A New Separation Algorithm for the Boolean Quadric and Cut Polytopes

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Abstract

A *separation algorithm* is a procedure for generating cutting planes. We present new separation algorithms for the *Boolean quadric* and *cut* polytopes, which are the polytopes associated with zero-one quadratic programming and the max-cut problem, respectively. Our approach exploits, in a non-trivial way, three known results in the literature: one on the separation of $\{0, \frac{1}{2}\}$-cuts, one on the symmetries of the polytopes in question, and one on the relationship between the polytopes. We remark that our algorithm for the cut polytope is the first combinatorial polynomial-time algorithm that is capable of separating over a class of valid inequalities that includes all odd bicycle wheel inequalities and all $(p, 2)$-circulant inequalities.

Key Words: cutting planes, separation, zero-one quadratic programming, max-cut problem.

1 Introduction

Consider the following problem of minimising a quadratic function of binary variables:

$$\min \left\{ x^T Q x : \ x \in \{0,1\}^n \right\},$$

where Q is a rational square matrix of order n. This problem, which is $\mathcal{NP}$-hard in the strong sense, has a surprising variety of practical applications (see, e.g., Deza & Laurent [11] or Boros & Hammer [6]).

Padberg [19] introduced a family of polytopes (bounded polyhedra) associated with this problem, which he called *boolean quadric polytopes.* He introduced several classes of facet-defining inequalities for these polytopes, including so-called *triangle*, *clique* and *cut* inequalities. Further classes of

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valid and facet-defining inequalities were introduced, for example, by Boros & Hammer [5] and Sherali *et al.* [21].

It was pointed out by Padberg [19], Barahona *et al.* [1] and De Simone [9], apparently independently, that the boolean quadric polytopes are equivalent to *cut polytopes* under a certain affine mapping. (Cut polytopes are the polytopes associated with the *max-cut problem*, a well-known combinatorial optimisation problem.) Using this fact, one can derive even more valid inequalities for boolean quadric polytopes from known valid inequalities for cut polytopes. An excellent survey of the literature on these polytopes can be found in Deza & Laurent [11].

A *separation algorithm* for a given polytope, and a given class of valid inequalities for that polytope, is an algorithm that takes as input a point that does not lie in the polytope, and outputs a violated inequality in that class, if one exists. Separation algorithms are at the heart of modern....

Although many classes of valid inequalities are known for the boolean quadric and cut polytopes, we have separation algorithms for only a few of them. The purpose of this paper is to present new separation algorithms for them.

The starting point is the application of a known result of Caprara and Fischetti, on so-called $\{0, \frac{1}{2}\}$-Chvátal-Gomory cuts, to the Boolean quadric polytope. Together with certain operations on linear inequalities, this yields a primitive separation algorithm for the Boolean quadric polytope. The next step is to exploit the so-called *switching symmetry* of the Boolean quadric polytope, in order to separate over a wider class of inequalities. By exploiting some structural properties of the Boolean quadric polytope, the running time of this algorithm can be brought down to $\mathcal{O}(n^4)$.

Finally, using the above-mentioned mapping, we obtain a separation algorithm for the cut polytope as well. This algorithm calls the one for the Boolean quadric polytope n times as a subroutine, and therefore runs in $\mathcal{O}(n^5)$ time. We show that the inequalities generated by our algorithm for max-cut include some well-known facet-inducing inequalities, known as odd bicycle wheel inequalities and $(p, 2)$-circulant inequalities. We remark that our algorithm is the first combinatorial polynomial-time algorithm that is capable of generating these inequalities.

The structure of the paper is as follows. Section 2 is the literature review. In Section 3, we describe some linear inequalities that are needed in our separation algorithm for the Boolean quadric polytope. This algorithm is presented in Section 4, along with the analysis of its running time. Section 5 presents the algorithm for max-cut. Some computational experiments are described in Section 6. Finally, some concluding remarks are made in Section 7.

2 Literature Review

In this section, we review the relevant literature. The boolean quadric and cut polytopes are dealt with in Subsections 2.1 and 2.2, respectively. Known separation routines are covered in Subsection 2.3, and $\{0, \frac{1}{2}\}$-Chvátal-Gomory cuts in Subsection 2.4.

