ON THE MEMBERSHIP PROBLEM FOR THE {0,1/2}-CLOSURE

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Abstract. In integer programming, $\{0,1/2\}$-cuts are Gomory-Chvátal cuts that can be derived from the original linear system by using coefficients of value 0 or 1/2 only. The separation problem for $\{0,1/2\}$-cuts is strongly NP-hard. We show that separation remains strongly NP-hard, even when all integer variables are binary.

1. Introduction

We consider rational polyhedra $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ with $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$. Inequalities of the form

$$(\lambda^\mathsf{T} A)x \leq \lfloor \lambda^\mathsf{T} b \rfloor, \tag{1}$$

with $\lambda \in \mathbb{R}^m$, $\lambda^\mathsf{T} A \in \mathbb{Z}^n$, and $\lambda^\mathsf{T} b \notin \mathbb{Z}$ are commonly referred to as Gomory-Chvátal cuts; they were first mentioned in the work of Gomory [13] and Chvátal [7]. Gomory-Chvátal cuts are valid for the integer hull, $P_I = \mathrm{conv}\{x \in \mathbb{Z}^n : Ax \leq b\}$, of P. It is well-known that it suffices to consider λ-vectors with small coefficients (see, e.g., [18]); more specifically,

$$P' := \{x : (\lambda^\mathsf{T} A)x \leq \lfloor \lambda^\mathsf{T} b \rfloor, \lambda \in \mathbb{R}^m, \lambda^\mathsf{T} A \in \mathbb{Z}^n\} = \{x : (\lambda^\mathsf{T} A)x \leq \lfloor \lambda^\mathsf{T} b \rfloor, \lambda \in [0,1]^m, \lambda^\mathsf{T} A \in \mathbb{Z}^n\},$$

and this rational polyhedron is commonly referred to as the first Gomory-Chvátal closure. Geometrically speaking, P' arises from P by considering all inequalities that are valid for P and pushing the associated hyperplanes towards P_I until they contain some integer point. In particular, P' is a stronger relaxation of P_I than P, i.e., $P_I \subseteq P' \subseteq P$. There are several prominent explicit examples of Gomory-Chvátal cuts in polyhedral combinatorics, including the blossom inequalities of the matching polytope [10, 7], the odd-cycle inequalities of the stable set polytope [17], the simple comb inequalities of the symmetric traveling salesman polytope [15, 4], and the simple Möbius ladder inequalities of the acyclic subdigraph polytope [14, 2], to name a few. Interestingly, the separation problem for all these families of inequalities (or classes containing them) can be solved in polynomial time. Moreover, all these cuts can be derived as in (1) with $\lambda \in \{0,1/2\}^m$. This prompted Caprara and Fischetti [2] to introduce the family of all $\{0,1/2\}$-cuts,

$$\mathcal{F}_{1/2}(A,b) := \{(\lambda^\mathsf{T} A)x \leq \lfloor \lambda^\mathsf{T} b \rfloor : \lambda \in \{0,1/2\}^m, \lambda^\mathsf{T} A \in \mathbb{Z}^n\},$$

and to analyze the computational complexity of the following problem: Given $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, and $\hat{x} \in \mathbb{Q}^n$ with $A\hat{x} \leq b$, does $\hat{x}$ violate an inequality in $\mathcal{F}_{1/2}(A,b)$? This problem is, of course, equivalent to the membership problem for the $\{0,1/2\}$-closure of $P = \{x \in \mathbb{R}^n : Ax \leq b\}$, which is defined by the points in P that satisfy all inequalities in $\mathcal{F}_{1/2}(A,b)$. Caprara and Fischetti showed that checking whether $\hat{x}$ violates some inequality in $\mathcal{F}_{1/2}(A,b)$ is, in general, strongly NP-complete (and, therefore, the membership problem is strongly coNP-complete). However, the polytopes of

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interest in combinatorial optimization oftentimes have vertices with coordinates 0 or 1; that is, $P \subseteq [0,1]^n$, which is not the case for the instances that occur in Caprara and Fischetti's proof. This provides the motivation for our work, in which we study the following problem:

> Given $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$ such that $\{x \in \mathbb{R}^n : Ax \le b\} \subseteq [0,1]^n$, and $\hat{x} \in \mathbb{Q}^n$ with $A\hat{x} \le b$, does $\hat{x}$ violate an inequality in $\mathcal{F}_{1/2}(A,b)$?

