Photoelectron counting in quantum optics

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Contents

Abstract

This tutorial gives an overview over photoelectron counting statistics in quantum optics. Many of the original ideas were developed in the 1950s and 1960s, and I will therefore start with Mandel's semiclassical counting formula that promotes a simple (short time) Fermi-Golden rule calculation to a (long-time) probability distribution. Much of the tutorial will be devoted to the quest for a quantum version of that formula, a quest that appears to have had a great importance for the development of quantum optics as a whole and which is characterised by the subtleties of theoretically describing sources, fields, and detectors in a consistent manner. I will explain and in part (re)-derive in detail the approaches by

Kelley/Kleiner [1], Scully/Lamb [2], and Ueda [3] whereby particular use is made of Scully and Lamb's photodetector model that describes the detector's backaction on the field. I will use a slightly modernised form of this model in order to briefly introduce calculational tools such as quantum master equations and Glauber's P-representation for single mode (cavity) fields without sources, before moving on to the more intriguing case of multimode fields with sources. I will then discuss practical questions such as how to obtain the counting distribution $p_n(t, t+T)$ from simplified master equations involving only sources. Here, motivated by single ion experiments, important contributions were made by several groups in the 1980s that lead towards the quantum jump (quantum trajectory) approach which in hindsight can be regarded as a 'by-product' of counting statistics. I will discuss resonance fluorescence and its conceptual similarities with quantum transport, such as Cook's early use in 1981 [4] of counting variables and generating functions in his counting statistics calculation.

Literature: introductory quantum optics texts such as Walls/Milburn [5] (some general stuff, spontaneous emission, resonance fluorescence, P-representation, correlation functions $g^{(1)}$ and $g^{(2)}$ etc.) or Carmichael [6] (master equations, photodetection, quantum trajectories, cf. my own lecture notes on quantum dissipation http://theoserv.phy.umist.ac.uk/∼brandes), Mandel/Wolf [7]. Also parts of the original literature, in particular Scully and Lamb [2] (introductory parts), Ueda [3] (part III), and Cook [4].

References

[1] P. L. Kelley and W. H. Kleiner, Phys. Rev. **136**, A316 (1964).

[2] M. O. Scully and W. E. Lamb, Jr., Phys. Rev. **179**, 368 (1969).

[3] M. Ueda, Phys. Rev. A **41**, 3875 (1990).

[4] R. J. Cook, Phys. Rev. A **23**, 1243 (1981).

[5] D. F. Walls and G. J. Milburn, *Quantum Optics* (Springer, Berlin, 1994).

[6] H. J. Carmichael, *An Open System Approach to Quantum Optics*, Vol. m 18 of *Lecture Notes in Physics* (Springer, Berlin, Heidelberg, 1993).

[7] L. Mandel, E. Wolf, *Quantum coherence and quantum optics* (Cambridge University Press, Cambridge, USA, 1995).

Part I
'Photons', detectors, and quantum dissipation

1 Introduction

Overview: photons, photon-counting, fluctuations

- Counting photons, but...
- ...'the eternal question: what is a photon'.
- 'What is light ?'

Einstein 1951: '...these days every fool pretends to know what a *photon* is. I have been thinking about this for the whole of my life, and I haven't found the answer'.

...cavity mode $H = \omega a^\dagger a$, n-photon eigenstate $|n\rangle$.

...photon as gauge-boson of QED .

- 'No photons' for the photoelectric effect.
- Quantum mechanics was discovered in its own classical limit.
- Big breakthrough: Hanbury Brown, Twiss experiment: intensity correlations, 'photon bunching'.
- Correlation functions ($a^\dagger$ creates cavity mode):

$$
\begin{aligned}
G^{(1)}(t, t+\tau) &= \langle a^\dagger(t) a(t+\tau)\rangle &(1.1)\\
G^{(2)}(t, t+\tau) &= \langle a^\dagger(t) a^\dagger(t+\tau) a(t+\tau) a(t)\rangle. &(1.2)
\end{aligned}
$$

- But not yet a complete triumph for quantum optics...

