Convex Relaxations of the Stable Set Problem[1]

Adam N. Letchford[2]
Lancaster University

Milan, March 2011

[1]With M. Giandomenico, F. Rossi & S. Smriglio, University of L'Aquila.

[2]Supported by the EPSRC under grant EP/D072662/1.

Outline

- Introduction
- Known linear programming relaxations
- Known semidefinite programming relaxations
- Our linear programming relaxation
- Our convex quadratic programming relaxation
- Conclusions

Introduction

Let $G = (V, E)$ be an undirected graph.

A set $S \subseteq V$ of pairwise non-adjacent vertices is called a *stable set*.

Problem (MSS)

The Maximum Stable Set Problem *(*MSS*) calls for a stable set of maximum cardinality.*

The optimal value of MSS is called the *stability number* and denoted by $\alpha(G)$.

Introduction (cont.)

MSS has applications, e.g., in:

- timetabling and scheduling
- cutting and packing
- facility location
- graph colouring
- map labelling
- general 0-1 integer programs
- ... and many other problems.

Introduction (cont.)

MSS is hard in both theory and practice:

- Strongly $\mathcal{NP}$-hard (Karp, 1972).
- Hard to approximate within $n^{1-\epsilon}$ for any $0 < \epsilon < 1$ (Hastad, 1999).
- Best known approximation algorithm gives only $n/\log^2 n$ factor (Boppana & Haldorsson, 1990).
- In practice, can be tough even for $n \approx 300$.

Compare with the TSP. Some $\mathcal{NP}$-hard problems are harder than others...

Introduction (cont.)

A variety of exact algorithms have been devised:

- Branch-and-bound with combinatorial relaxations (e.g., Balas & Xue, 1996; Sewell, 1998; Tomita & Kameda, 2007).
- Cut-and-branch / branch-and-cut (e.g., Nemhauser & Sigismondi, 1992; Borndörfer, 1998; Rossi & Smriglio, 2001; Rebennack *et al.*, 2009).
- Constraint programming (e.g., Régin, 2003).
- Dynamic programming (e.g., Tarjan & Trojanowski, 1977, achieve $\mathcal{O}(2^{n/3})$ time, and Robson, 2001, achieved $\mathcal{O}(2^{n/4})$ time.

Known LP Relaxations

The classical 0-1 Linear Programming formulation is:

$$\begin{array}{lll} \max & \sum_{i \in V} x_i & \\ \text{s.t.} & x_i + x_j \leq 1 & \forall \{i,j\} \in E \\ & x_i \in \{0,1\} & \forall i \in V. \end{array} \tag{1}$$

Inequalities (1) are called *edge* inequalities.
Padberg (1973) introduced the stronger *clique* inequalities:

$$\sum_{i \in C} x_i \leq 1 \qquad (\forall \text{ cliques } C \subset V). \tag{2}$$

Known LP Relaxations (cont.)

Padberg also introduced:

- *Odd hole inequalities* $\sum_{i \in H} x_i \leq \lfloor |H|/2 \rfloor$, for all $H \subseteq V$ inducing a chordless circuit of odd cardinality.
- *Odd antihole inequalities* $\sum_{i \in A} x_i \leq 2$, for all $A \subseteq V$ inducing an odd antihole (i.e., the complement of an odd hole).

These can make quite useful cutting planes.

In general however they do not define facets and need to be strengthened (*lifted*).

Known LP Relaxations (cont.)

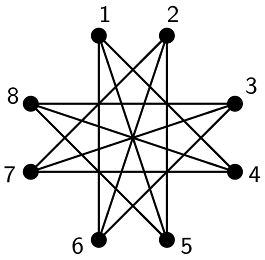
Trotter (1974) introduced:

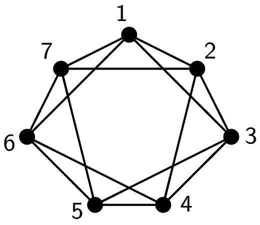
- *Web inequalities* $\sum_{i \in W} x_i \leq q$, for all $W \subseteq V$ inducing a (p,q)-web.
- *Antiweb inequalities* $\sum_{i \in A} x_i \leq \lfloor p/q \rfloor$, for all $A \subseteq V$ inducing a (p,q)-web.

These generalise the hole and antihole inequalities, and they too can sometimes be lifted.

Many other inequalities are known.

Known LP Relaxations (cont.)





$\sum_{i=1}^{8} x_i \leq 3$

$\sum_{i=1}^{7} x_i \leq 2$

Known LP Relaxations (cont.)

