Some Unbounded Convex Sets Arising in Non-Convex MIQP[1]

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Outline

Introduction

We consider *Mixed-Integer Quadratic Programs* (MIQPs) of the form:

$$\min \left\{ x^T Q x + c^T x : Ax = b, x \in \mathbb{R}_+^p \times \mathbb{Z}_+^q \right\}.$$

We assume w.l.o.g. that the matrix Q is symmetric.

We also assume that Q is not *positive semidefinite* (psd), which means that the objective function is *non-convex*.

Introduction (cont.)

Following a standard approach, we introduce the *matrix variable* $X = xx^T$. (Note that $X_{ij} = x_i x_j$ for all i and j.)

The problem can now be reformulated as:

$$\begin{aligned} \min \quad & Q \bullet X + c \cdot x \\ \text{s.t.} \quad & Ax = b \\ & x \in \mathbb{R}^p_+ \times \mathbb{Z}^q_+ \\ & X = xx^T \end{aligned}$$

The non-linearity and non-convexity have been 'transferred' to the quadratic constraint $X = xx^T$.

Introduction (cont.)

Notice that we can delete the constraints $Ax = b$, provided we penalise $||Ax - b||_2^2$ in the objective function.

This motivates us to study the following set of points:

$$\left\{(x, X) : \ x \in \mathbb{R}_+^p \times \mathbb{Z}_+^q, \ X = xx^T\right\},$$

along with its convex hull.

Valid linear (or conic) constraints for the convex hull can be used to construct useful linear (or conic) programming relaxations of non-convex MIQPs.

Introduction (cont.)

Remark 1

The convex hull is *unbounded* and turns out to be *open*.

So we look at the *closure* of the convex hull.

Remark 2

The closure of the convex hull is *not polyhedral*, even when $p = 0$.

So we use elements of *convex analysis* in addition to traditional polyhedral theory.

Literature Review

The idea of using matrix variables for quadratic problems has a long history (e.g., Lovász, 1979; Shor, 1987; Sherali & Adams, 1990; Ramana, 1993; Poljak *et al.*, 1995; Fujie & Kojima, 1997).

An important fact (probably first observed by Grötschel, Lovász & Schriver, 1988) is that, for any $x \in \mathbb{R}^n$, the augmented matrix

$$Y = \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T = \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix}$$

is psd.

This leads to the well-known *Semidefinite Programming* (SDP) relaxations of various quadratic problems.

Literature Review (cont.)

In most practical cases, x is constrained to be non-negative.

For such problems, we know that Y is *completely positive*, rather than merely psd.

Unfortunately, testing if a matrix is completely positive is co-$\mathcal{NP}$-complete (Murty & Kabadi, 1987).

Nevertheless, this observation has led to new convex relaxations of quadratic problems (e.g., Parrilo, 2000; De Klerk & Pasechnik, 2002; Bundfuss & Dur, 2009; Burer, 2009).

Literature Review (cont.)

The convex set associated with *pure 0-1* problems is:

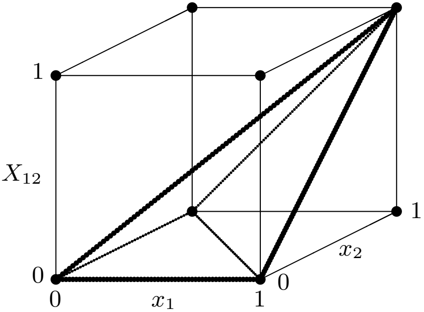
$$\text{conv}\left\{(x, X) : x \in \{0,1\}^n,\ X_{ij} = x_i x_j \ (1 \le i < j \le n)\right\}.$$

(There is no need to define X_{ij} when $i = j$, since binary variables are equal to their square.)

This is a polytope, called the *Boolean quadric polytope* (Padberg, 1989).

It has been studied in great depth (see Deza & Laurent, 1997).

Boolean Quadric Polytope when $n = 2$



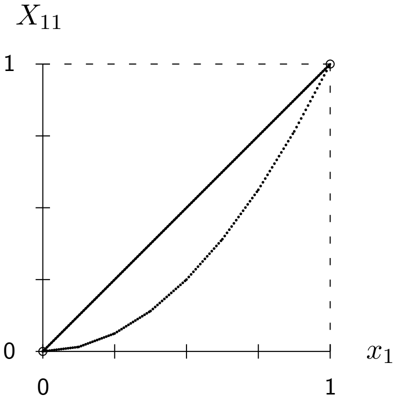
Literature Review (cont.)