Triumph came with *resonance fluorescence*: photon antibunching,

- Count photo-electrons instead of photons.
- Counting statistics: correct theory for

$$p_n(t, t+T) \quad \text{probability for } n \text{ photo-electrons in } [t, t+T).$$

- Detector back-action. System-bath problem 'with two baths'.
- ... no entirely trivial!

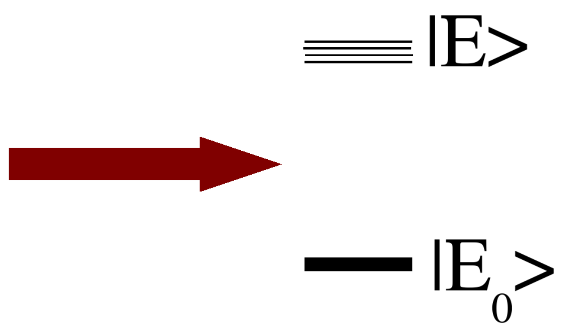
2 Photoelectric counting: classical field

2.1 Mandel formula

Semiclassical theory for $p_n(t, t+T)$: Mandel formula

Photodetector model: ionize single atom

- Classical electromagnetic field, vector potential $\mathbf{A}(\mathbf{r})e^{-i\omega t} + \mathbf{A}^*(\mathbf{r})e^{i\omega t}$.



Probability $p_1(t, t+\Delta t)$ of one count: Fermi's Golden Rule

$$
\begin{aligned}
p_1(t, t+\Delta t) &= \int_0^\infty dE\nu(E)\left|\langle E|\frac{e}{m}\mathbf{p}\mathbf{A}(\mathbf{r})|E_0\rangle\right|^2 D_{\Delta t}(E - E_0 - \omega)\\
&= \eta I(\mathbf{r})\Delta t, \quad I(\mathbf{r}) = |A(\mathbf{r})|^2(\text{intensity}). &(2.1)
\end{aligned}
$$

- $D_{\Delta t}(\varepsilon) \equiv \left([\sin\frac{1}{2}\varepsilon\Delta t]/[\frac{1}{2}\varepsilon]\right)^2$, $\Delta t \to 0$. Polarisation $\mathbf{A}(\mathbf{r}) = \vec{\varepsilon}A(\mathbf{r})$.

Mandel formula: many counts

How to obtain probability of n transitions $p_n(t,t+T)$

- Short-time probability $p_1(t,t+\Delta t)=\eta I(\mathbf{r})\Delta t$ for *single* electron transition ($\eta I(\mathbf{r})$ transition rate).
- Long-time probability of n transitions $p_n(t,t+T)\leftrightarrow n$ electrons.

- Individual transitions are statistically independent...
- $\rightsquigarrow$ Poisson distribution.
- Characterized by average $\bar{n}$ only $\rightsquigarrow$

$$p_n(t,t+T) = \frac{\bar{n}^n}{n!}e^{-\bar{n}}, \quad \bar{n}=\eta I(\mathbf{r})T. \tag{2.2}$$

- Markovian master equation for probabilities. $p_n(t)\equiv p_n(0,t)$,

$$p_n(t+dt) = p_n(t)\times[1-\eta I(\mathbf{r})dt]+p_{n-1}(t)\times\eta I(\mathbf{r})dt \tag{2.3}$$

$$\frac{d}{dt}p_n(t) = \eta I(\mathbf{r})[p_{n-1}(t)-p_n(t)]. \tag{2.4}$$

- Generating function $G(s,t)\equiv\sum_{n=0}^{\infty}s^n p_n(t)$, $\partial_t G(s,t)=\eta I(\mathbf{r})(s-1)G(s,t)$.
- Solve with $p_0(0)=1$, $p_n(0)=0$, $n>0$, $G(s,0)=1$.
- Thus $G(s,t)=\exp[\eta I(\mathbf{r})t(s-1)]=\sum_{n=0}^{\infty}s^n\frac{\bar{n}^n}{n!}e^{-\bar{n}}$, $\bar{n}=\eta I(\mathbf{r})t$.

