Linear Arrangement Polytopes

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Working paper, April 2011

Abstract

The *Linear Arrangement Problem* is a well-known, and strongly NP-hard, graph layout problem. In this paper, we define an associated family of integral polytopes, which turn out to be closely related to some other polyhedra explored in the literature. Several classes of facet-defining inequalities are presented, first for the case of complete graphs, and then for the case of general graphs.

Keywords: graph layout problems – linear arrangement – polyhedral combinatorics

1 Introduction

Given a graph $G = (V, E)$, an *arrangement* is simply a permutation of the vertex set V. The permutation is viewed as an arrangement of the vertices on the real line $\mathbb{R}$, in such a way that the distance between consecutive vertices is equal to 1. For a given arrangement π, and a pair of vertices $\{i, j\} \subset V$, the quantity $|\pi(i) - \pi(j)|$ can be thought of as the Euclidean distance between i and j in the arrangement.

Several important combinatorial optimisation problems, called *graph layout problems*, call for an arrangement that optimises a function of the pairwise distances. We refer the reader to the excellent survey by Díaz *et al.* [6] for details. Here, we are interested in one particular graph layout problem, the so-called *Linear Arrangement Problem* (LAP), in which the objective is to minimise $\sum_{\{i,j\}\in E} |\pi(i) - \pi(j)|$.

The LAP was originally proposed in [11], and proven to be strongly NP-hard in [9]. The survey paper [6] contains several references on exact and heuristic algorithms, polynomially-solvable special cases, and lower bounding techniques.

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P(G)	$= \text{conv}\left\{d \in \mathbb{R}_+^{\lvert E\rvert} :\ \exists \pi \in \Pi :\ d_{ij} = \lvert\pi(i) - \pi(j)\rvert\ \ \forall\{i,j\} \in E\right\}$	
D(G)	$= \text{conv}\left\{d \in \mathbb{R}_+^{\lvert E\rvert} :\ \exists \pi \in \Pi :\ d_{ij} \geq \lvert\pi(i) - \pi(j)\rvert\ \ \forall\{i,j\} \in E\right\}$ $= \text{P}(G) \oplus \mathbb{R}_+^{\lvert E\rvert}$	
Q(G)	$= \text{cl conv}\left\{d \in \mathbb{R}_+^{\lvert E\rvert} : \exists x \in \mathbb{R}^n :\ d_{ij} = \lvert x_i - x_j\rvert \geq 1\ \ \forall\{i,j\} \in E\right\}$ $= \text{P}(G) \oplus \text{CUT}(G)$	

Table 1: Three Families of Polyhedra

A promising approach to computing lower bounds for the LAP was initiated in Liu & Vannelli [13]. For each $\{i,j\} \in E$, they introduced a general integer variable d_{ij}, which represents the distance $|\pi(i) - \pi(j)|$. They then derived several classes of linear inequalities that are satisfied by all feasible d vectors. These linear inequalities were used in an LP-based cutting-plane algorithm to compute lower bounds for the LAP.

A few other papers presented valid linear inequalities for the feasible d vectors. (We will survey these papers in the next section.) Nevertheless, these papers do not address the question of whether these inequalities define facets of the convex hull of feasible d vectors. The goal of the present paper is to do exactly that. More specifically, we define the following integral polytope:

$$P(G) := \text{conv}\left\{d \in \mathbb{Z}_+^E : \exists \pi \in \Pi : d_{ij} = |\pi(i) - \pi(j)|\ (\forall \{i,j\} \in E)\right\},$$

where Π denotes the set of all permutations. We will derive several classes of linear inequalities that define *facets* of this polytope.

The structure of the paper is as follows. In Section 2, we briefly review some of the relevant literature. In Sections 3 and 4, we study the polytope $P(K_n)$, where K_n is the complete graph on n nodes. Section 3 is concerned with some inequalities of a 'combinatorial' flavour, having small integral coefficients. Section 4 is concerned with the so-called *hypermetric* inequalities, and variants of them, that can have quite complicated coefficients. In Section 5, we study $P(G)$ for a general graph G, which is more complicated. Finally, concluding remarks are given in Section 6.

