Generalised Network Design Polyhedra[1]

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Outline

1. Generalised network design problems
2. A family of polyhedra
3. Polyhedral results
4. Concluding remarks

The problems

Many important combinatorial optimisation problems can be written in the following form:

> *Given an undirected graph, a vector of node weights and/or a vector of edge weights, find a subgraph of minimum cost that satisfies a certain property.*

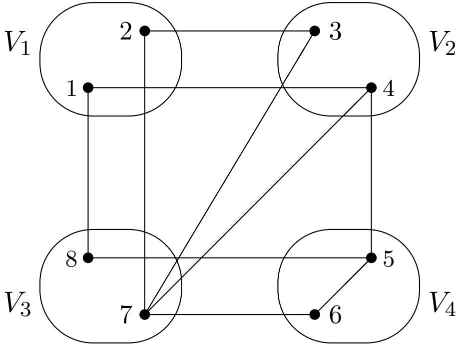
Well-known examples include the spanning tree problem, the perfect matching problem and the traveling salesman problem.

The problems (cont.)

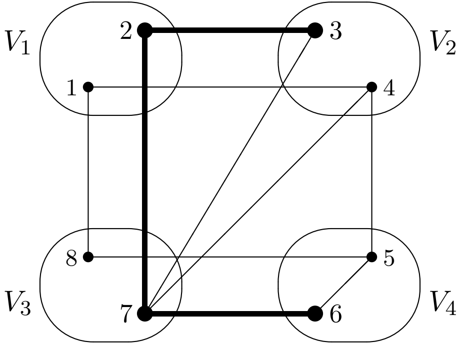
Given any such problem, we can define the following ‘generalised’ version:

> *Given an undirected graph whose set of nodes is partitioned into ‘clusters’, a vector of node weights and/or a vector of edge weights, find a subgraph of minimum cost that contain exactly one node from each cluster, and such that the graph obtained by shrinking each cluster into a single node satisfies a certain property.*

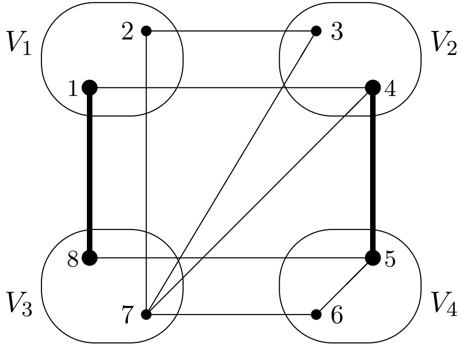
Graph partitioned into clusters



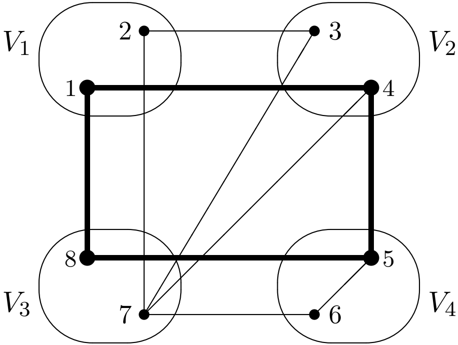
Generalised spanning tree



Generalised perfect matching



Generalised Hamiltonian circuit



The problems (cont.)

- The generalised traveling salesman problem was defined in Henry-Labordere, (1969), Srivastava *et al.* (1969) and Saskena (1970).
- The generalised minimum spanning tree problem was defined in Myung, Lee and Tcha (1995).
- Feremans, Labbé and Laporte (2003) defined the generalised shortest path problem and some other variants.

The problems (cont.)

Feremans *et al.* (2003) call all such problems generalised network design problems.

This is because the main source of practical applications is in the design of computer or telecommunications networks.

Normally, each cluster represents a geographical area (e.g., town, city, country) and the nodes in the cluster represent possible hub types and/or locations within that area.

