Rigidity of frameworks with a class of coordinated bars

Hattie Serocold

Lancaster University

(joint work with Bernd Schulze)

Bond-node Structures
9th June 2017

Standard Rigidity

Definition

Let a *framework* be (G, p), where $G = (V, E)$ is a simple graph with $|V| = n$, and $p \in \mathbb{R}^{dn}$ is a placement of the vertex set.

Standard Rigidity

Definition

Let a *framework* be (G, p), where $G = (V, E)$ is a simple graph with $|V| = n$, and $p \in \mathbb{R}^{dn}$ is a placement of the vertex set.

Definition

We consider p to be *generic* if the coordinates of the p_i are algebraically independent over $\mathbb{Q}$.

Standard Rigidity

Definition

(G, p) is *equivalent* to (G, p') if

$$\|p_i - p_j\| = \|p'_i - p'_j\| \quad \forall \{i, j\} \in E$$

Standard Rigidity

Definition

(G, p) is *equivalent* to (G, p') if

$$\|p_i - p_j\| = \|p'_i - p'_j\| \quad \forall \{i, j\} \in E$$

(G, p) is *congruent* to (G, p') if

$$\|p_i - p_j\| = \|p'_i - p'_j\| \quad \forall i, j \in V$$

Standard Rigidity

Definition

(G,p) is *equivalent* to (G,p') if

$$\|p_i - p_j\| = \|p'_i - p'_j\| \quad \forall \{i,j\} \in E$$

(G,p) is *congruent* to (G,p') if

$$\|p_i - p_j\| = \|p'_i - p'_j\| \quad \forall i,j \in V$$

Definition

(G,p) is *locally rigid* if there is a neighbourhood U around $p \in \mathbb{R}^{dn}$ where for all $p' \in U$ such that (G,p') is equivalent to (G,p), (G,p') is also congruent to (G,p).

Standard Rigidity

Definition

A *continuous motion* q of a framework (G, p) may be considered as a family of frameworks (G, q_t) for $t \in [0, 1)$ such that $(G, q_0) = (G, p)$ and each (G, q_t) is equivalent to (G, p).

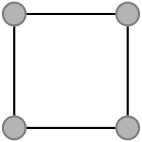
If not all (G, q_t) are congruent to (G, p), the motion is non-trivial.

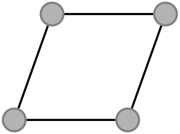
Standard Rigidity

Definition

A *continuous motion* q of a framework (G, p) may be considered as a family of frameworks (G, q_t) for $t \in [0, 1)$ such that $(G, q_0) = (G, p)$ and each (G, q_t) is equivalent to (G, p).

If not all (G, q_t) are congruent to (G, p), the motion is non-trivial.





Standard Rigidity

Definition

An *infinitesimal motion* of a framework (G, p) may be considered as $q \in \mathbb{R}^{dn}$ such that

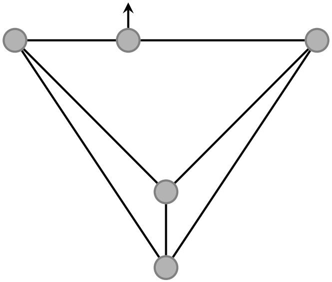
$$[p_i - p_j] \cdot [q_i - q_j] = 0 \quad \forall \{i, j\} \in E$$

Standard Rigidity

Definition

An *infinitesimal motion* of a framework (G, p) may be considered as $q \in \mathbb{R}^{dn}$ such that

$$[p_i - p_j] \cdot [q_i - q_j] = 0 \quad \forall \{i, j\} \in E$$

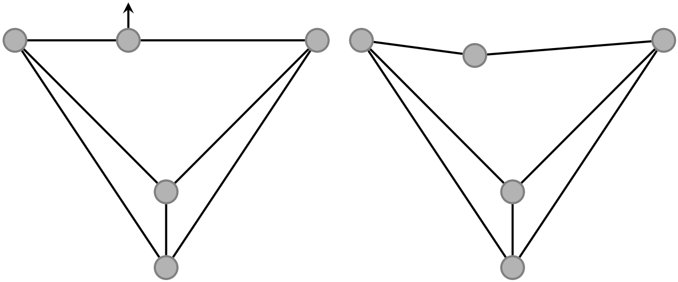


Standard Rigidity

Definition

An *infinitesimal motion* of a framework (G, p) may be considered as $q \in \mathbb{R}^{dn}$ such that

$$[p_i - p_j] \cdot [q_i - q_j] = 0 \quad \forall \{i, j\} \in E$$



Standard Rigidity

Definition

A framework (G, p) is *isostatic* if it is infinitesimally rigid and independent.