2 Literature Review

START BY DEFINING THE THREE FAMILIES OF POLYHEDRA

Notice that $P(K_n)$ 'lives' in $\mathbb{R}^{\binom{n}{2}}$, where n is the number of vertices, and that $P(G)$ is the projection of $P(K_n)$ onto the subspace $\mathbb{R}^{|E|}$.

As mentioned above, Liu & Vannelli [13] presented several classes of valid inequalities for $P(G)$. All of their inequalities were of the form:

$$\sum_{e \in F} d_e \geq r(F), \tag{1}$$

where $F \subset E$ is chosen in such a way that the appropriate right-hand side $r(F)$ can be computed easily. For the sake of brevity, we mention only one of their classes: the *star* inequalities. These take the form:

$$\sum_{i \in S} d_{ri} \geq \left\lfloor (|S|+1)^2/4 \right\rfloor, \tag{2}$$

where $r \in V$ and S is a set of vertices, all of which are adjacent to r in G.

Next, Even *et al.* [7] introduced the so-called *spreading metrics*. These are vectors $d \in \mathbb{R}^{\binom{n}{2}}$ that satisfy two sets of inequalities: the *triangle inequalities*

$$d_{ik} + d_{jk} \geq d_{ij} \qquad (\forall \{i,j,k\} \subset V), \tag{3}$$

and the *spreading* inequalities

$$\sum_{j \in S} d(i,j) \geq |S|(|S|+2)/4 \qquad (\forall i \in [n], \forall S \subseteq [n] \setminus \{i\}). \tag{4}$$

Note that the spreading inequalities are more general than the star inequalities, but have a slightly weaker right-hand side when n is odd. Spreading metrics were used in [14] to derive an approximation algorithm for the LAP with performance guarantee $\mathcal{O}(\log n)$.

In [3, 8], it was noted that one can get a tighter relaxation of graph layout problems by requiring the spreading metrics to satisfy the following *negative-type* inequalities:

$$\sum_{\{i,j\} \subset [n]} b_i b_j d(i,j) \leq 0 \qquad (\forall b \in \mathbb{R}^n : \sum_{i=1}^{n} b_i = 0). \tag{5}$$

The negative-type spreading metrics were used to obtain an approximation algorithm for MinLA with an improved performance guarantee of $\mathcal{O}(\sqrt{\log n} \log \log n)$.

Letchford *et al.* [12] study a family of unbounded polyhedra related to certain Euclidean metrics, and show that these polyhedra have $P(K_n)$ as a face.

In Caprara *et al.* [2], a family of unbounded polyhedra, the so-called *dominant* of $P(G)$, is studied.

Formalisation of L & V

Some more classes of valid inequalities of the same form were introduced by

Caprara *et al.* [2]. Again, for the sake of brevity, we mention only one class: the *circuit* inequalities. These take the form:

$$\sum_{e \in C} d_e \geq 2(|C| - 1), \tag{6}$$

where C is a set of edges that forms a circuit (i.e., a simple cycle) in G.

Caprara *et al.* [2] also pointed out that the following *metric* inequalities, which are not of the form (1), are valid for $P(G)$:

$$\sum_{e \in C \setminus \{f\}} d_e \geq d_f, \tag{7}$$

where C is again a circuit in G, and f is an arbitrary edge in C.

None of the above papers studies $P(K_n$ or $P(G)$ itself. In our companion paper... on facility layout problems [1], we presented various results on a class of polytopes generalizing $P(K_n$. Specialising these results to the case of MinLA, we obtain the following results:

Theorem 1 (Amaral & Letchford [1]) *$P(K_n)$ is of dimension $\binom{n}{2} - 1$, and its affine hull is defined by the implicit equation:*

$$\sum_{1 \leq i < j \leq n} d_{ij} = \binom{n}{3}. \tag{8}$$

Theorem 2 (Amaral & Letchford [1]) *The following clique inequalities define facets of $P(K_n)$:*

$$\sum_{\{i,j\} \subset C} d_{ij} \geq \binom{|C|+1}{3}, \tag{9}$$

for all $C \subseteq V$ such that $|C| \geq 2$.