Three papers are concerned with the convex set associated with the so-called *box-constrained case*:

$$\text{conv}\left\{(x, X) : \ x \in [0,1]^n, \ X = xx^T\right\}.$$

1. Yajima & Fujie (1998) showed that some valid inequalities for the Boolean quadric polytope are also valid in this case.
2. Burer & L. (2009) showed that, in fact, all inequalities from the Boolean quadric polytope are valid. They also characterised various facets, and some other maximal faces.
3. Anstreicher & Burer (2010) gave a complete description for $n = 2$, in terms of linear and conic constraints.

Box-Constrained Case when $n = 1$



Literature Review (cont.)

People have also explored the convex hulls associated with:

- The quadratic assignment problem (e.g., Padberg & Rijal, 1996; Jünger & Kaibel, 2001)
- The quadratic semi-assignment problem (e.g., Saito *et al.*, 2009)
- The quadratic knapsack problem (e.g., Helmberg, Rendl & Wolkowicz, 2000)
- Quadratic optimisation over a simplex (e.g., Anstreicher & Burer, 2010).

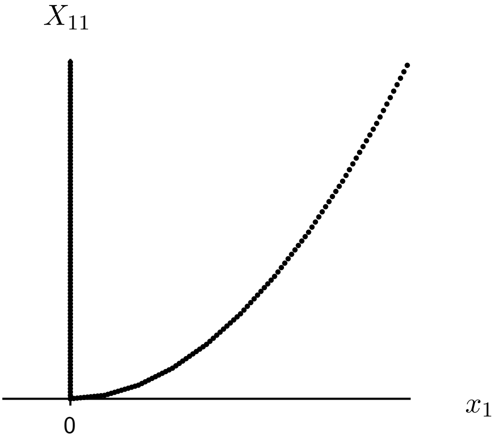
The Continuous Case

From now on we let $C(p,q)$ denote the closure of the convex hull of the set

$$\left\{(x,X):\ x \in \mathbb{R}^p_+ \times \mathbb{Z}^q_+,\ X = xx^T\right\}.$$

We begin with the (relatively) easy special case in which all variables are *continuous*, i.e., the case $q = 0$.

The Convex Set $C(1,0)$



The Continuous Case (cont.)

It is easy to show that:

$$C(p,0) = \left\{ (x,X) \in \mathbb{R}^{n+n^2} : \; Y \in \mathsf{CP}^{n+1} \right\},$$

where Y is the augmented matrix we saw before, and CP^d denotes the cone of completely positive matrices of order d.

This follows from the fact that the CP^d is the convex hull of the rank-one completely positive matrices of order d (Motzkin, 1952).

This implies of course that Y is *both psd and non-negative*.

The Continuous Case (cont.)

Since Y is psd, we have:

$$\begin{pmatrix}\beta\\ \alpha\end{pmatrix}^T \begin{pmatrix}1 & x^T\\ x & X\end{pmatrix}\begin{pmatrix}\beta\\ \alpha\end{pmatrix} \geq 0 \qquad (\forall \alpha \in \mathbb{R}^n, \beta \in \mathbb{R}),$$

which yields the following infinite family of valid linear inequalities (Ramana, 1993):

$$(2\beta)\alpha^T x + \alpha^T X \alpha \geq -\beta^2 \qquad (\forall \alpha \in \mathbb{R}^n, \beta \in \mathbb{R}).$$

We call these 'psd' inequalities.

The non-negativity of Y simply implies non-negativity of x and X.

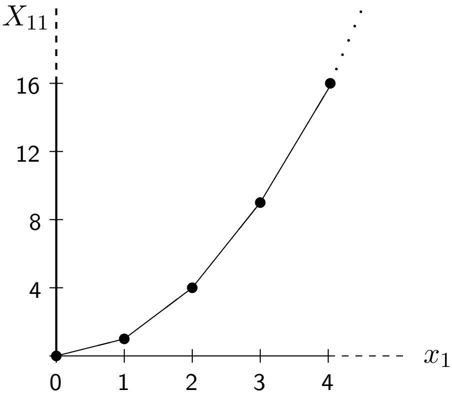
The General-Integer Case

Now we consider the special case in which all variables are *general integer*, i.e., the case $p = 0$.

