The Travelling Salesman Problem

Adam N. Letchford[1]

Department of Management Science
Lancaster University Management School

Swansea, April 2010

[1]Supported by the EPSRC under grant EP/D072662/1.

Outline of the Talk

1. Introducing the TSP
2. The origins of the TSP
3. Applications of the TSP
4. The hardness of the TSP
5. The influence of the TSP
6. Progress in solving the TSP
7. Conclusions

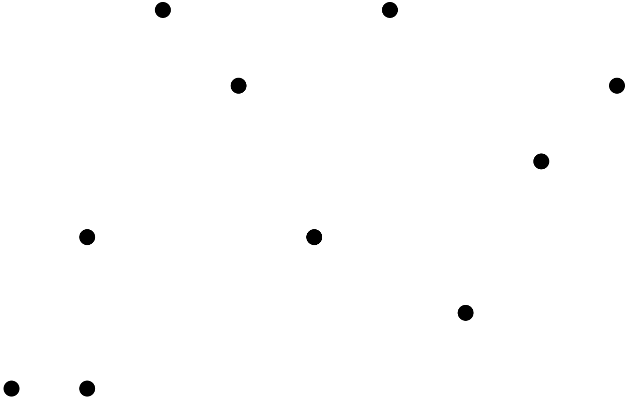
Introducing the TSP

An informal definition of the TSP is as follows:

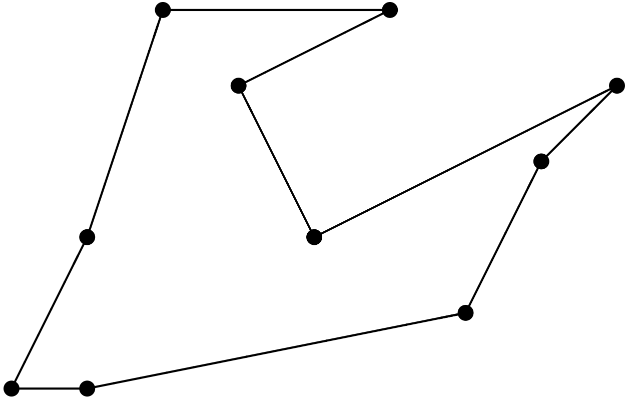
Definition (Informal Version)

Given a set of cities, and known distances between each pair of cities, the TSP is the problem of finding a tour that visits each city exactly once and that minimises the total distance travelled.

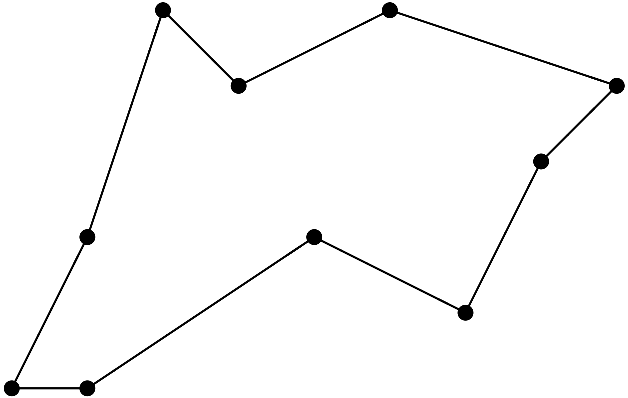
Example: ten randomly-placed cities



Example (cont.): a reasonable tour



Example (cont.): the optimal tour



Introducing the TSP (cont.)

To define the TSP formally, we use graph theoretic terms:

Definition (Formal Version)

Given an undirected graph, and a cost for each edge of that graph, find a Hamiltonian circuit of minimum total cost.

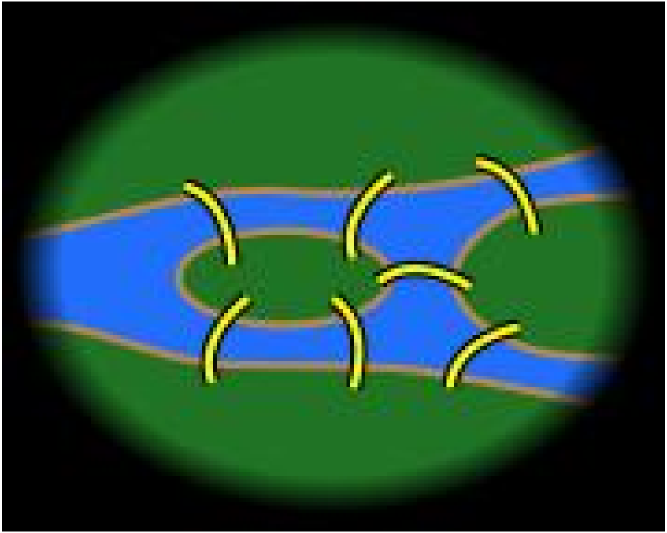
(One speaks of 'vertices' and 'edges', rather than 'cities' and 'roads'.)