2.1 The boolean quadric polytope

For a given positive integer n, let V_n denote $\{1, \ldots, n\}$ and let E_n denote $\{\{i,j\} : \{i,j\} \subset V_n\}$. The boolean quadric polytope of order n, denoted by BQP_n, is the convex hull of vectors $(x, X) \in \{0,1\}^{V_n + E_n}$ satisfying $X_{ij} = x_i x_j$ for all $\{i,j\} \in E_n$ (Padberg, [19]).

Observe that BQP_n is the convex hull of vectors $(x, X) \in \mathbb{Z}^{V_n + E_n}$ satisfying the following linear inequalities:

$$X_{ij} \geq 0 \quad (\{i,j\} \in E_n) \tag{1}$$

$$X_{ij} - x_i \leq 0 \quad (i \in V_n,\, j \in V_n \setminus \{i\}) \tag{2}$$

$$x_i + x_j - X_{ij} \leq 1 \quad (\{i,j\} \in E_n). \tag{3}$$

We will call these *trivial* inequalities.

Padberg [19] showed that the trivial inequalities (1)-(3) define facets of BQP_n, along with the following *triangle* inequalities:

$$-x_k - X_{ij} + X_{ik} + X_{jk} \leq 0 \quad (\{i,j\} \in E_n,\, k \in V_n \setminus \{i,j\}) \tag{4}$$

$$x_i + x_j + x_k - X_{ij} - X_{ik} - X_{jk} \leq 1 \quad (\{i,j,k\} \subset V_n). \tag{5}$$

He also introduced some other facet-defining inequalities, called cut, clique and generalized cut inequalities. Further facet-defining inequalities were presented by Boros & Hammer [5] and Sherali *et al.* [21]. Still more inequalities can be derived using the connection between BQP_n and the *cut polytope* described in the next subsection.

DEFINE SWITCHING FOR BQPn?

2.2 The cut polytope

Given any $S \subset V_n$, the set of edges

$$\{\{i,j\} \in E_n :\, i \in S,\, j \in V_n \setminus S\}$$

is called a *cut*. It is known that a vector $y \in \{0,1\}^{E_n}$ is the incidence vector of a cut if and only if it satisfies the following inequalities (also called *triangle* inequalities):

$$y_{ij} + y_{ik} + y_{jk} \leq 2 \quad (1 \leq i < j < k \leq n) \tag{6}$$

$$y_{ij} - y_{ik} - y_{jk} \leq 0 \quad (1 \leq i < j \leq n; k \neq i, j). \tag{7}$$

The *cut polytope*, which we will denote by CUT_n, is the convex hull in $\mathbb{R}^{\binom{n}{2}}$ of such incidence vectors [2].

It is known that CUT_{n+1} is the image of BQP_n under a certain affine transformation, now called the *covariance map* [1, 9, 11, 19]). Using this fact, any facet-defining inequality for CUT_{n+1} can be converted into a facet-defining inequality for BQP_n. For example, the trivial inequalities (1)-(3) and triangle inequalities (4)-(5) for BQP_n can be obtained in this way from the triangle inequalities (6)-(7) for CUT_n.

GIVE DETAILS OF THE COVARIANCE MAP?

DEFINE SWITCHING FOR CUTn?

A vast array of valid and facet-defining inequalities are known for CUT_n; see the survey in [11]. For brevity, we recall the definitions of only two classes of valid inequalities, which will be needed in what follows:

Proposition 1 (Barahona & Mahjoub [2]) *Let $k \geq 3$ be an odd integer. Let s, t and $v_1, \dots, v_k$ be distinct vertices in K_n. The following 'odd bicycle wheel' inequality defines a facet of CUT_n:*

$$\sum_{i=1}^{k} \left(X_{s,v_i} + X_{t,v_i} + X_{v_i,v_{i+1}} \right) \leq 2k, \tag{8}$$

where indices are taken modulo k.

Proposition 2 (Poljak & Turzik [20]) *Let k be a positive integer, and let $v_1, \dots, v_k$ be distinct vertices in K_n. The following '$(4k+1, 2)$'-circulant inequality defines a facet of CUT_n:*

$$\sum_{i=1}^{4k+1} \left(X_{v_i,v_{i+1}} + X_{v_i,v_{i+2}} \right) \leq 6k. \tag{9}$$

where indices are taken modulo $4k+1$.