The polyhedra

- The study of polyhedra has been the key to obtaining good algorithms for a variety of hard combinatorial optimisation problems.
- Researchers have studied the polyhedra associated with various specific GNDPs, but without any unifying framework.
- We define a class of ‘master’ polyhedra, that are associated with GNDPs in general.
- Inequalities that define facets of these ‘master’ polyhedra can be used as cutting planes for any specific GNDP.

The polyhedra (cont.)

We use the following notation:

- G is an undirected graph with vertex set V and edge set E.
- The clusters are labelled V_k, for $k \in K$.
- For each $v \in V$, we have a binary variable x_v.
- For each edge $e \in E$, we have a binary variable y_e.
- For any $S \subseteq V$, we let $x(S)$ denote $\sum_{v \in S} x_v$.
- For any $F \subset E$, we let $y(F)$ denote $\sum_{e \in F} y_e$.

The polyhedra (cont.)

We are interested in pairs of vectors $(x, y) \in \{0,1\}^{|V|+|E|}$ such that:

$$x(V_k) = 1 \qquad (k \in K) \tag{1}$$

$$y_{uv} \le x_u, y_{uv} \le x_v \quad (\{u, v\} \in E). \tag{2}$$

We let $P(G)$ denote the associated polyhedron.

We refer to the constraints (1) and (2) as *cluster* constraints and *variable upper bounds* (VUBs), respectively.

The polyhedra (cont.)

Remark

As well as being related to GNDPs, $P(G)$ is also closely related to the quadratic semi-assignment problem, and to several constraint satisfaction problems (CSPs) such as:

- `MAX CAPACITY REPRESENTATIVES` (Bellare, 1993)
- `UNIQUE GAMES` (Khot, 2002)
- `MAX 2-CONJ` and `MAX 2-CSP` (Serna, Trevisan & Xhafa, 2005).

Our results can be applied to those problems too.

Results for $P(G)$

Lemma

$P(G)$ is of dimension $|V| + |E| - m$. That is, the cluster constraints define the affine hull.

Theorem (Canonical Form)

Every non-trivial facet is defined by an inequality of the form

$$\sum_{e \in E} \beta_e y_e \leq \sum_{v \in V} \alpha_v x_v + \gamma,$$

where α, β and γ are all non-negative, and, in each cluster, there exists at least one node u for which $\alpha_u = 0$.

Results for $P(G)$ (cont.)

Lemma

For each $e \in E$, the lower bound $y_e \geq 0$ defines a facet of $P(G)$.

Lemma

The bounds $0 \leq x_v \leq 1$ for all v, and $y_e \leq 1$ for all e, do not define facets, except in degenerate cases.

Theorem

The VUBs do not define facets of $P(G)$, but the following strengthened VUBs (SVUBs) do so under mild conditions:

$$y(E(\{v\} : V_k)) \leq x_v \qquad (k \in K, v \in V \setminus V_k).$$

Results for $P(G)$ (cont.)

When $|V_k| \leq 2$ for all $k \in K$, optimising a linear function over $P(G)$ is essentially equivalent to *unconstrained boolean quadratic programming* (UBQP).

The polyhedron associated with UBQP is called the *boolean quadric polytope* (Padberg, 1989) and has been studied in depth (Deza & Laurent, 1997).

We found a general procedure for converting valid inequalities for the boolean quadric polytope into valid inequalities for $P(G)$ (for general G).

Example I

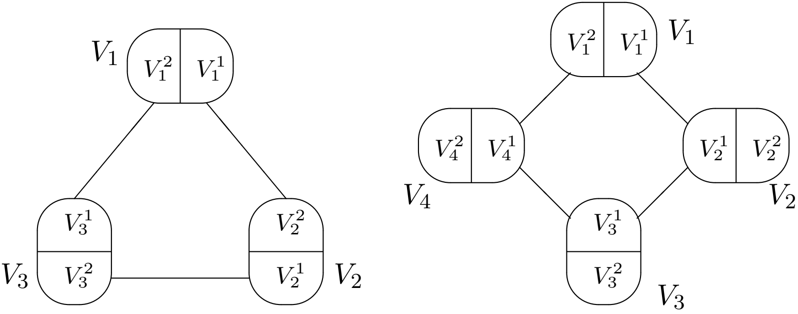


Figure: Odd ring inequalities, which correspond to Padberg's odd cycle inequalities.