The Origins of the TSP

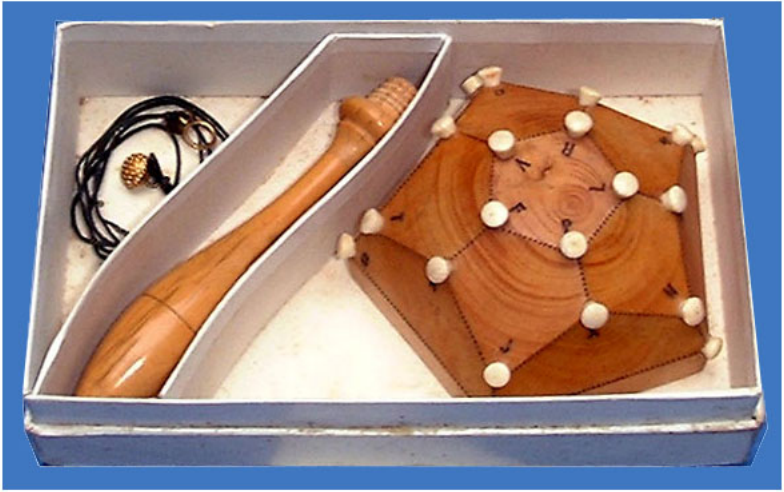
Ideas related to the TSP have been around for a long time:

- In 1736, Leonard Euler studied the problem of finding a round trip through seven bridges in Königsberg.
- In 1832, a handbook was published for German travelling salesmen, which included examples of tours.
- In the 1850s, Sir William Rowan Hamilton studied Hamiltonian circuits in graphs. He also marketed his 'Icosian Game', based on finding tours in a graph with 20 vertices and 30 edges.

The Seven Bridges of Königsberg (1736)



Hamilton's Icosian Game (1857)



The Origins of the TSP (cont.)

Moving to the 20th century...

- In the early 1930s, Karl Menger discussed the problem with colleagues in Vienna and Harvard.
- In the late 1930s, the problem reappeared at Princeton University. Hassler Whitney called it the TSP.
- In the mid-1940s, the TSP was studied by several statisticians.
- In the late 1940s and early 1950s, it was studied intensively by researchers at the RAND corporation.

More later...

Applications of the TSP

Obviously, the main application of the TSP is to *logistics*. One may wish to find good routes or schedules for:

- trucks (Dantzig & Ramser, 1959, and many others)
- order-pickers in a warehouse (Ratliff & Rosenthal, 1981)
- service engineers (Pante, Lowe & Chandrasekaran, 1987)
- aircraft (Boland, Jones & Nemhauser, 1994)
- tourists (Gentili, 2003)
- ...

Applications of the TSP

There are however some *less obvious* applications:

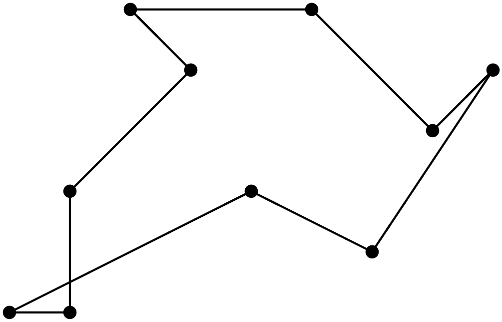
- scheduling jobs on machines
- controlling satellites, telescopes, microscopes, lasers...
- computing DNA sequences
- designing telecommunications networks
- designing and testing VLSI circuits
- x-ray crystallography
- clustering data arrays
- ...

The Hardness of the TSP

The TSP is a lot more complicated than it may appear!

Fact

If you just start at an arbitrary city and keep going to the city nearest to it, you can get a bad solution.



The Hardness of the TSP (cont.)

Why can't we just check all possible tours using a computer?

Fact

If there are n cities, the number of possible tours is $(n-1)!/2$.

No. cities	No. tours	Time
5	12	12 microsecs
8	2520	2.5 millisecs
10	181,440	0.18 secs
12	19,958,400	20 secs
15	87,178,291,200	12.1 hours
18	177,843,714,048,000	5.64 years
20	60,822,550,204,416,000	1927 years

The Hardness of the TSP (cont.)

In the 1950s and 1960s, great progress was made on various discrete optimisation problems. Efficient methods were found for:

- the transportation problem (Charnes & Cooper, 1954)
- the assignment problem (Kuhn, 1955)
- the maximum flow problem (Ford & Fulkerson, 1956)
- the minimum spanning tree problem (Prim, 1956)
- the matching problem (Edmonds, 1963).

But *nobody could solve the TSP!*

The Hardness of the TSP (cont.)

In 1972, Karp proved that the TSP is an '$\mathcal{NP}$-hard' problem.