Standard Rigidity

Definition

A framework (G, p) is *isostatic* if it is infinitesimally rigid and independent.

Definition

A *Laman graph* G satisfies $|E| = 2|V| - 3$ and $|D| \leq 2|V(D)| - 3$ for all $D \subseteq E$.
A subgraph H with $|E(H)| = 2|V(H)| - 3$ and and $|D| \leq 2|V(D)| - 3$ for all $D \subseteq E(H)$ may be referred to as a *rigid block*.

Standard Rigidity

Definition

A framework (G, p) is *isostatic* if it is infinitesimally rigid and independent.

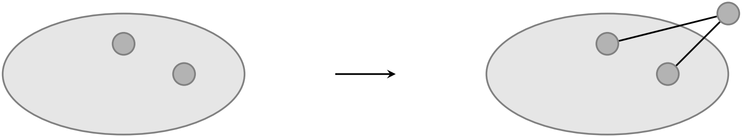
Definition

A *Laman graph* G satisfies $|E| = 2|V| - 3$ and $|D| \leq 2|V(D)| - 3$ for all $D \subseteq E$.
A subgraph H with $|E(H)| = 2|V(H)| - 3$ and and $|D| \leq 2|V(D)| - 3$ for all $D \subseteq E(H)$ may be referred to as a *rigid block*.

Theorem (Laman, 1970)

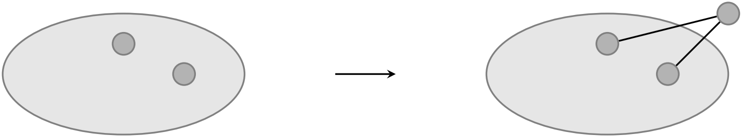
(G, p) is generically isostatic if and only if it is a Laman graph.

Henneberg Moves

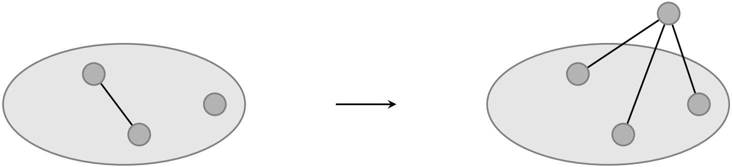


0-extension

Henneberg Moves



0-extension



1-extension

Coordinated Rigidity

Definition

Let $c : E \to \{\cdot, 1\}$ be a colouring of the edge set of G. We consider the edge set as partitioned into $E = E_0 \cup E_1$.

Coordinated Rigidity

Definition

Let $c : E \to \{\cdot, 1\}$ be a colouring of the edge set of G. We consider the edge set as partitioned into $E = E_0 \cup E_1$.

(G, c, p) is referred to as a 1-coloured framework.

Coordinated Rigidity

Definition

Let $c : E \to \{\cdot, 1\}$ be a colouring of the edge set of G. We consider the edge set as partitioned into $E = E_0 \cup E_1$.

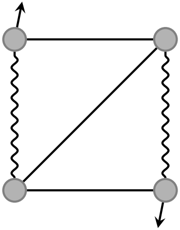
(G, c, p) is referred to as a 1-coloured framework.

Definition

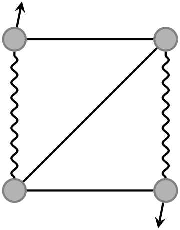
A *coordinated infinitesimal motion* of a 1-coloured framework (G, c, p) may be considered as $(q, c_1) \in \mathbb{R}^{dn+1}$ such that

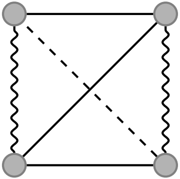
$$\begin{aligned} [p_i - p_j] \cdot [q_i - q_j] &= 0 \quad \forall \{i, j\} \in E_0 \\ [p_i - p_j] \cdot [q_i - q_j] &= c_1 \quad \forall \{i, j\} \in E_1 \end{aligned}$$

Coordinated Rigidity



Coordinated Rigidity





Coordinated Rigidity

Definition

A graph G is *Laman-plus-one* if $\exists e \in E$ such that $G - e$ is Laman.