2.3 Known separation routines

A *separation algorithm* for a given class of inequalities is a procedure that takes a point $(x^*, X^*) \in \{0,1\}^{V_n + E_n}$, and outputs an inequality in that class that is violated by (x^*, X^*), if one exists. A desirable feature of a separation algorithm is that it runs in polynomial time.

The separation problems for the trivial inequalities (1)-(3) and the triangle inequalities (4)-(5) can be solved in $\mathcal{O}(n^2)$ and $\mathcal{O}(n^3)$ time, respectively, by mere enumeration.

Gerards [13] presented a separation algorithm for the odd bicycle wheel inequalities (no switching). It involves $\mathcal{O}(n^2)$ shortest-path computations in graphs with $\mathcal{O}(n)$ nodes, leading to a total running time of $\mathcal{O}(n^4)$. (For sparse graphs, one can get it down to $\mathcal{O}(m(m + n \log n))$.)

Some additional valid inequalities can be derived using semidefinite programming, and their associated separation problem can be solved in polynomial time. See, e.g., section 28.4.1. of Deza & Laurent [11].

De Simone & Rinaldi [10], Helmberg & Rendl [15] and Gruber [14] presented heuristics for the separation of hypermetric inequalities and their switched versions. Helmberg & Rendl [15] also present a greedy separation heuristic for odd clique inequalities. (Mention our own result, that the separation problem for rounded psd inequalities can be reduced to that for hypermetric inequalities?)

Letchford [16] showed that one can separate over a class of inequalities that includes all odd bicycle wheel and $(p, 2)$-circulant inequalities, using lift-and-project techniques. However, this necessitates the solution of $\binom{n}{2}$ linear programs, each with $\mathcal{O}(n^3)$ variables and $\mathcal{O}(n^2)$ constraints, and so is impractical.

There are also several separation algorithms designed for max-cut instances on sparse graphs (see, e.g., [1, 2, 3, 8, 18]). For the sake of brevity, we do not give details.

2.4 $\{0, \frac{1}{2}\}$-Chvátal-Gomory cuts

We will need the following definitions:

Definition 1 (Gomory, 1958; Chvátal, 1973) *Let $Ax \leq b$ be a system of linear inequalities, where $A \in \mathbb{Z}^{p \times q}$, $x \in \mathbb{Z}^q$ and $b \in \mathbb{Z}^p$, and let*

$$P_I = conv\{x \in \mathbb{Z}^q : Ax \leq b\}$$

be the associated integral polyhedron. A 'Chvátal-Gomory cut' (or 'CG-cut' for short) is a valid inequality for P_I of the form $(\lambda^T A)x \leq \lfloor \lambda^T b \rfloor$, where $\lambda \in [0, 1)^p$ is such that $\lambda^T A$ has integral components, but $\lambda^T b$ is fractional.

Definition 2 (Caprara & Fischetti, 1996) *A '$\{0, \frac{1}{2}\}$-Chvátal-Gomory cut' (or '$\{0, \frac{1}{2}\}$-cut' for short) is a CG-cut such that $\lambda \in \{0, \frac{1}{2}\}^p$.*

Caprara and Fischetti observed that several important inequalities, for several well-known combinatorial optimisation problems, can be derived as $\{0, \frac{1}{2}\}$-cuts. They showed that the separation problem for $\{0, \frac{1}{2}\}$-cuts is $\mathcal{NP}$-hard in general, but solvable in polynomial time under certain conditions. For example, if the matrix A has at most two odd coefficients per row, then one can construct an auxiliary graph, with one node per variable and one edge per constraint, with the property that every odd cycle of weight less than 1 in the graph corresponds to a violated $\{0, \frac{1}{2}\}$-cut, and vice-versa.

For the case in which the linear system $Ax \leq b$ does not satisfy the above parity condition, Caprara and Fischetti suggest *weakening* the system in order to obtain a system that does meet the condition.

3 Three Weakened Linear Systems

Our new separation routine for BQP_n is based on the construction of a suitable system of valid inequalities with at most two odd left-hand-side coefficients per row, followed by an application of the Caprara-Fischetti scheme.

The crucial step is the construction of the linear system. In the following three subsections, we consider three candidates for the system, in increasing order of complexity.