- All known *exact* methods for $\mathcal{NP}$-hard problems run in *exponential* time.
- Nobody knows how to solve $\mathcal{NP}$-hard problems in *polynomial* time.
- Whether this can be done is a famous open question in theoretical computer science.

(Many other important problems in Operational Research and related fields are also $\mathcal{NP}$-hard.)

The Hardness of the TSP (cont.)

Actually, things are not as bad as they seem:

- Exact methods take exponential time only in the *worst case*, i.e., when run on the ‘hardest’ instances.
- Instances that arise in practice are unlikely to be among the ‘hardest’.
- There are many effective *heuristics* available, which quickly find solutions of acceptable quality.
- One heuristic, due to Christofides (1976), is *guaranteed* never to return a solution whose cost is more than 50% above optimal.

The Influence of the TSP

As well as having a vast array of applications, the TSP has had a profound influence on:

- Operational Research
- Discrete Mathematics
- Theoretical Computer Science
- Artificial Intelligence

The Influence of the TSP (cont.)

The following methods were all originally devised with the TSP in mind:

- Cutting planes (Dantzig *et al.*, 1954)
- Branch-and-bound (Little *et al.*, 1963)
- Lagrangian relaxation (Held & Karp, 1970)
- Simulated annealing (Kirkpatrick *et al.*, 1983)
- Branch-and-cut (Padberg & Rinaldi, 1987).

Progress in Solving the TSP

A great deal of progress has been made on the TSP. We now have:

- *Meta-heuristics* that can obtain upper bounds within 0.01% of optimal even when $n = 10^6$.
- *Cutting-plane methods* that can obtain lower bounds within 0.1% of optimal for n up to 10^5 or so.
- *Branch-and-cut* methods that can solve instances with n up to 10^4 to proven optimality.

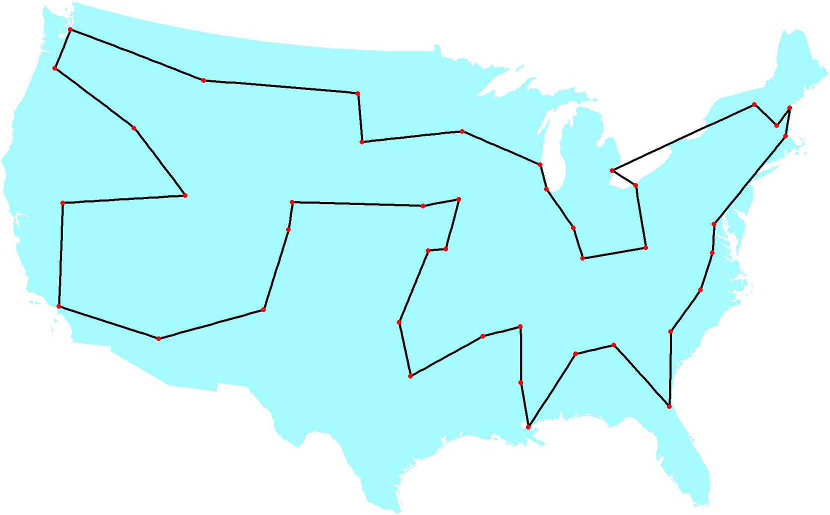
My own research has been on cutting-plane methods.

Progress in Solving the TSP (cont.)

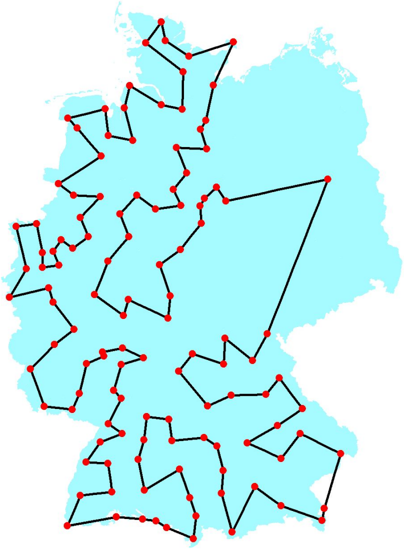
Cutting-plane methods work as follows:

- For each edge $\{i, j\}$, we define a binary variable that takes the value 1 if and only if the tour includes that edge.
- Each tour then corresponds to a point in $\mathbb{R}^{\binom{n}{2}}$.
- The convex hull of these points is a polyhedron.
- We iteratively generate linear inequalities that define facets of this polyhedron.
- We iteratively insert these inequalities into a linear program.
- Solving the sequence of linear programs yields a non-decreasing sequence of lower bounds on the cost of the optimal tour.