SUMMARY so far:

- Classical photo-electron counting formula (Mandel formula)

$$p_n(t,t+T)=\frac{\bar{n}^n}{n!}e^{-\bar{n}}, \quad \bar{n}=\eta I(\mathbf{r})T.$$

- Poisson process.
- Generating function $G(s,t)\equiv\sum_{n=0}^{\infty}s^n p_n(t)=\exp[\eta I(\mathbf{r})t(s-1)]$.
- Nothing said here about PHOTONS! This is a DETECTOR theory.

3 Photo-count formula in quantum optics

3.1 Mandel formula generalisation: discussion

'Quantum Mandel formulas'

Kelley-Kleiner, Carmichael, etc. version

- $p_n(t,t+T)=\langle:\frac{\hat{\Omega}^n}{n!}e^{-\hat{\Omega}}:\rangle$ with $\hat{\Omega}\equiv\xi\int_t^{t+T}dt'\hat{E}^-(t')\hat{E}^+(t')$.
- No backaction of detector on field.
- 'Non-absorbed photons escape, open system.'

- Typically many field degrees of freedom, field is a 'BIG QUANTUM SYSTEM'.

Mollow; Scully/Lamb; Srivinas/Davies; Ueda etc. version

- Backaction of detector leads to damping (continuous measurement) of the field.
- 'Eventually all photons absorbed, closed system.'
- Typically few field degrees of freedom, field is a 'SMALL QUANTUM SYSTEM'

Scully-Lamb photodetector

M. Scully, W. Lamb Jr., Phys. Rev. **179**, 368 (1969)

- 'Photon statistics' means (reduced) density operator $\rho(t)$ of a light field (more generally: boson field).
- 'Photon statistics' is inferred by photoelectric counting techniques.

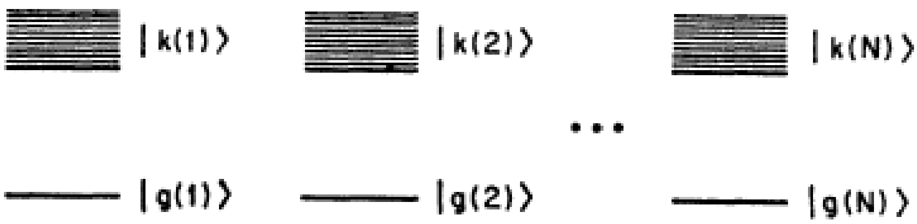


FIG. 1. Pictorial representation of photodetector consisting of N-independent atoms. Each atom in detector has a ground state $|g\rangle$ and continuum of excited states $|k\rangle$. Atoms are labeled by indexing atomic state with particle number, e.g., $|k(m)\rangle$.

4 Some quantum optics techniques

4.1 Master equations and quantum dissipation

System-bath theory

Divide 'total universe' into system S and bath B,

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_{\mathrm{S}} + \mathcal{H}_{\mathrm{B}} + \mathcal{H}_{\mathrm{SB}} \\ &\equiv \mathcal{H}_0 + V, \quad V \equiv \mathcal{H}_{\mathrm{SB}}. \end{aligned} \tag{4.1}$$

Total density matrix $\chi(t)$ obeys the Liouville-von-Neumann equation

$$\frac{d}{dt}\chi(t) = -i[\mathcal{H}, \chi(t)]. \tag{4.2}$$

Master equation

- Effective density matrix of the system $\rho(t) \equiv \mathrm{Tr}_B[\chi(t)]$.
- Interaction picture with respect to H_0,

$$\frac{d}{dt}\tilde{\rho}(t) = -i\mathrm{Tr}_B[\tilde{V}(t), \chi(t=0)] - \int_0^t dt' \mathrm{Tr}_B[\tilde{V}(t), [\tilde{V}(t'), \tilde{\chi}(t')]].$$

- Born approximation, $\tilde{\chi}(t') \approx R_0 \otimes \tilde{\rho}(t')$, R_0 bath density matrix.
- System-bath interaction as $V = \sum_k S_k \otimes B_k$,
- Bath correlation functions $C_{kl}(t,t') \equiv \mathrm{Tr}_B\left[\tilde{B}_k(t)\tilde{B}_l(t')R_0\right]$, $\mathrm{Tr}_B\tilde{B}_k(t)R_0 = 0$.