Theorem 3 (Amaral & Letchford [1]) *The following pure hypermetric inequalities are valid for $P(K_n)$:*

$$\sum_{i \in S, j \in T} d_{ij} - \sum_{\{i,j\} \subset S} d_{ij} - \sum_{\{i,j\} \subset T} d_{ij} \geq 0, \tag{10}$$

for all disjoint non-empty $S, T \subset V$ such that $|S| = |T| + 1$. They define facets if and only if $|S| + |T| \leq |V| - 2$.

Theorem 4 (Amaral & Letchford [1]) *The following strengthened pure negative-type (SPN) inequalities define facets of $P(K_n)$:*

$$\sum_{i \in S, j \in T} d_{ij} - \sum_{\{i,j\} \subset S} d_{ij} - \sum_{\{i,j\} \subset T} d_{ij} \geq |S|, \tag{11}$$

for all disjoint non-empty vertex sets S, T such that $|S| = |T|$.

Notice that the clique inequalities reduce to trivial lower bounds of the form $d_e \geq 1$ when $|C| = 2$, as do strengthened negative-type inequalities when $|S| = |T| = 1$. Moreover, the hypermetric inequalities reduce to triangle inequalities of the form $d_{ik} + d_{jk} - d_{ij} \geq 0$ when $|S| = 2$ and $|T| = 1$.

Theorem 5 (Amaral & Letchford [1]) *The star inequalities do not in general define facets of $P(K_n)$.*

MENTION CANONICAL FORM. NOTE THAT THE IDEA, BUT NOT THE TERM ITSELF, APPEARS IN OUR SRFLP PAPER.

We will need the following definition, taken from [1]:

Definition 1 (Amaral & Letchford, 2008) *Let $\alpha^T d \geq \beta$ be a linear inequality, where $\alpha, d \in \mathbb{R}^{\binom{n}{2}}$. The inequality is said to be 'canonical' if:*

$$\min_{\emptyset \neq S \subset [n]} \sum_{i \in S} \sum_{[n] \setminus S} \alpha_{ij} = 0. \tag{12}$$

In [1], it is shown that every facet of $P(n, \ell)$ is defined by a canonical inequality. The following lemma is the analogous result for $P(G)$:

Lemma 1 *Every facet of $P(K_n)$ is defined by a canonical inequality.*

Proof. We can always add or subtract a suitable multiple of the implicit equation. □

Theorem 6 *Suppose that the inequality $\alpha^T d \geq \beta$ is valid for $P(K_n)$ and defines a proper face, and suppose that it is canonical. Then the 'zero-lifted' inequality*

$$\sum_{1 \leq i < j \leq n} \alpha_{ij} d_{ij} \geq \beta \tag{13}$$

is valid for $P(K'_n)$ and defines a proper face of it, for all $n' > n$.

WE WILL SHOW IN SUBSECTION... THAT ZERO-LIFTING A FACET-DEFINING CANONICAL INEQUALITY does not always produce a facet-defining inequality!

3 Some Combinatorial Inequalities for $P(K_n)$

In this section, we present four new classes of valid inequalities for $P(K_n)$. We call them 'combinatorial' inequalities because they are derived using combinatorial arguments, and involve graph-theoretic structures, such as cycles, wheels and stars.

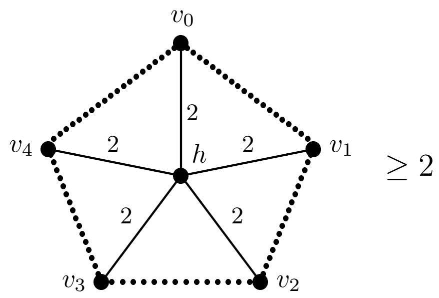

Figure 1: An odd wheel inequality with $c = 5$.

3.1 Odd wheel inequalities

Our first class of inequalities is described in the following proposition:

Proposition 1 *Let $c \geq 3$ be an odd integer, and let $v_0, \ldots, v_c$ and h be distinct vertices. Then the following 'odd wheel' inequality is valid for $P(K_n)$:*

$$2\sum_{i=0}^{c-1} d_{v_i,h} - \sum_{i=0}^{c-1} d_{v_i,v_{i+1}} \geq 2, \tag{14}$$

where indices are taken modulo c (see Figure 1 for an illustration).