Our main result is that this problem is still strongly NP-complete, and we give two different proofs for it, each of which is interesting in its own right. One proof is a careful modification of Caprara and Fischetti's proof, and the other one shows a slightly stronger result in that the corresponding 0/1-polytopes that arise from our reduction are of set-packing type. That is, P is of the form $P = \{x \in \mathbb{R}^n_+ : Ax \le \mathbb{1}\}$, where $A \in \{0,1\}^{m\times n}$ is a binary matrix, and $\mathbb{1}$ denotes the all-one vector with m entries. Before we present the first proof in Section 2 and the second proof in Section 3 below, we briefly discuss some other work related to $\{0,1/2\}$-cuts.

Related Work. Caprara and Fischetti's original proof of their own hardness result (published in [3]) was, in some sense, stronger than that in [2], because it actually showed that checking whether a point violates some inequality in $\mathcal{F}_{1/2}(A,b)$ is strongly NP-hard, even when the nonnegativity constraints $-x_i \le 0$ for all $i = 1,\ldots,n$ are part of the system $Ax \le b$. A similar proof was later given in [5]. "Maximally violated" $\{0,1/2\}$-cuts, however, can always be separated in polynomial time; in fact, this is true more generally for mod-k cuts, for any given $k \ge 2$ [4]. A mod-k cut is an inequality of the form (1) in which each component of λ is a multiple of $1/k$. A mod-k cut is maximally violated by a given point if the difference between its left-hand side and its right-hand side is equal to $(k-1)/k$. Earlier, Caprara and Fischetti [2, 3] had given a couple of sufficient conditions under which the separation problem for $\{0,1/2\}$-cuts can be solved in polynomial time. Letchford [16] later introduced a superclass of $\{0,1/2\}$-cuts, so-called binary clutter inequalties, which allowed him to describe broader classes of special cases in which $\{0,1/2\}$-cuts can be separated efficiently. However, the computational complexity of separating {0,1/2}-cuts for systems $Ax \le b$ with $P = \{x : Ax \le b\} \subseteq [0,1]^n$ has remained open and is the subject of this paper.

2. A reduction from Decoding of Linear Codes

The following problem, known as Decoding of Linear Codes, is NP-complete [12, Problem MS7]:

> Given a matrix $Q \in \{0,1\}^{r\times t}$, a vector $d \in \{0,1\}^r$, and a positive integer K, is there a $z \in \{0,1\}^t$ with no more than K entries equal to one such that $Qz \equiv d \mod 2$?

Theorem 2.1. *The membership problem for the $\{0,1/2\}$-closure of polytopes contained in the 0/1-hypercube is strongly coNP-complete.*

Our reduction carefully modifies the proof of Caprara and Fischetti in [2] so as to ensure that $P \subseteq [0,1]^n$.

Proof. Membership testing is clearly in coNP. We give a reduction from Decoding of Linear Codes to show completeness. Let Q, d, and K describe an instance of Decoding of Linear Codes. We construct the following instance of the membership problem for the $\{0,1/2\}$-closure:

$$A := \begin{pmatrix} Q^\top & \\ d^\top & 2I_{t+1} \\ 2I_r & 0 \\ -2I_r & 0 \\ 0 & -3I_{t+1} \end{pmatrix}, \quad b := \begin{pmatrix} 2\cdot\mathbf{1}^t \\ 1 \\ 2\cdot\mathbf{1}^r \\ \mathbf{0}^r \\ \mathbf{0}^{t+1} \end{pmatrix}, \quad \hat{x} := \begin{pmatrix} \mathbf{0}^r \\ \mathbf{1}^t - \frac{1}{2}w \\ \frac{1}{2} \end{pmatrix},$$

where I_l is the identity matrix in dimension l, $\mathbf{0}^l$ is the all-zero vector in dimension l, $\mathbf{1}^l$ is the l-dimensional all-one vector, $w := \frac{1}{K+1}\cdot\mathbf{1}^t$, and the 0's in the matrix represent all-zero submatrices of appropriate dimension. We first show that $P = \{x \mid Ax \leq b\} \subseteq [0,1]^{r+t+1}$. Consider row l of $Ax \leq b$:

Case 1: $(t+1)+1 \leq l \leq (t+1)+r$. We obtain the inequality $2x_{l-(t+1)} \leq 2$ and, therefore, $x_{l-(t+1)} \leq 1$. Put differently, $x_l \leq 1$ for all $1 \leq l \leq r$.

Case 2: $(t+1)+r+1 \leq l \leq (t+1)+2r$. We have $-2x_{l-(t+1+r)} \leq 0$ and, therefore, $x_{l-(t+1+r)} \geq 0$. We obtain $x_l \geq 0$ for all $1 \leq l \leq r$.

The first $t+1$ rows of A correspond to inequalities of the form $\sum_{j=1}^{r} q_{ji}x_j + 2x_{r+l} \leq 2$ for $1 \leq l \leq t$ and $\sum_{j=1}^{r} d_j x_j + 2x_{r+l} \leq 1$ for $l = t+1$. The nonnegativity of the coefficients of Q and d together with $x_j \geq 0$ for all $j \in \{1,\ldots,r\}$ implies $x_{r+l} \leq 1$ for all $1 \leq l \leq t+1$.

Finally, consider row l with $(t+1)+2r+1 \leq l \leq 2(t+1)+2r$. The corresponding inequalities are of the form $-3x_{r+l-((t+1)+2r)} \leq 0$ and, therefore, $x_{r+l} \geq 0$ for all $1 \leq l \leq t+1$. It follows that $P \subseteq [0,1]^{r+t+1}$. Moreover, $b - A\hat{x} = (w_1,\ldots,w_t, 0, 2\cdot\mathbf{1}^r, \mathbf{0}^r, 3-\frac{3}{2}w_1,\ldots,3-\frac{3}{2}w_t,\frac{3}{2})^\mathsf{T}$. In particular, $\hat{x} \in P$.

Note that $\hat{x}$ violates a $\{0,1/2\}$-cut if and only if there exists $\mu \in \{0,1\}^{2(t+1)+2r}$ such that $\mu^\mathsf{T}A \equiv 0 \pmod 2$, $\mu^\mathsf{T}b \equiv 1 \pmod 2$, and $\mu^\mathsf{T}(b - A\hat{x}) < 1$. To have $\mu^\mathsf{T}A \equiv 0 \pmod 2$, it is necessary that $\mu_l = 0$ for $(t+2)+2r \leq l \leq 2(t+1)+2r$. Furthermore, $\mu^\mathsf{T}b \equiv 1 \pmod 2$ if and only if $\mu_{t+1} = 1$. Consequently, there exists a $\mu \in \{0,1\}^{2(t+1+r)}$ with $\mu^\mathsf{T}A \equiv 0 \pmod 2$ and $\mu^\mathsf{T}b \equiv 1 \pmod 2$ if and only if there exists a $z \in \{0,1\}^t$ such that $Qz \equiv d \pmod 2$ with $z \in \{0,1\}^t$. Indeed, $z_l = \mu_l$ for $1 \leq l \leq t$, and the remaining μ_l for the reverse direction can be chosen arbitrarily for those rows of A that are equal to zero (mod 2).