$$\begin{aligned}\frac{d}{dt}\tilde{\rho}(t) &= -\int_0^t dt' \sum_{kl}\Big[C_{kl}(t-t')\left\{\tilde{S}_k(t)\tilde{S}_l(t')\tilde{\rho}(t') - \tilde{S}_l(t')\tilde{\rho}(t')\tilde{S}_k(t)\right\} \\ &+ C_{lk}(t'-t)\left\{\tilde{\rho}(t')\tilde{S}_l(t')\tilde{S}_k(t) - \tilde{S}_k(t)\tilde{\rho}(t')\tilde{S}_l(t')\right\}\Big]. \end{aligned} \tag{4.3}$$

4.2 Application: microscopic field-detector theory

Scully-Lamb Photodetector

Detector model

- *System*: single photon mode a and N detector single level 'quantum dots' j with one ($|1\rangle_j$) or zero ($|0\rangle_j$) electrons.
- Photon absorption empties dots into *bath*: leads j, $c^\dagger_{\alpha j}|vac\rangle$.

$$\mathcal{H}_{\mathrm{SB}} = \sum_{\alpha j}\left(V^j_\alpha c^\dagger_{\alpha j}|0\rangle_j\langle 1|a + \bar{V}^j_\alpha c_{\alpha j}|1\rangle_j\langle 0|a^\dagger\right) \equiv \sum_k S_k \otimes B_k. \tag{4.4}$$

Master equation: trace out the leads

- Terms $C_{kl}(t-t')\tilde{S}_k(t)\tilde{S}_l(t')\tilde{\rho}(t')$; $C_{kl}(t-t') = \langle\tilde{B}_k(t)\tilde{B}_l(t')\rangle$.
- 'Broadband detection' at all energies, $\sum_\alpha |V^j_\alpha|^2\delta(\varepsilon - \varepsilon_{\alpha j}) = \nu$.

$$\frac{d}{dt}\tilde{\rho}_t = -\pi\nu\sum_j\left\{|1\rangle_j\langle 1|a^\dagger a\tilde{\rho}_t + \tilde{\rho}_t a^\dagger a|1\rangle_j\langle 1| - 2|0\rangle_j\langle 1|a\tilde{\rho}_t a^\dagger|1\rangle_j\langle 0|\right\}.$$

State with m excitations

- Detector states $|m;\lambda\rangle \equiv \hat{\Pi}_\lambda |0\rangle_1 ... |0\rangle_m |1\rangle_{m+1} ... |1\rangle_N$. Permutations
- m-resolved field 'pseudo' density matrix $\tilde{\rho}_t^{(m)} \equiv \sum_\lambda \langle m;\lambda|\tilde{\rho}_t|m;\lambda\rangle$.

$$
\begin{aligned}
\frac{d}{dt}\tilde{\rho}_t &= -\pi\nu \sum_j \left\{ |1\rangle_j\langle 1|a^\dagger a\tilde{\rho}_t + \tilde{\rho}_t a^\dagger a|1\rangle_j\langle 1| - 2|0\rangle_j\langle 1|a\tilde{\rho}_t a^\dagger|1\rangle_j\langle 0| \right\} \\
& \sum_j \langle m;\lambda|\tilde{\rho}_t|1\rangle_j\langle 1|m;\lambda\rangle = (N-m)\langle m;\lambda|\tilde{\rho}_t|m;\lambda\rangle \\
& \sum_j \langle m;\lambda|0\rangle_j\langle 1|\tilde{\rho}_t|1\rangle_j\langle 0|m;\lambda\rangle = \sum_{\lambda'}^{m\text{terms}} \langle m-1;\lambda'|\tilde{\rho}_t|m-1;\lambda'\rangle \\
\frac{d}{dt}\tilde{\rho}_t^{(m)} &= -\pi\nu \left\{ (N-m)\left[a^\dagger a\tilde{\rho}_t^{(m)} + \tilde{\rho}_t^{(m)} a^\dagger a \right] - 2(N-m+1)a\tilde{\rho}_t^{(m-1)}a^\dagger \right\}.
\end{aligned}
$$

$N \gg m$, $\gamma \equiv 2\pi N\nu \rightsquigarrow$

$$
\frac{d}{dt}\rho_t^{(m)} = -i[\mathcal{H}_{\mathrm{F}}, \rho_t^{(m)}] - \frac{\gamma}{2}\left(a^\dagger a\rho_t^{(m)} + \rho_t^{(m)} a^\dagger a - 2a\rho_t^{(m-1)}a^\dagger \right). \tag{4.5}
$$

- Now counting statistics as $p_m(t) \equiv \mathrm{Tr}\rho_t^{(m)}$!