3.1 A simple weakened linear system

As a starting point, we consider the following simple linear system:

$$\begin{aligned}
-X_{ij} &\leq 0 \quad (\{i,j\} \in E_n) && (10)\\
-x_i + X_{ij} &\leq 0 \quad (i \in V_n,\ j \in V_n \setminus \{i\}) && (11)\\
x_i &\leq 1 \quad (i \in V_n) && (12)\\
x_i + x_j - 2X_{ij} &\leq 1 \quad (\{i,j\} \in E_n). && (13)
\end{aligned}$$

We call this 'system I'. Note that the inequalities (13) are a weakened version of the trivial inequalities (3). Note also that, although the inequalities (12) are dominated by the trivial inequalities (2) and (3), they are not dominated by the other inequalities in system I.

The following theorem characterises the valid inequalities for BQP_n that can be derived as $\{0, \frac{1}{2}\}$-cuts from system I:

Theorem 1 *Let P^1 be the polytope defined by the inequalities in system I, together with all inequalities that can be derived as $\{0, \frac{1}{2}\}$-cuts from system I. A complete and non-redundant linear description of P^1 is given by the trivial inequalities (1)–(3) and the triangle inequalities (4)–(5).*

Proof. If we multiply the inequalities $x_i + x_j - 2X_{ij} \leq 1$, $x_i \leq 1$ and $x_j \leq 1$ by $\frac{1}{2}$ and sum them together, we obtain the inequality $x_i + x_j - X_{ij} \leq \frac{3}{2}$. This shows that the trivial inequality (3) is a $\{0, \frac{1}{2}\}$-cut.

Similarly, if we multiply the inequalities $-x_i + X_{ik} \leq 0$, $-x_k + X_{ik} \leq 0$, $-x_j + X_{jk} \leq 0$, $-x_k + X_{jk} \leq 0$ and $x_i + x_j - 2X_{ij} \leq 1$ by $\frac{1}{2}$ and sum them together, we obtain the inequality $-x_k - X_{ij} + X_{ik} + X_{jk} \leq \frac{1}{2}$. This shows that the triangle inequality (4) is a $\{0, \frac{1}{2}\}$-cut.

Moreover, if we multiply the inequalities $x_i + x_j - 2X_{ij} \leq 1$, $x_i + x_k - 2X_{ik} \leq 1$ and $x_j + x_k - 2X_{jk} \leq 1$ by $\frac{1}{2}$ and sum them together, we obtain the inequality $x_i + x_j + x_k - X_{ij} - X_{ik} - X_{jk} \leq \frac{3}{2}$. This shows that the triangle inequality (5) is a $\{0, \frac{1}{2}\}$-cut.

The above three results show that all points in P^1 satisfy the trivial and triangle inequalities. Now, it was shown by Boros *et al.* [4] that the only non-redundant inequalities that can be derived as CG-cuts from the system (1)–(3) are the triangle inequalities. Therefore, every point satisfying the trivial and triangle inequalities must lie in P^1. □

Theorem 1 is rather disappointing, since one can easily solve the separation problem for the triangle inequalities by mere enumeration, without invoking the machinery of $\{0, \frac{1}{2}\}$-cuts. To obtain more interesting $\{0, \frac{1}{2}\}$-cuts, we must enlarge our linear system.

3.2 A more sophisticated linear system

We now present two more classes of valid inequalities, each having only two odd left-hand side coefficients, which will turn out to be very useful. First, by summing together one triangle inequality of the form (4) and one trivial inequality of the form (2), we obtain:

$$-2x_k - X_{ij} + X_{ik} + 2X_{jk} \leq 0. \tag{14}$$

Second, by summing together two triangle inequalities of the form (4) and two trivial inequalities of the form (1), we obtain:

$$-2x_i + X_{ij} + X_{ik} + 2X_{i\ell} - 2X_{j\ell} - 2X_{k\ell} \leq 0. \tag{15}$$

We can now form a new linear system, called 'system II', by adding the inequalities (14) and (15) to those already present in system I.

It follows from Theorem 1 that the trivial inequalities (3) and the triangle inequalities (4) and (5) can be derived as $\{0, \frac{1}{2}\}$-cuts from system II. The following proposition shows that some other facet-defining inequalities can be obtained.