Proof. Note that a triangle inequality of the form $d_{v_i,h} + d_{v_{i+1},h} \geq d_{v_i,v_{i+1}}$ can be satisfied at equality if and only if vertex h lies between vertices v_i and v_{i+1}, otherwise it has a slack at least 2. Since c is odd, at most $c-1$ of the triangle inequalities of this form can be satisfied at equality. □

Note that the odd wheel inequalities are canonical.

Computational experiments with PORTA suggest that the odd wheel inequalities define facets when $5 \leq c \leq n-2$.

We remark that Chopra & Rao [4] introduced some inequalities called 'odd wheel' inequalities for a different combinatorial optimisation problem, called the *clique partitioning problem*. Using a transformation given in Deza *et al.* [5], those inequalities can be converted into valid inequalities for the cut polytope. The resulting inequalities are however invalid for both the cut cone and $P(K_n)$.

3.2 2-chorded cycle inequalities

Our second class of inequalities is described in the following proposition:

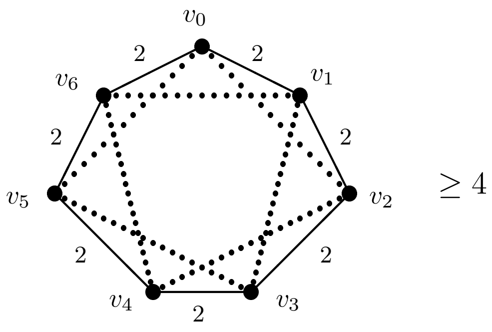

Figure 2: A 2-chorded cycle inequality with $c = 7$.

Proposition 2 *Let $c \geq 4$ be an integer, and let $v_0, \ldots, v_c$ be distinct vertices. Then the following '2-chorded cycle' inequality is valid for $P(K_n)$:*

$$2\sum_{i=0}^{c-1} d_{v_i,v_{i+1}} - \sum_{i=0}^{c-1} d_{v_i,v_{i+2}} \geq 4, \tag{15}$$

where indices are taken modulo c (see Figure 2 for an illustration).

Proof. Note that a triangle inequality of the form $d_{v_i,v_{i+1}} + d_{v_{i+1},v_{i+2}} \geq d_{v_i,v_{i+2}}$ can be satisfied at equality if and only if vertex v_{i+1} lies between vertices v_i and v_{i+2}, otherwise it has a slack at least 2. Moreover, at most $c-2$ of the triangle inequalities of this form can be satisfied at equality. □

Note that the 2-chorded cycle inequalities are also canonical.

Computational experiments with PORTA suggest that the 2-chorded cycle inequalities always define facets.

We remark that Grötschel & Wakabayashi [10] introduced some inequalities called '2-chorded cycle' inequalities for the clique partitioning problem. Again, however, the corresponding valid inequalities for the cut polytope are invalid for both the cut cone and $P(K_n)$.

3.3 Modified 2-chorded cycle inequalities

Next, we present some inequalities that can be obtained from 2-chorded cycle inequalities by simply adjusting three of the left-hand coefficients, along with the right-hand side:

Proposition 3 *The following 'modified 2-chorded cycle' inequalities are valid for $P(K_n)$:*

$$2\sum_{i=1}^{c-2} d_{v_i,v_{i+1}} + d_{v_0,v_1} + d_{v_0,v_{c-1}} - \sum_{i=0}^{c-2} d_{v_i,v_{i+2}} \geq 2, \tag{16}$$

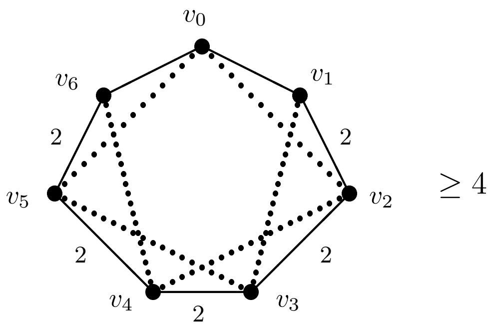

Figure 3: A modified 2-chorded cycle inequality, with $c = 7$.

where $v_0, \ldots, v_c$ and h are vertices, $c \geq 4$, and indices are taken modulo c (see Figure 3 for an illustration).