Jump super-operator J, $J\rho \equiv \gamma a\rho a^\dagger$, time evolution generator $\mathcal{L}_0$

- Define $\mathcal{L}_0\rho \equiv Y\rho + \rho Y^\dagger$ with $Y \equiv -i\mathcal{H}_{\mathrm{F}} - \frac{\gamma}{2}a^\dagger a$.

$$
\boxed{\dot{\rho}_t^{(m)} = \mathcal{L}_0\rho_t^{(m)} + J\rho_t^{(m-1)}.} \tag{4.6}
$$

Summary: counting statistics in Scully-Lamb detector model

m-resolved field density matrix
$\dot{\rho}_t^{(m)} = \mathcal{L}_0\rho_t^{(m)} + J\rho_t^{(m-1)}$.

- Counting statistics as $p_m(t) \equiv \mathrm{Tr}\rho_t^{(m)}$!

Generating operator $\hat{G}(s,t)$

- Define $\hat{G}(s,t) \equiv \sum_{m=0}^\infty s^m \rho_t^{(m)}$, s: *counting variable.*
- Usually s complex, e.g. $s = e^{i\phi}$ with real ϕ.
- Infinite set of master equations now becomes a single equation,

$$
\boxed{\frac{\partial}{\partial t}\hat{G}(s,t) = (\mathcal{L}_0 + sJ)\hat{G}(s,t).} \tag{4.7}
$$

4.3 Quantum optics techniques: P-representation

Solve $\frac{d}{dt}\hat{G} = -i[\mathcal{H}_{\rm F}, \hat{G}] - \frac{\gamma}{2}\left(a^\dagger a\hat{G} + \hat{G}a^\dagger a - 2sa\hat{G}a^\dagger\right)$

P-representation in harmonic oscillator Hilbert space

- Glauber introduced coherent states $|z\rangle$, $a|z\rangle = z|z\rangle$.
- Glauber-Sudarshan representation of operators such as $\hat{G}$ as $\hat{G} = \int d^2zP(\hat{G}; z, z^*)|z\rangle\langle z|$.
- z and z^* independent variables. Short form $P(z)$ instead $P(\hat{G}; z, z^*)$.
- Rules $a\hat{G}a^\dagger \leftrightarrow zz^*P(z)$, $a^\dagger a\hat{G} \leftrightarrow (z^* - \partial_z)P(z)$, $\hat{G}a^\dagger a \leftrightarrow (z - \partial_{z^*})P(z)$.

PDE for P-function of generating operator

- Field Hamiltonian $\mathcal{H}_{\rm F} = \Omega a^\dagger a$.

$$
\begin{aligned}
\frac{\partial}{\partial t}P_s(z,t) &= \left[-yz\partial_z - y^*z^*\partial_{z^*} + \gamma(1 + |z|^2(s-1))\right]P_s(z,t) \\
y &\equiv -i\Omega - \frac{\gamma}{2}.
\end{aligned} \tag{4.8}
$$

Solve $\frac{\partial}{\partial t}P_s = \left[-yz\partial_z - y^*z^*\partial_{z^*} + \gamma(1 + |z|^2(s-1))\right]P_s$

Case $s = 1$: simply damped harmonic oscillator

- 1st order PDE's are solved by method of characteristics

$$
P_1(z,t) = e^{\gamma t}P^{(0)}\left(ze^{i(\Omega - i\gamma/2)t}\right) \tag{4.9}
$$

Example 1 *($G(s, t = 0) \equiv \rho^{(0)}(t = 0) = |z_0\rangle\langle z_0|$).*

$$
\begin{aligned}
P_1(z, t = 0) &= \delta^{(2)}(z - z_0) \rightsquigarrow \\
P_1(z, t > 0) &= e^{\gamma t}\delta^{(2)}\left(ze^{i(\Omega - i\gamma/2)t} - z_0\right) = \delta^{(2)}\left(z - z_0e^{-i(\Omega - i\gamma/2)t}\right)
\end{aligned} \tag{4.10}
$$

(two-dimensional Delta-function!). State spirals into the origin.