Coordinated Rigidity

Definition

A graph G is *Laman-plus-one* if $\exists e \in E$ such that $G - e$ is Laman.

A graph G is a *rigidity circuit* if $G - e$ is Laman for all $e \in E$.

Coordinated Rigidity

Definition

A graph G is *Laman-plus-one* if $\exists e \in E$ such that $G - e$ is Laman.

A graph G is a *rigidity circuit* if $G - e$ is Laman for all $e \in E$.

Note

Every Laman-plus-one graph contains a unique rigidity circuit.

Result

Theorem

Let (G, c, p) be a 1-coloured framework, with generic $p : V \to \mathbb{R}^2$. The following are equivalent:

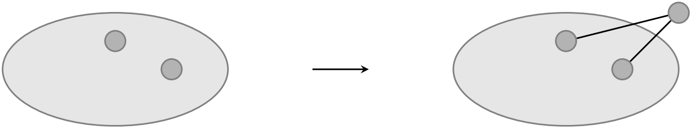
1. (G, c, p) is isostatic;
2. G is Laman-plus-one, and for all $D \subseteq E_0$

$$|D| \leq 2|V(D)| - 3;$$

3. (G, c) may be constructed from K_4 with at least one coloured edge, by a sequence of coloured 0-extensions and 1-extensions.