Proof. Note that a triangle inequality of the form $d_{v_i,v_{i+1}} + d_{v_{i+1},v_{i+2}} \geq d_{v_i,v_{i+2}}$, where $0 \leq i \leq c-2$, can be satisfied at equality if and only if vertex v_{i+1} lies between vertices v_i and v_{i+2}, otherwise it has a slack at least 2. Moreover, at most $c-2$ of the triangle inequalities of this form can be satisfied at equality. □

Note that the modified 2-chorded cycle inequalities are canonical as well.

Computational experiments with PORTA suggest that these inequalities define facets when $c < n$.

3.4 Strong star inequalities

As mentioned in Section 2, the *star* inequalities (2) are valid for $P(K_n)$, for all $r \in V$ and all $S \subseteq V \setminus \{r\}$. Note however that they are not canonical, and therefore they cannot define facets of $P(K_n)$ in general.

The following proposition presents some inequalities that dominate the star inequalities, and are also canonical:

Proposition 4 *The following* strong star *inequalities are valid for $P(K_n)$:*

$$(|S|-1)\sum_{i\in S} d_{hi} - \sum_{\{i,j\}\subset S} d_{ij} \geq \left\lfloor (|S|+1)^2(|S|-1)/12 \right\rfloor . \qquad (17)$$

for all $r \in V$ and all $S \subseteq V \setminus \{r\}$ with $|S| \geq 2$. Together with the clique inequality on $S \cup \{i\}$, the strong star inequality dominates the star inequality.

Proof. Validity is easy to show when $V = \{r\} \cup S$: it suffices to multiply the star inequality (2) by $|S|$ and subtract the implicit equation (8). It is also

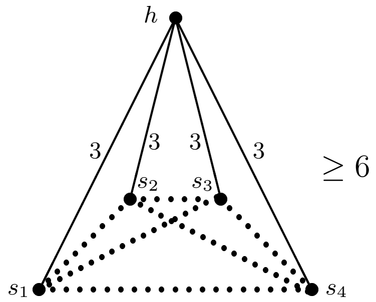

Figure 4: A strong star inequality with $|S| = 4$.

easy to show that the inequality meets the condition for lifting presented in Theorem 6. Thus, it is valid also when $\{r\} \cup S$ is a strict subset of V. Finally, the star inequality can be obtained by summing together the strengthened star inequality and the clique inequality, and dividing by $|S|$. □

The conditions for strong star inequalities to define facets of $P(K_n)$ turn out to be remarkably complicated, as demonstrated in the following theorem:

Theorem 7 *Strong star inequalities define facets of $P(K_n)$ if and only if any of the following conditions hold:*

- $|S| = 2$ *and* $n \geq 5$,
- $|S| \geq 3$ *and odd,*
- $|S| \geq 6$ *and even, and* $n \neq |S| + 2$.

Proof. For the sake of brevity, we only sketch the proof. If $|S| = 2$, the strong star inequality reduces to a triangle inequality, which we already know to be facet-defining if and only if $n \geq 5$. If $|S| = 4$, then the strong star inequality is easily shown to be dominated by four strong star inequalities having three 'satellites' each. If $|S| \geq 6$ and even, and $n = |S| + 2$, then every extreme point of $P(K_n)$ satisfying the strong star inequality at equality corresponds to an arrangement in which $\pi(r) \in \{n/2, n/2 + 1\}$. Thus, each such extreme point satisfies the equation

$$\sum_{i \in V \setminus \{r\}} d_{ri} = (|S| + 2)^2/4,$$

and the inequality is not facet-defining. In all other cases, a sequence of exchange arguments (similar to those used in [1]) demonstrates that the inequality is facet-defining. □

Remark: The strong star inequality with $|S| = 6$ defines a facet of $P(K_7)$, but not of $P(K_8)$. This shows that zero-lifting a facet-defining canonical inequality does not always produce a facet-defining inequality!