Example II

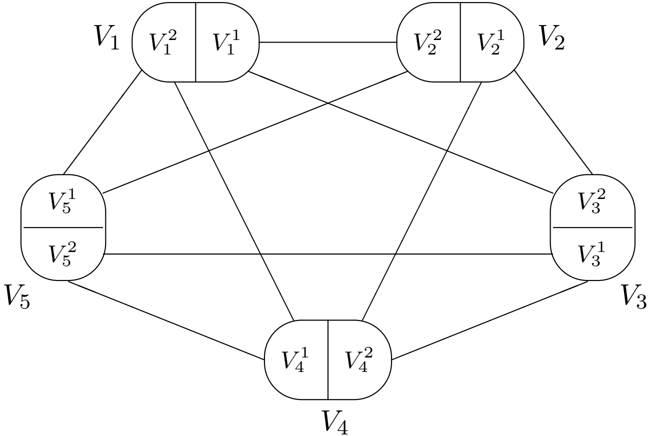


Figure: Odd clique inequalities, which correspond to Padberg's clique inequalities.

Results for $P(G)$ (cont.)

Theorem

Odd ring inequalities define facets of $P(G)$*.*

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Odd clique inequalities define facets of $P(G)$*.*

Remark

Inequalities similar to odd ring inequalities appeared in Feremans, Labbé & Laporte (2004) and Koster, Van Hoesel & Kolen (1998). The odd clique inequalities are however entirely new.

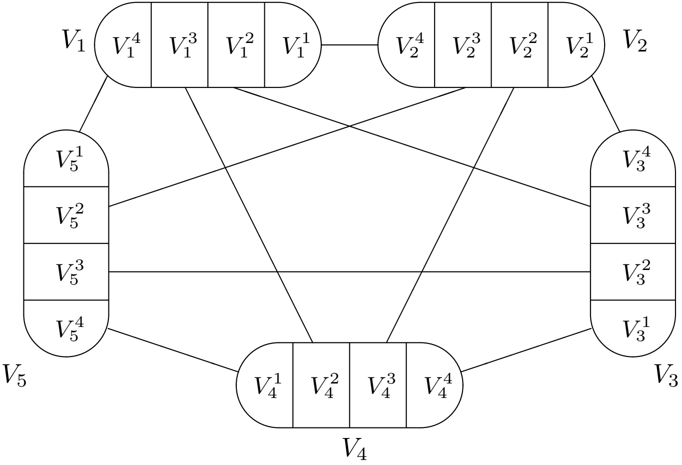
Results for $P(G)$ (cont.)

We also found another large class of facet-defining inequalities, called *odd circulant* inequalities.

These inequalities involve partitions of clusters into more than two sub-clusters, and therefore do not come from the boolean quadric polytope.

They generalise the *odd clique matching* and *odd cycle* inequalities, introduced by Feremans *et al.* (2004) in the context of the generalised minimum spanning tree problem.

Odd circulant inequalities



Related polyhedra

We also considered two other (families of) polyhedra:

- $P^{\leq}(G)$, obtained by requiring the subgraph to contain *at most* one node per cluster;
- $P^{\geq}(G)$, obtained by requiring the subgraph to contain *at least* one node per cluster.

$P^{\leq}(G)$ has similar properties to $P(G)$, but $P^{\geq}(G)$ is significantly more complex.

Concluding remarks

- Before now, people studied GNDPs more or less independently.
- We have shown that it is possible, to some extent, to study all of them simultaneously.
- We have defined a wide array of valid and facet-defining inequalities, along with a general procedure to derive further ones.
- These inequalities can be used within branch-and-cut algorithms for specific GNDPs.
- Some research is needed however to derive effective separation routines.