Note that $w^\mathsf{T}z < 1$ if and only if no more than K entries of z are equal to 1. Thus, it remains to show that $\mu^\mathsf{T}(b - A\hat{x}) < 1$ if and only if $w^\mathsf{T}z < 1$ with z and μ as above. Assume first that $\mu^\mathsf{T}(b - A\hat{x}) < 1$. Recall that $b - A\hat{x} = (w_1,\ldots,w_t,0,2\cdot\mathbf{1}^r,\mathbf{0}^r,3-\frac{3}{2}w_1,\ldots,3-\frac{3}{2}w_t,\frac{3}{2})^\mathsf{T}$. Therefore, $w^\mathsf{T}z < 1$ for $z \in \{0,1\}^t$ with $z_l = \mu_l$ for $1 \leq l \leq t$. Conversely, let $w^\mathsf{T}z < 1$ for some $z \in \{0,1\}^t$. Define $\mu \in \{0,1\}^{2(t+1)+2r}$ by $\mu_l := z_l$ for $1 \leq l \leq t$, $\mu_{t+1} := 1$, and $\mu_l := 0$ otherwise. Then $1 > w^\mathsf{T}z = \mu^\mathsf{T}(w_1,\ldots,w_t,0,2\cdot\mathbf{1}^r,\mathbf{0}^r,3-\frac{3}{2}w_1,\ldots,3-\frac{3}{2}w_t,\frac{3}{2})^T = \mu^\mathsf{T}(b - A\hat{x})$. So there is a violated $\{0,1/2\}$-cut if and only if there is a solution to Decoding of Linear Codes. □

3. Reduction from Exact 3-Cover

For a given $n \times m$ 0/1-matrix A, the intersection graph is an undirected graph with vertex set $V = \{1,\ldots,n\}$, and an edge $\{i,j\}$ if and only if there is at least one row of A with a '1' in the ith and jth columns [17]. The edge $\{i,j\}$ represents the fact that x_i and x_j cannot take the value 1 simultaneously. The set packing problem amounts to the problem of finding a maximum weight stable set (set of pairwise non-adjacent vertices) in the intersection graph. Padberg [17] showed that every clique C (i.e., every set of pairwise adjacent vertices) in the intersection graph yields a valid clique inequality $\sum_{j\in C} x_j \leq 1$ for the set packing polytope, and that such an inequality induces a facet of that polytope if and only if the clique is maximal.

In general, there may be many facet-inducing clique inequalities which are not represented in the system $Ax \leq \mathbf{1}$. Indeed, the number of maximal cliques can be exponential in n and m. If, however, there is a one-to-one correspondence between the rows of A and the maximal cliques of the intersection graph (i.e., the system $Ax \leq \mathbf{1}$ consists of the facet-inducing clique inequalities), then A is said to be a *clique matrix*.

We will find it helpful to write the $\{0,1/2\}$-cuts of a clique matrix in a certain explicit form. Let $t \geq 1$ be an odd integer, and let $C_1,\ldots,C_t$ be maximal cliques whose associated clique inequalities are to be used (receive a multiplier of 1/2) in the derivation of the cut. For $i = 1,\ldots,n$, let ϕ_i represent the number of these cliques which contain i. That is, $\phi_i = |\{k \in \{1,...,t\} : i \in C_k\}|$. Then, we must use (set the multiplier to 1/2 for) a non-negativity inequality $-x_i \leq 0$ for each

$i \in V$ such that ϕ_i is odd. Thus, the cut takes the form:

$$\sum_{i=1}^{n} \lfloor \phi_i/2 \rfloor x_i \leq \lfloor t/2 \rfloor.$$

Multiplying by two, we see that this is equivalent to

$$\sum_{k=1}^{t} \sum_{i \in C_k} x_i - \sum_{\phi_i \text{ odd}} x_i \leq t-1.$$

Following [2], we define the slack variables $s_k := 1 - \sum_{i \in C_k} x_i$ for $k = 1, \ldots, t$. The cut can then be written as

$$\sum_{k=1}^{t} s_k + \sum_{\phi_i \text{ odd}} x_i \geq 1.$$

Thus, we see that the $\{0, 1/2\}$-cut derived using cliques $C_1, \ldots, C_t$ is violated by a given $\hat{x}$ if and only if

$$\sum_{k=1}^{t} \hat{s}_k + \sum_{\phi_i \text{ odd}} \hat{x}_i < 1, \tag{2}$$

where $\hat{s}_k$ equals the slack of the kth clique inequality, computed with respect to $\hat{x}$.