Interestingly, we will show in Subsection 5.3 that (unstrengthened) star inequalities can sometimes define facets of $P(G)$.

4 Hypermetric Inequalities and Variants for $P(K_n)$

We know already that pure hypermetric and strong star inequalities define facets under mild conditions, and we know how to strengthen the pure negative-type inequalities to obtain facets. In this section, we try to generalise those results.

4.1 Strong bounded hypermetric inequalities

Here, show how to compute the correct RHS for the case in which the components of the vector sum to 1 and all components are bounded from above by 1.

4.2 Strong bounded negative-type inequalities

Here, do the same for the case in which the components of the vector sum to 0 and all components are bounded from above by 1.

5 $P(G)$ for general graphs

We now turn our attention to general graphs.

5.1 Dimension

Above, we saw that $P(K_n)$ is not full-dimensional. Remarkably, $P(G)$ is full-dimensional for almost all graphs, as stated in the following theorem:

Theorem 8 *If G is connected, but not complete, then $P(G)$ is of dimension $|E|$, i.e., is full-dimensional.*

Proof. Suppose that G is connected but not complete, and suppose that $\sum_{e \in E} \alpha_e d_e = \beta$ is an implicit equation, i.e., is satisfied by all extreme points of $P(G)$. Since G is not complete, there exists at least one vertex that is not adjacent to every other vertex. Let i be such a vertex and let j and k be distinct vertices such that $\{i, j\} \in E$ but $\{i, k\} \notin E$. Let π^1 be any arrangement such that j and k are in the first two positions, and let π^2

be the arrangement obtained from π^1 by exchanging vertices j and k. A comparison of the two arrangements shows that:

$$\sum_{e \in \delta(j)} \alpha_e = \sum_{e \in \delta(k)} \alpha_e. \tag{18}$$

Now let π^3 be any arrangement in which the first three positions are occupied by vertices i, j and k, respectively, and let π^4 be the arrangement obtained from π^3 by exchanging vertices j and k. A comparison of these two arrangements shows that:

$$\sum_{e \in \delta(j)} \alpha_e - 2\alpha_{ij} = \sum_{e \in \delta(k)} \alpha_e. \tag{19}$$

Together with (18) this implies that $\alpha_{ij} = 0$. Thus, if i is any vertex that is not adjacent to all other vertices, $\alpha_e = 0$ for all $e \in \delta(i)$.

Now let S be the set of vertices that are adjacent to all other vertices. Since we are assuming that G is not complete, $|S| \leq |V| - 2$. Moreover, if $|S| \leq 1$, the implicit equation reduces to the vacuous equation $0 = 0$. So we assume that $2 \leq |S| \leq |V| - 2$. Let i be a vertex in S and j a vertex not in S. Let π^5 be an arrangement in which i is in the first position and j is in the second, and let π^6 be the arrangement obtained from π^5 by exchanging the positions of i and j. A comparison of these two arrangements shows that:

$$\sum_{r \in S \setminus \{i\}} \alpha_{ir} = 0 \qquad (\forall i \in S). \tag{20}$$

Finally, let i and j be vertices in S and let k be a vertex not in S. Let π^7 be an arrangement in which vertices i, j and k occupy the first three positions, in order, and let π^8 be the arrangement obtained from π^7 by moving i, j and k to the second, third and first position, respectively. A comparison of these two arrangements shows that:

$$\sum_{r \in S \setminus \{i,j\}} \alpha_{ir} + \sum_{r \in S \setminus \{i,j\}} \alpha_{jr} = 0 \qquad (\forall \{i,j\} \subset S). \tag{21}$$

Together with (20), this implies that $\alpha_{ij} = 0$ for all $\{i,j\} \subset S$. Thus, the implicit equation is again vacuous. □

5.2 'Borrowing' facets of $P(K_n)$

The following proposition shows that inequalities that define facets of $P(K_n)$ can also define facets of $P(G)$:

Proposition 5 *If an inequality defines a facet of $P(K_n)$, and its left-hand-side coefficients are zero for all pairs $\{i,j\} \notin E$, then it defines a facet of $P(G)$ as well.*