Proposition 3 *Suppose that $n \geq 4$. The following facet-defining inequality, called a 'cut' inequality in [19], can be derived as a $\{0, \frac{1}{2}\}$-cut from system II:*

$$-x_3 - x_4 - X_{12} - X_{34} + X_{13} + X_{14} + X_{23} + X_{24} \leq 0.$$

Proof. If we sum together the inequalities $X_{23} + 2X_{13} - 2x_3 - X_{12} \leq 0$, $X_{24} + 2X_{14} - 2x_4 - X_{12} \leq 0$, $X_{23} - x_3 \leq 0$, $X_{24} - x_4 \leq 0$ and $x_3 + x_4 - 2X_{34} \leq 1$, we obtain the inequality:

$$-2x_3 - 2x_4 - 2X_{12} - 2X_{34} + 2X_{13} + 2X_{14} + 2X_{23} + 2X_{24} \leq 1.$$

Dividing the inequality by two and rounding down the right-hand side yields the cut inequality. □

More interestingly, one can obtain an exponentially large family of facet-defining inequalities, as explained in the following theorem:

Theorem 2 *Let $C \subset E_n$ be the edge set of a simple cycle of odd length, let h be a node not in the cycle, and let S be the set of 'spokes', i.e., edges connecting each node in the cycle to h. The following inequality, which we call an 'odd wheel' inequality, defines a facet of BQP_n and can be obtained $\{0, \frac{1}{2}\}$-cut with respect to system II:*

$$-\left\lfloor \frac{|C|}{2} \right\rfloor x_h - \sum_{e \in C} X_e + \sum_{e \in S} X_e \leq 0 \tag{16}$$

Proof. Let c denote $|C|$. Without loss of generality, assume that the cycle passes through nodes $1, \ldots, c$, in order. If we sum together the following $\lfloor \frac{c}{2} \rfloor$ inequalities of type (15):

$$\begin{gathered} -2x_h + X_{1h} + 2X_{2h} + X_{3h} - 2X_{12} - 2X_{23} \leq 0 \\ -2x_h + X_{3h} + 2X_{4h} + X_{5h} - 2X_{34} - 2X_{45} \leq 0 \\ \cdots \\ -2x_h + X_{c-2,h} + 2X_{c-1,h} + X_{ch} - 2X_{c-2,c-1} - 2X_{c-1,c} \leq 0 \end{gathered}$$

together with the following inequalities from system I:

$$\begin{gathered} -x_1 + X_{1h} \leq 0 \\ -x_c + X_{ch} \leq 0 \\ x_1 + x_c - 2X_{1c} \leq 1 \end{gathered}$$

we obtain:

$$-(c-1)x_h + 2\sum_{i=1}^{c} X_{ih} - 2\sum_{i=1}^{c} X_{i,i+1} \leq 1.$$

Dividing this inequality by 2 and rounding down the right-hand side, we obtain the odd wheel inequality.

We will show in the next section that the odd wheel inequality for BQP_n can be obtained from an odd bicycle wheel inequality for CUT_{n+1} using the covariance map. In light of...., this shows that the odd wheel inequality defines a facet. □

3.3 A linear system that is closed under switching

System II has a rather undesirable feature: the class of inequalities that can be derived as $\{0, \frac{1}{2}\}$-cuts from it is not closed with respect to the switching operation. For example, if we take the odd wheel inequality (16) and switch on node h, we obtain the inequality:

$$\left\lfloor \frac{c}{2} \right\rfloor x_h + \sum_{i \in V(C)} x_i - \sum_{e \in C \cup S} X_e \leq \left\lfloor \frac{c}{2} \right\rfloor, \tag{17}$$

where $V(C)$ is the set of nodes in the cycle C. Observe that the point (x^*, X^*) with $x_i^* = \frac{1}{2}$ for all i and $X_e^* = 0$ for all e satisfies all inequalities in system II, but violates the inequality (17) by $(c+1)/4$, which exceeds $1/2$ since $c \geq 3$. This shows that the inequality (17) is not a CG-cut with respect to system II, and in particular not a $\{0, \frac{1}{2}\}$-cut.