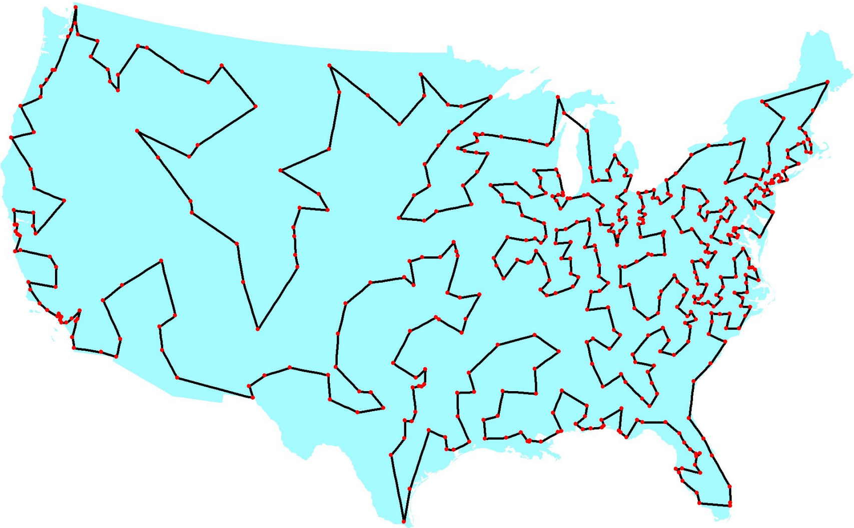
Progress: 42 cities (Dantzig *et al.*, 1954)



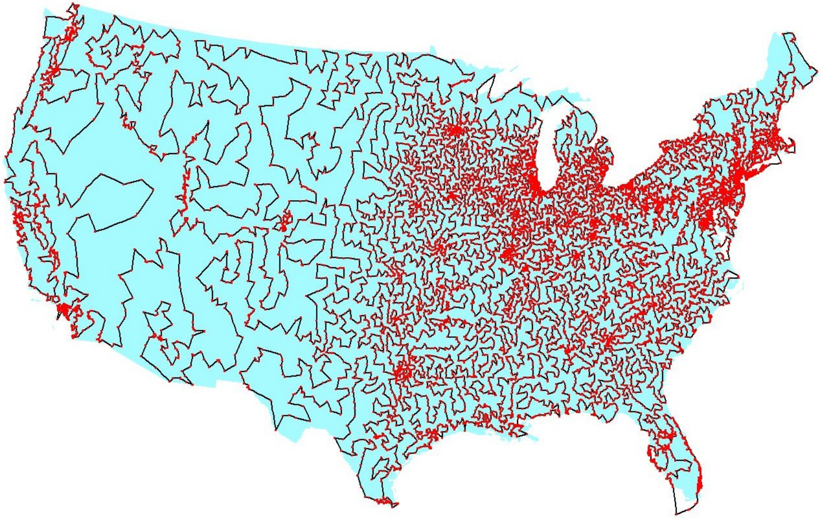
Progress: 120 cities (Grötschel, 1977)



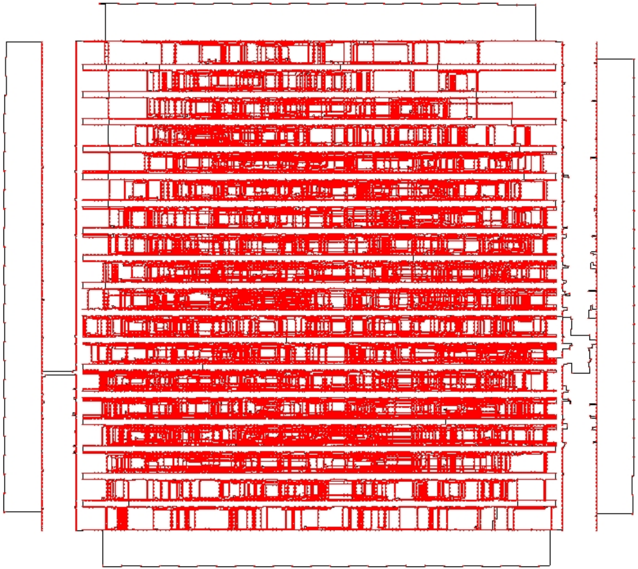
Progress: 532 cities (Padberg & Rinaldi, 1987)



Progress: 13509 cities (Applegate *et al.*, 1998)



Progress: 85900 cities (Cook *et al.*, 2006)



For Further Reading

There are three *excellent books* on the TSP:

- Lawler, Lenstra, Rinnooy-Kan & Schmoys (1985) The Traveling Salesman Problem: a Guided Tour of Combinatorial Optimisation.
- Gutin & Punnen (2002) The Traveling Salesman Problem and its Variations.
- Applegate, Bixby, Chvátal & Cook (2007) The Traveling Salesman Problem: A Computational Study.

See also the web site `http://www.tsp.gatech.edu` and my own web site `http://www.lancs.ac.uk/staff/letchfoa`.