We recall the definition of the NP-complete decision problem EXACT 3-COVER [12, Problem SP2]:

> Let s be a multiple of three and let $S_1, \ldots, S_q \subset \{1, \ldots, s\}$ be such that $|S_k| = 3$ for $k = 1, \ldots, q$. Is there some $R \subseteq \{1, \ldots, q\}$ with $|R| = s/3$ such that $\bigcup_{k \in R} S_k = \{1, \ldots, s\}$?

Theorem 3.1. *Testing whether a given $\hat{x} \in P = \{x \mid Ax \leq b\}$ violates a $\{0, 1/2\}$-cut is strongly NP-complete, even when the corresponding integer linear program is a set packing problem, the matrix A is a clique matrix, and the intersection graph of A contains only $O(n)$ maximal cliques.*

Proof. Given an instance of EXACT 3-COVER, we construct a graph with $2s + 2 + q$ vertices and $2q + 3$ maximal cliques (see Figure 1). For $i = 1, \ldots, s$, we have two vertices u_i and v_i. For $k = 1, \ldots, q$ we have a vertex w_k. We also add two further vertices u^* and v^*. Edges are put into the graph so that there are $2q + 3$ maximal cliques, as follows. The vertices of type 'u' will be mutually adjacent and form the *u-clique*. The vertices of type 'v' will likewise be mutually adjacent and form the *v-clique*. The two vertices u^* and v^* will also be connected by an edge, forming the *2-clique*. For $k = 1, \ldots, q$, we connect w_k to the three u-vertices representing S_k, thus forming q cliques of cardinality 4. We will call these the *upper 4-cliques*. Finally, for $k = 1, \ldots, q$, we connect w_k to the three v-vertices representing S_k, thus forming q more cliques of cardinality 4. We will call these the *lower 4-cliques*.

We now let A equal the clique matrix of this graph. (Note that A has $2q + 3$ rows and $2s + 2 + q$ columns.) We define a vector $\hat{x} \in P$ as follows. For $i = 1, \ldots, s$, we set the component of $\hat{x}$ corresponding to u_i to $2/(3s + 3)$, and we do the same for v_i. We set the component of $\hat{x}$ corresponding to u^* to $(s + 3)/(3s + 3)$, and we do the same for v^*. Finally, for $k = 1, \ldots, q$, we set the component of $\hat{x}$ corresponding to w_k to $(3s - 6)/(3s + 3)$.

It is readily checked that the u-clique and the v-cliques have slack zero, the 2-clique has slack $(s - 3)/(3s + 3)$, and each of the upper and lower 4-cliques have slack $3/(3s + 3)$.

If the ϕ coefficient of a given vertex is odd, then we say that the vertex is *exposed*. Each w vertex is contained in exactly two cliques (an upper 4-clique and a lower 4-clique). An exposed w vertex contributes $(3s - 6)/(3s + 3)$ to the left-hand side of (2). Thus, there is at most one exposed w vertex.

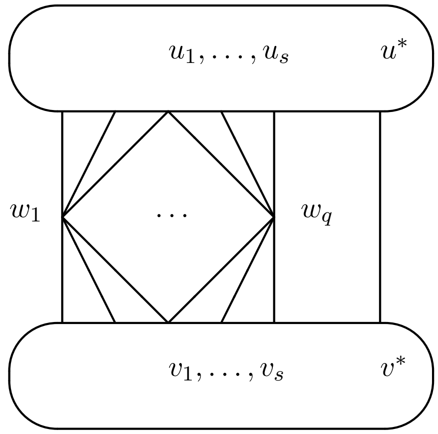

FIGURE 1. Graph used in proof

Suppose there was *exactly one* exposed w vertex. As each upper and lower 4-clique used contributes $3/(s+3)$ to the left-hand side of (2), at most two of them could be used in the Gomory-Chvátal derivation. In fact, exactly one would have to be used, otherwise there would be either zero or two exposed w vertices. Moreover, the 2-clique could not be used either, because it would contribute $(s-3)/(3s+3)$ to the left-hand side of (2). Only the u and v cliques remain, and the $\{0, 1/2\}$-cut becomes vacuous. Therefore, there are no exposed w vertices.