Fortunately, this problem can be resolved by expanding the linear system. We start by introducing the following additional variables:

$$
\begin{aligned}
x'_i &= 1 - x_i && (\forall i \in V_n) \\
X'_{ij} &= x_i x'_j = x_i - X_{ij} && (\forall i \in V_n, j \in V_n \setminus \{i\}) \\
X''_{ij} &= x'_i x'_j = 1 - x_i - x_j + X_{ij} && (\forall \{i,j\} \in E_n).
\end{aligned}
$$

Note that the X' variables are 'directed', in the sense that X'_{ij} is not equal to X'_{ji}. Note also that the constraints $X'_{ij} = x_i x'_j$ and $X''_{ij} = x'_i x'_j$ are quadratic, so we cannot insert them into our linear system. Also, the constraints $X'_{ij} = x_i - X_{ij}$ and $X''_{ij} = 1 - x_i - x_j + X_{ij}$ have more than two odd left-hand side coefficients, so they cannot be inserted either. Nevertheless, we can derive useful additional linear inequalities that can be so inserted. In particular, we can insert:

- inequalities of the form $-X'_{ij} \leq 0$ and $-X''_{ij} \leq 0$, which are analogous to the inequalities (10);
- inequalities of the form $X'_{ij} \leq x_i$, $X'_{ij} \leq x'_j$, $X''_{ij} \leq x'_i$ and $X''_{ij} \leq x''_j$, which are analogous to (11);
- inequalities of the form $x'_i \leq 1$, which are analogous to (12);
- inequalities of the form $x_i + x'_j \leq 1 + 2X'_{ij}$ and $x'_i + x'_j \leq 1 + 2X''_{ij}$, which are analogous to (13).

In exactly the same way, one can insert 'switched' versions of the inequalities (14) and (15).

Remember to call this extended linear system 'System III'.

Now, consider any inequality that can be derived as a $\{0, \frac{1}{2}\}$-cut from system II. In order to obtain a particular switching of this inequality, we just apply the same switching to all the original inequalities that are used to generate the cut. For example, to derive the switched odd wheel inequality (17), we use the inequalities:

$$
\begin{gathered}
-2x'_h + X'_{1h} + 2X'_{2h} + X'_{3h} - 2X_{12} - 2X_{23} \leq 0 \\
-2x'_h + X'_{3h} + 2X'_{4h} + X'_{5h} - 2X_{34} - 2X_{45} \leq 0 \\
\cdots \\
-2x'_h + X'_{c-2,h} + 2X'_{c-1,h} + X'_{ch} - 2X_{c-2,c-1} - 2X_{c-1,c} \leq 0
\end{gathered}
$$

together with the inequalities

$$
\begin{aligned}
-x_1 + X'_{1h} &\leq 0 \\
-x_c + X'_{ch} &\leq 0 \\
x_1 + x_c - 2X_{1c} &\leq 1.
\end{aligned}
$$

This yields the following $\{0, \frac{1}{2}\}$-cut:

$$
-\left\lfloor \frac{c}{2} \right\rfloor x'_h + \sum_{i=1}^{c} X'_{ih} - \sum_{i=1}^{c} X_{i,i+1} \leq 0.
$$

Expressing this $\{0, \frac{1}{2}\}$-cut in terms of the original variables, we obtain the inequality (17).

Using system III, then, leads to a much richer class of $\{0, \frac{1}{2}\}$-cuts, and in particular one that is closed with respect to the switching operation.

4 The Algorithm

Note that system II contains $\mathcal{O}(n^4)$ inequalities...

Lemma 1 *Given the vector (x^*, X^*), one can extract a family of $\mathcal{O}(n^3)$ inequalities in system III such that, if any $\{0, \frac{1}{2}\}$-cut is violated, there is a most-violated $\{0, \frac{1}{2}\}$-cut that uses only inequalities in that family in the CG derivation. This family can be constructed in $\mathcal{O}(n^4)$ time.*

Caprara and Fischetti observe that, for the purposes of separation, one can exclude from consideration all but $\mathcal{O}(q^2)$ constraints. Indeed, let S_{ij} be the set of constraints in the linear system that have odd coefficients for the ith and jth variables, and let k^* be a constraint in S_{ij} of minimum slack with respect to x^*. Then one can impose $\lambda_k = 0$ for all $k \in S_{ij} \setminus \{k^*\}$, without losing a most-violated $\{0, \frac{1}{2}\}$-cut (if any exists). This observation can enable one to eliminate parallel edges from the auxiliary graph mentioned above. Then... we only need to retain $\mathcal{O}(n^3)$ of them for the purposes of separation. Specifically, for each ordered triplet (i, j, k), among all the inequalities having odd coefficients for the edges $\{i, j\}$ and $\{i, k\}$, we need only retain the inequality with smallest slack...