Proof. The inequality is clearly valid for $P(G)$, since $P(G)$ is the projection of $P(K_n)$ onto $\mathbb{R}^{|E|}$. Now, suppose that it does not define a facet of $P(G)$. Then it is dominated by two or more inequalities that do define facets of $P(G)$. Since these inequalities also involve only the edges in E, they are valid for $P(K_n)$ as well. This contradicts the fact that the original inequality defines a facet of $P(K_n)$. $\square$

Thus, the clique, pure hypermetric, SPN, strengthened star, odd wheel and 2-chorded cycle inequalities all define facets of $P(G)$, whenever G contains subgraphs with a suitable structure.

5.3 Facets that do not come from $P(K_n)$

Proposition 5 tells us that facets of $P(K_n)$ can yield facets of $P(G)$. It is however possible for an inequality to define a facet of $P(G)$, yet not define a facet of $P(K_n)$. Before introducing some of these inequalities, we will need the following refinement of Proposition 5:

Proposition 6 *Let $G^1 = (V, E^1)$ and $G^2 = (V, E^2)$ be two graphs, such that $E^2 \subset E^1$. If an inequality defines a facet of $P(G^1)$, and its left-hand-side coefficients are zero for all pairs $\{i, j\} \notin E^2$, then it defines a facet of $P(G^2)$ as well.*

Proof. Similar to that of Proposition 5. $\square$

The following theorem states that most of the metric inequalities define facets of $P(G)$:

Theorem 9 *Metric inequalities define facets of $P(G)$ if $|C| \leq n-2$.*

Proof. For the sake of brevity, we only sketch the proof. Let $c = |C|$. We assume without loss of generality that $f = \{1, c\}$ and that the other edges in the circuit are of the form $\{i, i+1\}$ for $i = 1, \ldots, c-1$. From Proposition 6, we can assume that each vertex in $\{c+1, \ldots, n\}$ is adjacent to all other vertices. Now let F be the face of $P(G)$ defined by the metric inequality, and let $\sum_{e \in E} \alpha_e d_e = \beta$ be an equation satisfies by all points in F. We use a series of exchange arguments to show that the equation is unique (up to scaling).

First, note that the identity arrangement yields a distance vector that lies in F, as does any arrangement obtained from it by permuting the vertices in $\{c+1, \ldots, n\}$. Just as in the proof of Theorem 1, this shows that α_{ij} is a constant for all $\{i, j\} \subset \{c+1, \ldots, n\}$. Let us call this constant γ.

Now consider the arrangement obtained from the identity arrangement by shifting vertices $2, \ldots, n-2$ two places to the right, and placing vertices $n-1$ and n in the second and third position. This arrangement also yields a distance vector that lies in F, as does any arrangement obtained from it by permuting the vertices in the first three positions. This shows that $\alpha_{1,n-1} = \alpha_{1,n} = \gamma$. By symmetry, $\alpha_{1k} = \gamma$ for all $k \in \{c+1, \ldots, n\}$.

Now consider the arrangement obtained from the identity arrangement by shifting vertices $3, \ldots, n-2$ two places to the right, and placing vertices $n-1$ and n in the third and fourth position. This arrangement also yields a distance vector that lies in F, as does any arrangement obtained from it by permuting the vertices in the second, third and fourth positions. This shows that $\alpha_{2,n-1} = \alpha_{2,n} = \gamma$. By symmetry, $\alpha_{2k} = \gamma$ for all $k \in \{c+1, \ldots, n\}$.

Repeating this argument, one shows that $\alpha_{ij} = \gamma$ for $i = 1, \ldots, c$ and for $j = c+1, \ldots, n$.

Now consider the identity arrangement again, and note that the arrangement obtained from it by shifting vertices $2, \ldots, n-1$ one place to the right, and placing vertex n in the first position, also lies in F. This fact can be used to show that $\gamma = 0$.