Inductive Moves

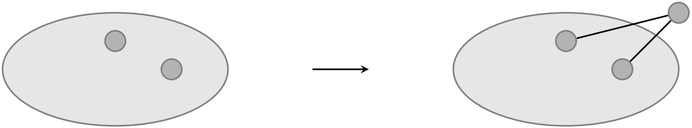
0-extensions



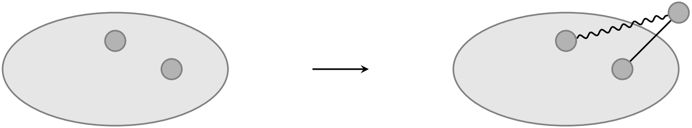
Standard

Inductive Moves

0-extensions



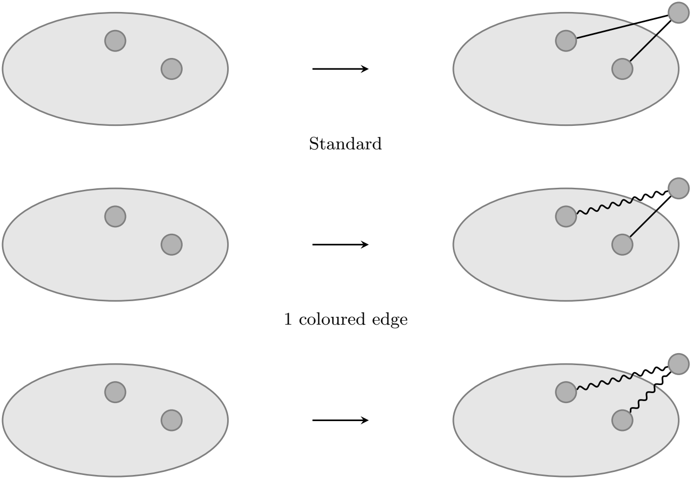
Standard



1 coloured edge

Inductive Moves

0-extensions



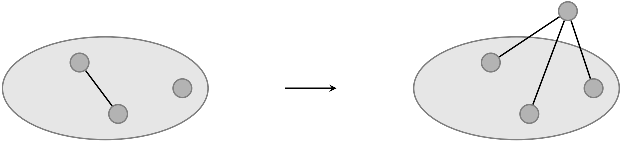
Standard

1 coloured edge

2 coloured edges

Inductive Moves

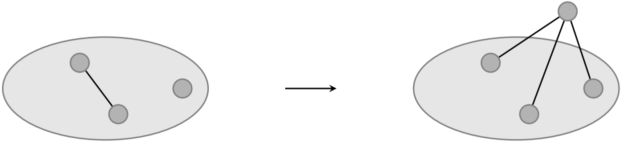
1-extensions



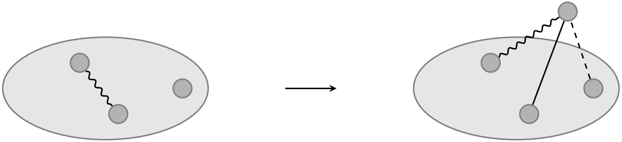
Standard

Inductive Moves

1-extensions



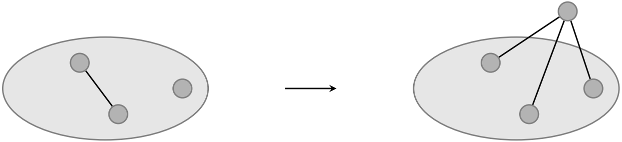
Standard



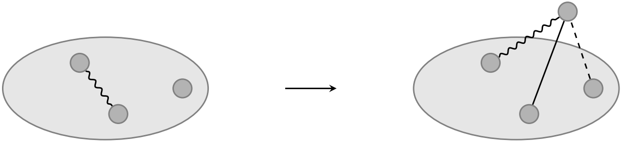
1 coloured edge

Inductive Moves

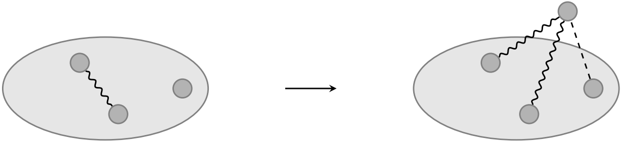
1-extensions



Standard



1 coloured edge



2 coloured edges

$1 \Rightarrow 2$

(G, c, p) is isostatic $\Rightarrow$ G is Laman-plus-one, and for all $D \subseteq E_0$, $|D| \leq 2|V(D)| - 3$.

We require the following result to prove that the conditions are necessary:

$1 \Rightarrow 2$

(G, c, p) is isostatic $\Rightarrow$ G is Laman-plus-one, and for all $D \subseteq E_0$, $|D| \leq 2|V(D)| - 3$.

We require the following result to prove that the conditions are necessary:

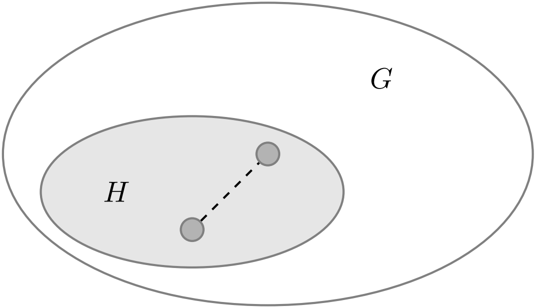
Lemma

Let (G, c) be a 1-coloured graph and $p : V \to \mathbb{R}^2$ be a generic embedding. The following are equivalent:

1. (G, c, p) is isostatic;
2. (G, c, p) is independent and $|E| = 2|V| + 1 - \binom{3}{2}$;
3. (G, c, p) is infinitesimally rigid and $|E| = 2|V| + 1 - \binom{3}{2}$;
4. (G, c, p) is infinitesimally rigid, and is minimally rigid as a 1-coloured framework.

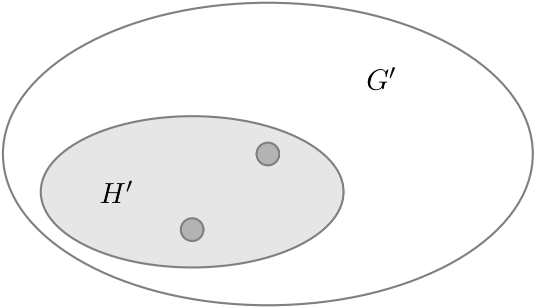
$1 \Rightarrow 2$

(G, c, p) is isostatic $\Rightarrow$ G is Laman-plus-one, and for all $D \subseteq E_0$, $|D| \leq 2|V(D)| - 3$.



$1 \Rightarrow 2$

(G, c, p) is isostatic $\Rightarrow$ G is Laman-plus-one, and for all $D \subseteq E_0$, $|D| \leq 2|V(D)| - 3$.



$2 \Rightarrow 3$

G is Laman-plus-one, and for all $D \subseteq E_0$, $|D| \leq 2|V(D)| - 3$, $\Rightarrow$ (G, c) is constructable from K_4.