Now explain that only $\mathcal{O}(n)$ odd cycle computations are needed, since it suffices to run Dijkstra only from nodes corresponding to x variables.

Lemma 2 *If any $\{0, \frac{1}{2}\}$-cut is violated, then there is a most-violated $\{0, \frac{1}{2}\}$-cut that corresponds to an odd cycle that passes through an x-node in the auxiliary graph.*

This yields our desired result:

Theorem 3 *The separation problem for the* $\{0, \frac{1}{2}\}$*-cuts derived from system III can be solved in* $\mathcal{O}(n^4)$ *time.*

This is so since the parity graph contains $\mathcal{O}(n^2)$ nodes and $\mathcal{O}(n^3)$ edges...

Some Additional Implementational Details:

If an odd cycle found by the algorithm is not simple (i.e., not a 'circuit'), one can easily extract a simple odd cycle with no larger weight, in linear time. In terms of $\{0, \frac{1}{2}\}$-cuts, this amounts to replacing the original $\{0, \frac{1}{2}\}$-cut with a stronger one.

It also helps to add a small quantity $\epsilon > 0$ to the edge-weights in the parity graph. This biases the algorithm in favour of smaller cycles, which tend to correspond to cuts that are both stronger and sparser. (We chose $\epsilon = 0.01$. This reduced the running time by about 50%, without worsening the bound.)

if a violated or near-violated $\{0, \frac{1}{2}\}$-cut is produced from the weakened system, it may be possible to strengthen it *post hoc*, by replacing the weakened constraints in the $\{0, \frac{1}{2}\}$-derivation with their original, stronger counterparts. Then, it sometimes happens that the same non-weakened inequality contributes to more than one of the weakened inequalities. Then, we can obtain a stronger 0-1/2 cut by taking the fractional part of its CG multiplier. We call this 'post-hoc strengthening'.

For large instances, one may not want to generate so many inequalities. To get around this, one can maintain a list of the 100 odd cycles with smallest weights. When this list is full, the largest weight of a cycle in the list can be used as an upper bound during the shortest path computations, to skip any paths with larger weights.

Another possible improvement: as soon as a node becomes permanently labelled in a Dijkstra call, we can check whether its copy has also been permanently labelled. If so, we can check if the sum of the two labels is less than 1. If so, this yields a violated cut.

Note: In the next section, we will see that we could potentially get more cutting planes by running our algorithm an additional $n-1$ times, on vectors that are modified using the covariance map.

5 Application to the Cut Polytope

We can apply our separation routine to the cut polytope, since the covariance mapping maps points outside of CUT_n to points outside of BQP_{n-1}.

Now we consider the $(2p+1,2)$-circulant inequalities. Applying the covariance map to the inequality (9) and eliminating index $2p+1$ we obtain the following:

Lemma 3 *Let $p \geq 2$ be an even integer, let $1,\ldots,2p$ be distinct indices, and let $2p+1=1$. The inequality*

$$\sum_{i=1}^{2p} 2x_i - \sum_{i=1}^{2p} X_{i,i+1} - \sum_{i=1}^{2p-2} X_{i,i+2} \leq \frac{3}{2}p \tag{18}$$

defines a facet of BQP_n*, $n \geq 2p$.*

Using the additional variables introduced above, we can obtain inequality (18) as a $\{0,\frac{1}{2}\}$-cut:

Theorem 4 *The inequality (18) is a $\{0,\frac{1}{2}\}$-cut with respect to the extended linear system III.*