Interestingly, $C(0, q)$ is not a polyhedron.

Indeed, a polyhedron is defined by a *finite* number of facets, whereas $C(0, q)$ has an infinite (though countable) number of facets, even for $q = 1$.

The Convex Set $C(0,1)$



The General-Integer Case (cont.)

Here are some (fairly) trivial results:

- $C(0,q)$ is full-dimensional.
- Non-negativity inequalities of the form $x_i \geq 0$ and $X_{ij} \geq 0$ (with $i \neq j$) define facets.
- Non-negativity inequalities of the form $X_{ii} \geq 0$ never define facets.
- The Boolean quadric polytope is (an affine image of) a face of $C(0,q)$.

The General-Integer Case (cont.)

Here are a few less obvious results:

- Each facet of the Boolean quadric polytope can be 'lifted', 'rotated' and 'translated' to yield an infinite family of facets for $C(0, q)$.
- When $q > 1$, each extreme point of $C(0, q)$ lies on an infinite (but countable) number of facets.
- The projection of $C(0, q)$ into X-space is CP_n.

(This last result may seem a bit counter-intuitive.)

The General-Integer Case (cont.)

Now consider any *split disjunction* of the form

$$(v^T x \leq -s - 1) \vee (v^T x \geq -s),$$

where $v \in \mathbb{Z}^n$ and $s \in \mathbb{Z}$.

From this we deduce the quadratic inequality

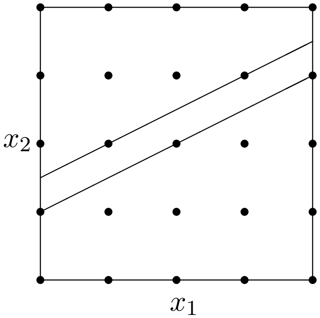
$$(v^T x + s)(v^T x + s + 1) \geq 0.$$

Linearising, we obtain the following 'split' inequality:

$$v^T X v + (2s + 1) v^T x + s(s + 1) \geq 0.$$

(Analogous inequalities were introduced by Boros & Hammer, 1993, for the boolean quadric polytope.)

A Split Disjunction



The General-Integer Case (cont.)

Theorem

Split inequalities induce facets of $C(0, q)$ if the non-zero components of v are relatively prime and not all of the same sign, or if they are relatively prime and all of the same sign, and s is sufficiently large.

Lemma

$C(0, 1)$ is completely described by split inequalities and the non-negativity inequality $x_1 \geq 0$.

Lemma

$C(0, 2)$ is not completely described by split and non-negativity inequalities!

The General-Integer Case (cont.)

Consider the two lines in $\mathbb{R}^2$ defined by the equations $x_1 + x_2 = 3$ and $x_1 + 2x_2 = 4$.

All points in $\mathbb{Z}_+^2$ are either above both lines, or below both.

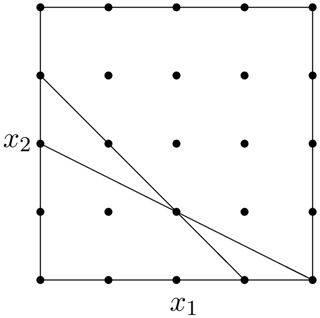
This implies the quadratic inequality $(x_1 + x_2 - 3)(x_1 + 2x_2 - 4) \geq 0$.

Linearising, we obtain the linear inequality

$$-7x_1 - 9x_2 + X_{11} + 3X_{12} + 2X_{22} \geq 12$$

that induces a facet of $C(0, 2)$.

The General-Integer Case (cont.)



The Mixed-Integer Case

For the mixed-integer case, we have:

- $C(p, q)$ is full-dimensional.
- Non-negativity inequalities of the form $x_i \geq 0$ and $y_{ij} \geq 0$ (with $i \neq j$) still define facets.
- The projection of $C(0, q)$ into X-space is still CP_n.
- Split inequalities remain valid and facet-defining, provided v has zero components for all continuous variables.

Concluding Remarks

- These convex sets are remarkably complex.
- One needs convex analysis as well as polyhedral theory to handle them properly.
- Yet, they seem worthy of further study.
- There is still room for improved theory, algorithms and software.
- We still don't have a complete description of either $C(1,1)$ or $C(0,2)$!