Finally, for $k = 2, \ldots, c$, we can take the identity arrangement, shift vertices $k, \ldots, n-1$ one position to the right, and place vertex n in the kth position, and still obtain a distance vector that lies in F. This fact can be used to show that $\alpha_{k,k+1} = -\alpha_{1n}$ for $k = 2, \ldots, c$. Thus, the equation is of the form:

$$\alpha \sum_{e \in C \setminus \{f\}} d_e - \alpha d_f = \beta,$$

for some constant α. It is therefore unique (up to scaling). □

The situation with the star inequalities is more complicated. At the moment, we have the following conjecture:

Conjecture 1 *Star inequalities define facets of $P(G)$ if the following conditions all hold:*

1. *S is not a clique,*
2. *$V \setminus (S \cup \{r\})$ is not a clique,*
3. *$|S| \geq 3$.*

Notice that, when $|S| = 1$, the star inequality reduces to a trivial lower bound, and is therefore facet-defining from Theorem 2. Moreover, when $|S| = 2$, the star inequality is the sum of two trivial lower bounds, and therefore not facet-defining.

6 FURTHER NOTES

As in the SRFLP paper:

- The separation problems for clique, bounded strong hypermetric and bounded strong negative-type inequalities are likely to be NP-hard.
- The separation problem for triangle inequalities can be solved in $\mathcal{O}(n^3)$ time by mere enumeration.
- The separation problem for star inequalities (without strengthening) can be solved in polynomial time by a simple greedy algorithm.

It is possible that the separation problem for strong star inequalities (a special kind of bounded strong hypermetric inequalities) could be solved in polynomial time, by some kind of reduction to minimisation of a quadratic submodular function.

It seems likely that the separation problem for odd wheel and 2-chorded cycle inequalities are polynomially-solvable. One would construct an auxiliary graph in which each node represents a triangle inequality, and each edge represents a pair of triangle inequalities that have two nodes in common, and then perform some kind of shortest path or shortest circuit computation.

Acknowledgement: The research of the first author was supported by CNPq-Brazil Grant `500212/02-3`. The research of the second author was supported by the Engineering and Physical Sciences Research Council under grant `EP/D072662/1`.

References

[1] A.R.S. Amaral & A.N. Letchford (2008) A polyhedral approach to the single-row facility layout problem. Submitted.

[2] A. Caprara, A.N. Letchford & J.J. Salazar (2011) Decorous lower bounds for minimum linear arrangement. *INFORMS J. Comput.*, 23, 26–40.

[3] M. Charikar, M.T. Hajiaghayi, H. Karloff & S. Rao (2010) l_2^2 spreading metrics for vertex ordering problems. *Algorithmica*, 56, 577–604.

[4] S. Chopra, M.R. Rao (1993) The partition problem. *Math. Program.*, 59, 87–115.

[5] M. Deza, M. Grötschel & M. Laurent (1992) Clique-web facets for multicut polytopes. *Math. Oper. Res.*, 17, 981–1000.

[6] J. Díaz, J. Petit & M. Serna (2002) A survey of graph layout problems. *ACM Computing surveys*, 34, 313–356.

[7] G. Even, J. Naor, S. Rao & B. Schieber (2000) Divide-and-conquer approximation algorithms via spreading metrics. *J. of the ACM*, 47, 585–616.

[8] U. Feige & J.R. Lee (2007) An improved approximation ratio for the minimum linear arrangement problem. *Inf. Proc. Lett.*, 101, 26–29.

[9] M.R. Garey, D.S. Johnson & L.J. Stockmeyer (1976). Some simplified NP-complete graph problems. *Theor. Comp. Sci.*, 1, 237–267.

[10] M. Grötschel & Y. Wakabayashi (1990) Facets of the clique partitioning polytope. *Math. Program.*, 47, 367-387.

[11] L.H. Harper (1964) Optimal assignment of numbers to vertices. *SIAM Journal*, 12, 131–135.

[12] A.N. Letchford, G. Reinelt, H. Seitz & D.O. Theis (2010) On a class of metrics related to graph layout problems. *Lin. Alg. Appl.*, 433, 1760–1777.

[13] W. Liu & A. Vannelli (1995) Generating lower bounds for the linear arrangement problem. *Discr. Appl. Math.*, 59, 137–151.

[14] S. Rao & A.W. Richa (2005) New approximation techniques for some linear ordering problems. *SIAM J. on Computing*, 34, 388–404.