Thus, we have shown that if an upper 4-clique is used, the corresponding lower 4-clique must be used as well. That is, the 4-cliques come in *pairs*. Then, in order for the number of cliques used to be odd, we must use either one or three of the u-, v- and 2-cliques.

Suppose we use the u-clique but not the v- or 2-cliques. The vertex u^* is exposed, contributing $(s+3)/(3s+3)$ to the left-hand side of (2). Suppose we use K *pairs*. Each pair contributes $6/(3s+3)$ to the left-hand side. Moreover, the number of exposed u_i vertices is at least $s - 3K$ and each contributes $2/(3s+3)$ to the left-hand side. Thus, the left-hand side is at least $(s + 3 + 6K + 2s - 6K)/(3s+3) = 1$, and the cut is not violated. By symmetry, we cannot use the v-clique without using the u- and 2-cliques. Moreover, we cannot use the 2-clique without using the u- and v-cliques because this would immediately contribute 1 to the left-hand side of (2).

In order to obtain a violated cut, then, we must use the u-, v- and 2-cliques, together with a number of *pairs*. Suppose we use K *pairs*. Each *pair* contributes $6/(3s+3)$ to the left-hand side of (2) and the 2-clique contributes $(s-3)/(3s+3)$. Moreover, the number of exposed u vertices is at least $\max\{0, s - 3K\}$, and the same holds for the number of exposed v vertices. Thus, the left-hand side of (2) is at least

$$6K/(3s+3) + (s-3)/(3s+3) + \max\{0, 4s - 12K\}/(3s+3).$$

It is readily checked that this is less than one if and only if $K = s/3$. Thus, there is a violated $\{0, 1/2\}$-cut if and only if $K = s/3$ and there are no exposed vertices at all. This is true if and only if, for $i = 1, \ldots, s$, vertex u_i appears in exactly one of the $s/3$ upper 4-cliques and vertex v_i appears in exactly one of the $s/3$ lower 4-cliques. Thus, there is a violated $\{0, 1/2\}$-cut if and only if there is a solution to EXACT 3-COVER. □

4. CONCLUDING REMARKS

It is not difficult to see that finding a stable set of maximum weight in graphs of the type used in the proof of Theorem 3.1 can be performed in polynomial time (by enumerating over all possible choices of a u-vertex, and all possible choices of a v-vertex). Therefore, the hardness result holds even if the associated integer linear program itself is polynomially solvable. On the other hand, Caprara and Salazar [6] consider an interesting class of NP-hard set packing problems for which the separation of $\{0, 1/2\}$-cuts is polynomially solvable. So the complexity of a class of integer linear programs is not related to the complexity of the separation problem for the associated $\{0, 1/2\}$-cuts. See also Caprara and Letchford [5] and Cornuéjols and Li [9].

It is worth pointing out that the hardness proof of Section 3 can easily be adapted to set partitioning and set covering problems. This is interesting because Bienstock and Zuckerberg [1] have recently shown that, in the case of set covering, one can separate over *all* Gomory-Chvátal-cuts to an arbitrary fixed precision in polynomial time.

Naturally, our results imply that it is NP-hard to optimize a linear function over the $\{0, 1/2\}$-closure of a polyhedron $P \subseteq [0, 1]^n$. This provides an interesting contrast to the fact that one can optimize in polynomial time over the elementary closures associated with lift-and-project, Sherali-Adams, Lovász-Schrijver, and Lasserre cuts (see, e.g., [8]).

For Caprara and Fischetti's second proof of their hardness result (in [2]), it is not difficult to see that the $\{0, 1/2\}$-closure and the Gomory-Chvátal closure coincide [11]. In particular, testing membership (or separation) over the Gomory-Chvátal closure is NP-hard in general. However, in spite of the results provided herein, it remains unknown whether testing membership for the Gomory-Chvátal closure remains NP-hard for rational polytopes contained in the unit cube.

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