Non-degenerate graphs

We call a graph G *non-degenerate* in $\mathbb{R}^d$ if every generic realization (G, p) of G in $\mathbb{R}^d$ admits a non-degenerate stress.

Lemma (T.J. and S. Tanigawa, 2017)

Non-degeneracy is a generic property in $\mathbb{R}^d$ for all $d \geq 1$.

Conjecture (T.J. and S. Tanigawa, 2017)

Every globally rigid graph G in $\mathbb{R}^d$ is non-degenerate in $\mathbb{R}^d$.

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Non-degenerate stresses and vertex splitting

Theorem (T.J. and S. Tanigawa, 2017)

Let G be obtained from H by a nontrivial vertex splitting at v_1 with respect to partition $\{U_{12}, U_1, U_2\}$ of $N_G(v_1)$, where $U_{12} = \{u_1, \ldots, u_{d-1}\}$. Suppose that a generic framework (H, p) in $\mathbb{R}^d$ admits a full rank stress ω. Then
(a) If ω is not degenerate with respect to $\{\{u_1\}, \ldots, \{u_{d-1}\}, U_2\}$, then some generic framework (G, p') admits a full rank stress.
(b) Moreover, if ω is non-degenerate, then (G, p') admits a full rank non-degenerate stress.

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Theorem (T.J. and S. Tanigawa, 2017)

Let H be a globally rigid graph in $\mathbb{R}^d$ with maximum degree at most $d+2$ and let G be obtained from H by a sequence of nontrivial $(d-1)$-vertex splitting operations. Then G is globally rigid in $\mathbb{R}^d$.

Theorem (T.J. and S. Tanigawa, 2017)

Suppose that G can be obtained from K_{d+2} by a sequence of non-trivial $(d-1)$-vertex splitting operations. Then G is globally rigid in $\mathbb{R}^d$.

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Inductive construction

Let $G = (V, E)$ be a triangulation and let $a, b \in V$ be a pair of non-adjacent vertices. Then $G + ab$ is called a *uni-braced triangulation rooted at* (a, b). Let $H = G + ab$ be a 4-connected uni-braced triangulation. We say that an edge $e = uv$ *avoids* the vertex pair (a, b) if $\{u, v\} \cap \{a, b\} = \emptyset$. An edge e is said to be *contractible* in H if e avoids (a, b) and H/e is a 4-connected uni-braced triangulation rooted at (a, b).

Theorem (T.J. and S. Tanigawa, 2017)

Let $H = G + ab$ be a 4-connected uni-braced triangulation rooted at (a, b). Then either
(i) H has a contractible edge not induced by $N_G(a) \cap N_G(b)$, or
(ii) G is a double pyramid with poles (a, b).

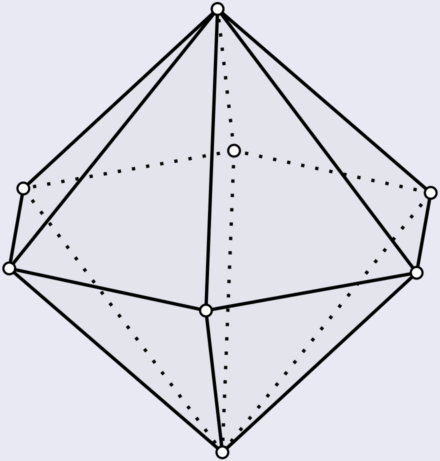
Inductive construction

Let $G = (V, E)$ be a triangulation and let $a, b \in V$ be a pair of non-adjacent vertices. Then $G + ab$ is called a *uni-braced triangulation rooted at* (a, b). Let $H = G + ab$ be a 4-connected uni-braced triangulation. We say that an edge $e = uv$ *avoids* the vertex pair (a, b) if $\{u, v\} \cap \{a, b\} = \emptyset$. An edge e is said to be *contractible* in H if e avoids (a, b) and H/e is a 4-connected uni-braced triangulation rooted at (a, b).

Theorem (T.J. and S. Tanigawa, 2017)

Let $H = G + ab$ be a 4-connected uni-braced triangulation rooted at (a, b). Then either
(i) H has a contractible edge not induced by $N_G(a) \cap N_G(b)$, or
(ii) G is a double pyramid with poles (a, b).

Double pyramid



A double pyramid

The inductive construction

Theorem (T.J. and S. Tanigawa, 2017)

Let $H = G + ab$ be a 4-connected uni-braced triangulation. Then H can be obtained from K_5 by a sequence of non-trivial 2-vertex splitting operations.

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Every 4-connected uni-braced triangulation is globally rigid in $\mathbb{R}^3$.

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Redundant edges

Let $H = (V, E + B)$ be a braced triangulation and let $e \in B$ be a designated bracing edge. The unique maximal 4-connected subgraph containing e is called the 4-*block* of e in H.

Theorem (T.J. and S. Tanigawa, 2017)

Let $H = (V, E + B)$ be a braced triangulation and let $e \in E + B$ be a designated edge. Then $H - e$ is rigid if and only if e belongs to the 4-block of some bracing edge in H.

It follows that H is redundantly rigid if and only if every edge belongs to the 4-block of some bracing edge.

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Further corollaries

Theorem (T.J. and S. Tanigawa, 2017)

Let G be a triangulation and let $\{u, v\}$ be a pair of non-adjacent vertices of G. Then $\{u, v\}$ is globally loose.

Theorem (Whiteley, 1988)

Let H be a graph with a quadrilateral hole and a quadrilateral block, obtained from a triangulation by removing an edge and adding a new edge between adjacent triangles. Then H is rigid in $\mathbb{R}^3$ if and only if there exist 4 vertex-disjoint paths between the hole and the block.

Conjecture (T.J. and S. Tanigawa, 2017)

Let $G = (V, E)$ be a 5-connected braced triangulation with $|E| \geq 3|V| - 4$. Then $G - e$ is globally rigid in $\mathbb{R}^3$ for all $e \in E$.

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Globally rigid triangulations of other surfaces

Theorem (Barnette, 1990)

(a) Every 4-connected triangulation of the projective plane can be obtained from K_6 or K_7 minus a triangle by nontrivial vertex-splitting operations.
(b) Every 4-connected triangulation of the torus can be obtained from one of 21 graphs by nontrivial vertex-splitting operations.

Theorem (T.J. and S. Tanigawa, 2017)

Suppose that G is a 4-connected triangulation of the torus or the projective plane. Then G is globally rigid in $\mathbb{R}^3$.

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Thank you.