Proof. When $p=2$, the inequality (18) coincides with a 4-clique inequality with $\alpha=2$, which can be obtained by switching from a 3-wheel inequality. So we will assume here that $p \geq 4$. Add the following inequalities:

$$\begin{gathered} x'_1 + x'_{2p} - 2X''_{1,2p} \leq 1 \\ -x'_1 + X'_{3,1} \leq 0 \\ -x'_{2p} + X'_{2p-2,2p} \leq 0 \\ -2x_3 + X'_{3,1} + X'_{3,4} + 2X'_{3,2} - 2X''_{1,2} - 2X''_{2,4} \leq 0 \\ -2x_{2p-2} + X'_{2p-2,2p-3} + X'_{2p-2,2p} + 2X'_{2p-2,2p-1} - 2X''_{2p-3,2p-1} - 2X''_{2p-1,2p} \leq 0 \\ -2x_3 + 2X'_{3,5} + X'_{3,4} - X''_{4,5} \leq 0 \\ -2x_6 + 2X'_{6,4} + X'_{6,5} - X''_{4,5} \leq 0. \end{gathered}$$

If $p \geq 6$ also add the following inequalities, for $i = 5, 9, \ldots, 2p-7$,

$$\begin{gathered} -2x'_i + 2X'_{i+2,i} + X'_{i+1,i} - X_{i+1,i+2} \leq 0 \\ -2x'_{i+3} + 2X'_{i+1,i+3} + X'_{i+2,i+3} - X_{i+1,i+2} \leq 0, \\ -2x_{i+2} + 2X'_{i+2,i+4} + X'_{i+2,i+3} - X''_{i+3,i+4} \leq 0 \\ -2x_{i+5} + 2X'_{i+5,i+3} + X'_{i+5,i+4} - X''_{i+3,i+4} \leq 0. \end{gathered}$$

Again, division by 2 and truncation of the right-hand side yields the following (admittedly complicated) $\{0,\frac{1}{2}\}$-cut

$$\begin{aligned}
-2x_3 - \sum_{k=2}^{p/2-1} \left(x'_{4k-3} + x_{4k-2} + x_{4k-1} + x'_{4k}\right) - 2x_{2p-2} \\
-X''_{1,2p} - X''_{1,2} + X'_{3,1} + X'_{3,2} - X''_{2,4} + X'_{3,4} + X'_{3,5} - X''_{4,5} + X'_{6,4} \\
+ \sum_{k=2}^{p/2-1} \big(X'_{4k-2,4k-3} + X'_{4k-1,4k-3} - X_{4k-2,4k-1} + X'_{4k-2,4k} \\
+X'_{4k-1,4k} + X'_{4k-1,4k+1} - X''_{4k,4k+1} + X'_{4k+2,4k}\big) \\
+X'_{2p-2,2p-3} - X''_{2p-3,2p-1} + X'_{2p-2,2p-1} + X'_{2p-2,2p} - X''_{2p-1,2p} \quad \leq \quad 0.
\end{aligned}$$

By expressing this inequality in the original variables we obtain the inequality (18). □

I think that the choice of the 'special' node that we eliminate when we go from the cut polytope to the Boolean quadric polytope affects the cuts that we can get. E.g., take an odd bicycle wheel inequality with a wheel of size 5, and designate a node outside the bicycle wheel as 'special'. The resulting inequality for the Boolean quadric polytope involves 7 nodes, and I don't think it can be derived as a 0-1/2 cut from our system.

This suggests that one could run our algorithm n times, choosing a different 'special' node each time, to get more general cuts.

Two thoughts:

- A $\{0,\frac{1}{2}\}$-cut for BQP_{n-1} does not necessarily correspond to a $\{0,\frac{1}{2}\}$-cut for CUT_n. Indeed, it is mentioned in the Deza and Laurent book that the triangle inequalities for CUT_n that do not involve a given node cannot be derived as CG-cuts from the triangle inequalities that do involve the node.
- We now know that we can get all odd bicycle wheel and $(p,2)$-circulant inequalities using our approach.

Emphasize that our algorithm has the same running time as the one by Gerards, yet generates more cuts. It is also much faster than the one by Letchford.

6 Computational Experiments

We could report the results of some computational experiments. It makes sense to separate first over the trivial and triangle inequalities, before invoking the heavy machinery of the $\{0,\frac{1}{2}\}$-cut separation routine.

Suitable test problems might be those given in the 'BiqMac' library.

7 Conclusion

We could adapt our algorithm to the case of sparse instances. One option would be to make the instance complete by inserting all missing variables and constraints. A better option might be to follow the strategy of Gruber [14], who adds only enough edges to his max-cut instances to make the graph *chordal.* (One can cover the edges of a chordal graph with a linear number of cliques.)

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