The smallest simple Laman-plus-one graph is K_4. Requiring that $|D| \leq 2|V(D)| - 3$ for all $D \subseteq E_0$ gives that $|E_1| \geq 1$, as condition 3 requires.

$2 \Rightarrow 3$

G is Laman-plus-one, and for all $D \subseteq E_0$, $|D| \leq 2|V(D)| - 3$, $\Rightarrow$ (G, c) is constructable from K_4.

The smallest simple Laman-plus-one graph is K_4. Requiring that $|D| \leq 2|V(D)| - 3$ for all $D \subseteq E_0$ gives that $|E_1| \geq 1$, as condition 3 requires.

Let (G, c) be Laman-plus-one with $|D| \leq 2|V(D)| - 3$ for all $D \subseteq E_0$ and $|V| \geq 5$.

$2 \Rightarrow 3$

G is Laman-plus-one, and for all $D \subseteq E_0$, $|D| \leq 2|V(D)| - 3$, $\Rightarrow$ (G, c) is constructable from K_4.

The smallest simple Laman-plus-one graph is K_4. Requiring that $|D| \leq 2|V(D)| - 3$ for all $D \subseteq E_0$ gives that $|E_1| \geq 1$, as condition 3 requires.

Let (G, c) be Laman-plus-one with $|D| \leq 2|V(D)| - 3$ for all $D \subseteq E_0$ and $|V| \geq 5$.

The average degree within G is $4 - \frac{4}{|V|} < 4$, so there will be a vertex of degree 2 or degree 3.

$2 \Rightarrow 3$

G is Laman-plus-one, and for all $D \subseteq E_0$, $|D| \leq 2|V(D)| - 3$, $\Rightarrow$ (G, c) is constructable from K_4.

Let $X = (V(X), E(X))$ be the subgraph outside the rigidity circuit C, and let F be the edges from $V(X)$ to $V(C)$.

$2 \Rightarrow 3$

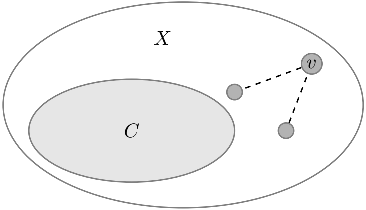
G is Laman-plus-one, and for all $D \subseteq E_0$, $|D| \leq 2|V(D)| - 3$, $\Rightarrow$ (G, c) is constructable from K_4.

Let $X = (V(X), E(X))$ be the subgraph outside the rigidity circuit C, and let F be the edges from $V(X)$ to $V(C)$.

If $V(X) \neq \emptyset$, $|F| \geq 3$ and the average degree within X is $4 - \frac{|F|}{|V(X)|} < 4$, so there is a vertex of degree 2 or 3.

$2 \Rightarrow 3$

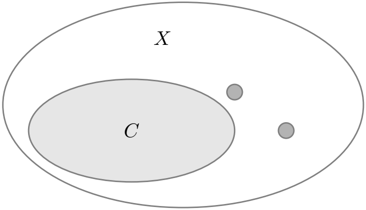
G is Laman-plus-one, and for all $D \subseteq E_0$, $|D| \leq 2|V(D)| - 3$, $\Rightarrow$ (G, c) is constructable from K_4.



Degree 2

$2 \Rightarrow 3$

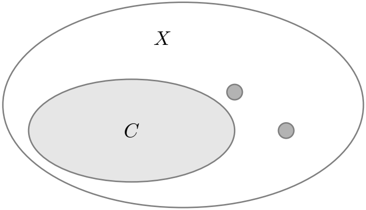
G is Laman-plus-one, and for all $D \subseteq E_0$, $|D| \leq 2|V(D)| - 3$, $\Rightarrow$ (G, c) is constructable from K_4.



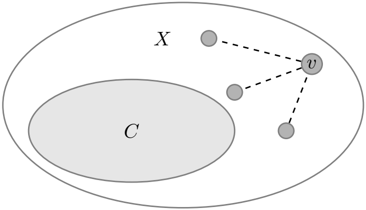
Degree 2

$2 \Rightarrow 3$

G is Laman-plus-one, and for all $D \subseteq E_0$, $|D| \leq 2|V(D)| - 3$, $\Rightarrow$ (G, c) is constructable from K_4.