- Separation for odd hole inequalities can be solved in polynomial time (Barahona & Mahjoub, 1986).
- Same applies to antiweb inequalities with bounded q (Cheng & de Vries, 2002) and some other inequalities.
- For most inequalities, separation is $\mathcal{NP}$-hard.
- Good heuristics for clique and lifted odd hole separation were devised by Hoffman & Padberg (1991), Nemhauser & Sigismondi (1992) and Borndoerfer (2000).
- Rossi & Smriglio (2001) give a separation heuristic for rank inequalities.
- More powerful separation results can be obtained using SDP...

Semidefinite Programming Relaxations

Lovász (1972) introduced a bound based on SDP, now called the *theta number* and denoted by $\theta(G)$.

It can be derived in various ways. We follow the development of Grötschel, Lovász & Schrijver (1988).

We start by reformulating MSS as the following *non-convex quadratically-constrained program*:

$$\begin{array}{lll} \max & \sum_{i \in V} x_i & \\ \text{s.t.} & x_i = x_i^2 & (i \in V) \\ & x_i x_j = 0 & (\{i,j\} \in E). \end{array}$$

SDP Relaxations (cont.)

We then introducing the *matrix variable* $X = xx^T$, along with the augmented matrix

$$Y := \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix}.$$

We observe that Y is *positive semidefinite*. This leads to the SDP relaxation:

$$\begin{array}{lll} \max & \sum_{i \in V} x_i & \\ \text{s.t.} & x_i = X_{ii} & (i \in V) \\ & X_{ij} = 0 & (\{i,j\} \in E) \\ & Y \in \mathcal{S}^+_{n+1}, & \end{array}$$

where $\mathcal{S}^+_{n+1}$ is the cone of symmetric psd matrices of order $n+1$.

SDP Relaxations (cont.)

The projection of the feasible region of the theta relaxation onto the subspace of the original variables is called the *theta body* and denoted by $TH(G)$.

- Grötschel *et al.* showed that $TH(G)$ satisfies all clique inequalities.
- One can optimise over $TH(G)$ to arbitrary precision in polynomial time...
- ... even though clique separation is $\mathcal{NP}$-hard!
- This is behind the proof that MSS is polynomially-solvable on perfect graphs.

SDP Relaxations (cont.)

Here's a nice result:

Theorem (Juhász, 1982)

On a random graph in which each edge is present with probability $1/2$*, then almost surely*

- $\alpha(G)$ *is roughly* $2\log_2 n$,
- $TH(G)$ *is roughly* $\sqrt{n}$,
- *and the bound using clique inequalities is roughly* $\frac{n}{2\log_2 n}$.

So SDP performs much better than LP here.

SDP Relaxations: adding cutting planes

The above SDP can be strengthened by adding cutting planes:

- Schrijver (1979) added $X_{ij} \geq 0 \ \ \forall \{i,j\} \notin E$.
- Lovász & Schrijver (1991) add

$$\begin{aligned} X_{ik} + X_{jk} &\leq x_k & (\{i,j\} \in E, k \neq i,j), \\ x_i + x_j + x_k &\leq 1 + X_{ik} + X_{jk} & (\{i,j\} \in E, k \neq i,j). \end{aligned}$$

  They showed that the projection now satisfies all odd hole and antihole inequalities, plus some special lifted odd hole inequalities called wheel inequalities.
- Giandomenico & Letchford (2006) showed that, in fact, it satisfies all web inequalities, along with a huge class of lifted versions.

SDP Relaxations: more cutting planes

- Gruber & Rendl (2003) derive a stronger relaxation than that of Lovász & Schrijver by adding the triangle inequalities:

$$
\begin{aligned}
X_{ik} + X_{jk} &\le x_k + X_{ij} && (\forall \text{ stable}\{i,j,k\}), \\
x_i + x_j + x_k &\le 1 + X_{ij} + X_{ik} + X_{jk} && (\forall \text{ stable}\{i,j,k\}).
\end{aligned}
$$

- Even stronger SDP relaxations, and even hierarchies of relaxations, have been proposed by Lovász & Schrijver, Lasserre, Laurent, etc.

Our LP Relaxation

In Giandomenico *et al.* (2009), we explored a different approach, based on the Lovász-Schrijver $M(K, K)$ operation.

Given a pair of inequalities $\alpha^T x - \beta \geq 0$ and $(\alpha')^T x - \beta' \geq 0$, the product is:

$$\begin{pmatrix} -\beta & \alpha^T \end{pmatrix} Y \begin{pmatrix} -\beta' \\ \alpha' \end{pmatrix} \geq 0.$$

The $M(K, K)$ operation applies this to every pair of inequalities in a 0-1 LP to produce an extended formulation.

Our LP Relaxation (cont.)