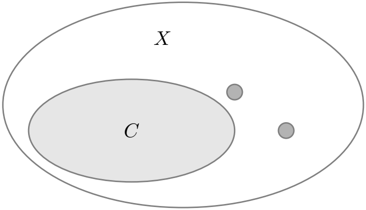
Degree 2



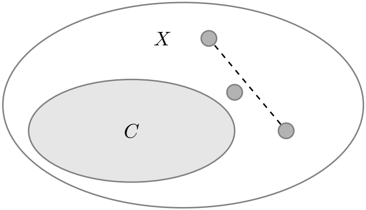
Degree 3

$2 \Rightarrow 3$

G is Laman-plus-one, and for all $D \subseteq E_0$, $|D| \leq 2|V(D)| - 3$, $\Rightarrow$ (G, c) is constructable from K_4.



Degree 2



Degree 3

$2 \Rightarrow 3$

G is Laman-plus-one, and for all $D \subseteq E_0$, $|D| \leq 2|V(D)| - 3$, $\Rightarrow$ (G, c) is constructable from K_4.

Suppose there are no degree 2 or 3 vertices in $V(X)$, and hence $G = C$.

$2 \Rightarrow 3$

G is Laman-plus-one, and for all $D \subseteq E_0$, $|D| \leq 2|V(D)| - 3$, $\Rightarrow$ (G, c) is constructable from K_4.

Suppose there are no degree 2 or 3 vertices in $V(X)$, and hence $G = C$.

When G is a 3-connected rigidity circuit, we may make use of the following result:

Theorem (Berg-Jordán, 2003)

Let $G = (V, E)$ be a 3-connected generic circuit with $|V| \geq 5$. Then G has either four, or three pairwise non-adjacent, degree 3 vertices, at which there exists a 1-reduction resulting in a 3-connected circuit.

$2 \Rightarrow 3$

G is Laman-plus-one, and for all $D \subseteq E_0$, $|D| \leq 2|V(D)| - 3$, $\Rightarrow$ (G, c) is constructable from K_4.

If G is 2-connected, we may consider it as the union of two Laman graphs H_1, H_2 with a cut pair considered as the pair of common vertices $a, b \in V(H_1) \cap V(H_2)$ (so $\{a, b\} \notin E$). We will have $deg(a), deg(b) \geq 4$.

$2 \Rightarrow 3$

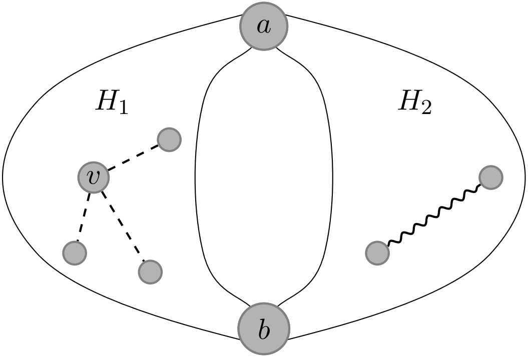
G IS LAMAN-PLUS-ONE, AND FOR ALL $D \subseteq E_0$, $|D| \leq 2|V(D)| - 3$, $\Rightarrow$ (G, c) IS CONSTRUCTABLE FROM K_4.

If G is 2-connected, we may consider it as the union of two Laman graphs H_1, H_2 with a cut pair considered as the pair of common vertices $a, b \in V(H_1) \cap V(H_2)$ (so $\{a, b\} \notin E$). We will have $deg(a), deg(b) \geq 4$.

Suppose $e \in E_1(H_2)$. Since H_1 is Laman, there will be $v \in V(H_1)$ such that $deg_{H_1}(v) = deg_G(v) = 3$. We may apply a 1-reduction at v to reduce H_1, and hence also reduce the graph G.

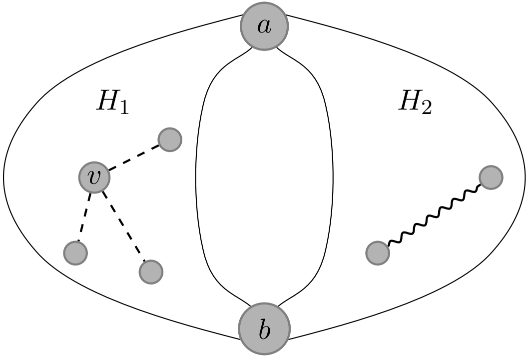
$2 \Rightarrow 3$

G IS LAMAN-PLUS-ONE, AND FOR ALL $D \subseteq E_0$, $|D| \leq 2|V(D)| - 3$, $\Rightarrow$ (G, c) IS CONSTRUCTABLE FROM K_4.



$2 \Rightarrow 3$

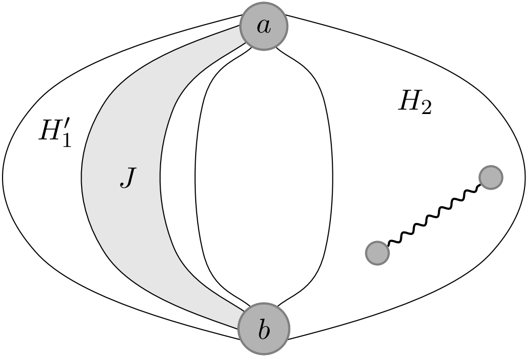
G IS LAMAN-PLUS-ONE, AND FOR ALL $D \subseteq E_0$, $|D| \leq 2|V(D)| - 3$, $\Rightarrow$ (G, c) IS CONSTRUCTABLE FROM K_4.



The reduced circuit will be $J \cup H_2$, where $J \subset H_1$ is the minimal Laman subgraph containing both a and b.

$2 \Rightarrow 3$

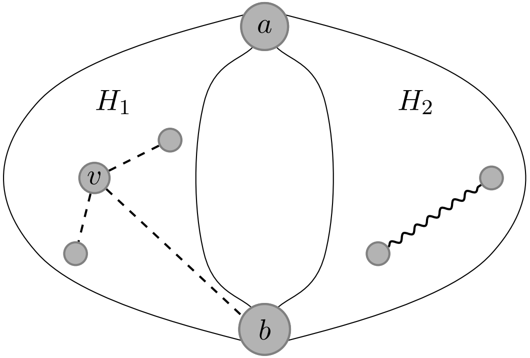
G is Laman-plus-one, and for all $D \subseteq E_0$, $|D| \leq 2|V(D)| - 3$, $\Rightarrow$ (G, c) is constructable from K_4.



The reduced circuit will be $J \cup H_2$, where $J \subset H_1$ is the minimal Laman subgraph containing both a and b.

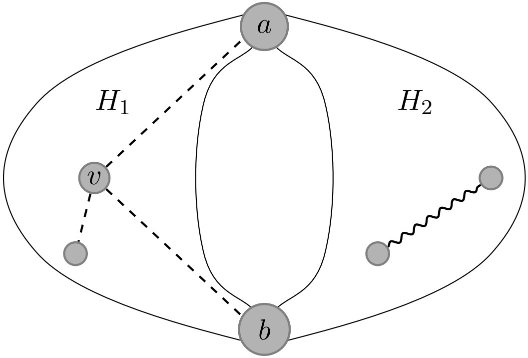
$2 \Rightarrow 3$

G IS LAMAN-PLUS-ONE, AND FOR ALL $D \subseteq E_0$, $|D| \leq 2|V(D)| - 3$, $\Rightarrow$ (G, c) IS CONSTRUCTABLE FROM K_4.



$2 \Rightarrow 3$

G is Laman-plus-one, and for all $D \subseteq E_0$, $|D| \leq 2|V(D)| - 3$, $\Rightarrow$ (G, c) is constructable from K_4.



$3 \Rightarrow 1$

(G, c) is constructable from K_4 $\Rightarrow$ (G, c, p) is isostatic.

(K_4, c) with $|E_1| \geq 1$ will satisfy the Lemma stated earlier to be isostatic.

$3 \Rightarrow 1$

(G, c) IS CONSTRUCTABLE FROM $K_4 \Rightarrow (G, c, p)$ IS ISOSTATIC.

(K_4, c) with $|E_1| \geq 1$ will satisfy the Lemma stated earlier to be isostatic.

It is straightforward to prove that when the coloured inductive moves are applied to an isostatic framework (G, c, p), (G', c', p') will be isostatic, hence anything built using these moves from an isostatic base graph (such as K_4 with at least one coloured edge) will also be isostatic.

Thank you