Let Ω denote the set of all maximal cliques of G.

The polytope QSTAB(G) is defined by:

$$\begin{aligned} 1 - \sum_{i \in C} x_i \geq 0 \qquad & (C \in \Omega) \\ x_i \geq 0 \qquad & (i \in V). \end{aligned}$$

We consider the polyhedron M(QSTAB(G),QSTAB(G)).

Our LP Relaxation (cont.)

Applying the $M(K,K)$ operation to QSTAB(G) yields:

$$\begin{array}{lll} CVIs: & \sum_{j \in C:\{i,j\} \in \bar{E}} x_{ij} - x_i \leq 0 & (C \in \Omega, i \in V \setminus C) \\ CPIs: & \sum_{i \in C \cup C'} x_i - \sum_{\{i,j\} \in \bar{E}(C:C')} x_{ij} \leq 1 & (C, C' \in \Omega) \\ & x_{ij} = 0 & (\{i,j\} \in E), \\ & x_{ij} \geq 0 & (\{i,j\} \in \bar{E}), \end{array}$$

- The separation problems for CVIs and CPIs are strongly NP-hard (Giandomenico, 2007).
- But one can *approximately* solve this relaxation...

Our LP Relaxation: Theoretical Results

We denote by $N(K, K)$ the projection of $M(K, K)$ onto the subspace of the original variables.

Theorem

$N(QSTAB(G), QSTAB(G))$ *satisfies all antiweb inequalities.*

Theorem

$N(QSTAB(G), QSTAB(G))$ *satisfies all web inequalities.*

Moreover, $N(QSTAB(G), QSTAB(G))$ satisfies all inequalities which can be obtained from these by two sequential lifting procedures.

Our LP Relaxation: The Algorithm

A cutting plane algorithm performs badly.
The following approach gives an effective implementation:

- Clique Selection: A collection Ω of maximal cliques is built by a cutting plane algorithm in the original space.
- Core Selection: A subset of CVIs and CPIs is chosen by a subgradient algorithm on the Lagrangian relaxation.
- Optimization: CPLEX 9.1 Barrier algorithm to solve the core LP.

Only LP techniques are used.

Our LP Relaxation: Results

Graph name	$\lvert V\rvert$	$\lvert E\rvert$	# var.	$\alpha(G)$	$\theta(G)$	UB_{MKK}	% gap closed	θ time	MKK time
g150_4	150	459	10,866	58	62.40	60.62	40.45	4.1	4.5
g150_5	150	556	10,769	55	58.01	56.61	46.51	4.7	36.7
g170_3	170	451	14,084	70	73.51	70.00	100.00	4.7	22.8
g200_2	200	420	19,680	93	94.77	93.00	100.00	3.8	15.1
g200_3	200	603	19,497	80	83.63	81.07	70.52	4.8	30.6
g300_2	300	905	44,245	121	128.10	124.93	44.65	11.9	102.2
g350_2	350	1,206	60,219	132	141.94	140.34	16.10	23.0	151.8
g400_1	400	816	79,384	187	191.42	187.00	100.00	19.5	100.3

Sparse random graphs

- UB_{MKK} is significantly stronger than $\theta(G)$.
- The approach is faster than some SDP methods, but still rather slow.

The Dilemma!

We can't seem to get strong LP bounds without moving to quadratic space, which makes the LP hard to solve.

We can get pretty strong bounds from SDP, but SDP software is still rather slow and hard to embed within a branch-and-bound scheme.

This is what led us to search for some kind of compromise...

A Possible Solution

In our 2011 IPCO paper, we prove the following result:

Theorem

One can compute in polynomial time a convex quadratic inequality of the form

$$x^T Q x \le \lambda^T x$$

such that:

- $TH(G) \subset \left\{x \in \mathbb{R}^n : x^T Q x \le \lambda^T x\right\}$.
- $\theta(G) = \max\left\{\sum_{i \in V} x_i : x^T Q x \le \lambda^T x\right\}$.

A Possible Solution

In other words, we can efficiently construct an *ellipsoid* that contains the theta body, such that optimising over the ellipsoid still gives us the theta number.

This result can be used to construct two different kinds of branch-and-cut algorithms:

- Use the ellipsoid directly, leading to convex quadratically-constrained programming relaxations.
- Use the ellipsoid to generate linear inequalities (by taking tangents), leading to traditional LP relaxations.

Concluding Remarks

We are currently experimenting with a cutting plane algorithm in which linear inequalities derived from the ellipsoid are strengthened using a kind of sequential lifting procedure.

The results so far (included in the IPCO paper), are quite promising.

We are still refining the procedure and hope to obtain still better results soon.