Arbitrary s: $P_s(z,t) = e^{\gamma t}P^{(0)}\left(ze^{i(\Omega - i\gamma/2)t}\right)\exp\{-|z|^2(s-1)(1 - e^{\gamma t})\}$

- Now ${\rm Tr}\hat{G}(s,t) \equiv \sum_{m=0}^\infty s^m{\rm Tr}\rho_t^{(m)}$, read off *photoelectron counting distribution* $p_m(t) \equiv {\rm Tr}\rho_t^{(m)}$.

$$
\begin{aligned}
{\rm Tr}\hat{G}(s,t) &= \int d^2zP_s(z,t) = \int d^2zP^{(0)}(z)e^{-|z|^2(s-1)(e^{-\gamma t}-1)} \\
&= \sum_{m=0}^\infty s^m\int d^2zP^{(0)}(z)\frac{\left(|z|^2\eta_t\right)^m}{m!}e^{-|z|^2\eta_t}, \quad \eta_t \equiv 1 - e^{-\gamma t}.
\end{aligned}
$$

- Use normal ordering property of P-representation,

$$
\boxed{p_m(t) = {\rm Tr}\rho(0) : \frac{(a^\dagger a\eta_t)^m}{m!}e^{-a^\dagger a\eta_t} :, \quad \eta_t \equiv 1 - e^{-\gamma t}} \tag{4.11}
$$

Single-mode counting formula: discussion of $p_m(t) = \mathrm{Tr}\rho(0) : \frac{(a^\dagger a \eta_t)^m}{m!} e^{-a^\dagger a \eta_t} :, \quad \eta_t \equiv 1 - e^{-\gamma t}$

- Coherent state $\rho(0) = |z_0\rangle\langle z_0| \rightsquigarrow$

$$p_m(t) = \frac{(\langle n\rangle \eta_t)^m}{m!} e^{-\langle n\rangle \eta_t}.$$

  - Poisson-distribution.
  - Average $\langle n\rangle \equiv \langle a^\dagger a\rangle = |z_0|^2$.
  - Coincides with semiclassical Mandel formula for $\gamma t \ll 1$.

- Fock-state $\rho(0) = |n\rangle\langle n| \rightsquigarrow$

$$p_m(t) = \binom{n}{m} \eta_t^m (1-\eta_t)^{n-m}, \quad n \geq m.$$

  - Bernoulli-distribution.
  - m successful events (counts), $n-m$ failures (no counts) regardless of order.

Summary part 1

Done so far

- Photon counting: photo-electron counting.
- Semiclassical Mandel formula.
- Photo-detector theory: Scully/Lamb.
- Some techniques: quantum master equations, $P-$representation, counting variables and generating functions/operators.

Still to do

- More general situations.
- Sources, fields, and detectors.

Part II
Photo(electron) counts, quantum jumps and trajectories

Revision: towards a counting formula in quantum optics

- Mandel (Poissonian)

$$p_n(t, t+T) = \frac{\bar{n}^n}{n!} e^{-\bar{n}}, \quad \bar{n} = \eta I(\mathbf{r}) T.$$

  - Classical field with intensity $I(\mathbf{r})$. Golden rule (photo-electric effect).

- Mollow, Scully-Lamb single mode

$$p_n(0,t) = \mathrm{Tr}\rho(0) : \frac{1}{n!}\left(a^\dagger a \eta_t\right)^n \exp(-a^\dagger a \eta_t) :, \quad \eta_t \equiv 1 - e^{-\gamma t}.$$

  – Correctly describes detector backaction. 'Closed system'. Free cavity fields only, no sources.

- 'Quantum Mandel', Kelley-Kleiner

$$p_n(t, t+T) = \langle : \frac{\hat{\Omega}^n}{n!} e^{-\hat{\Omega}} : \rangle.$$

  – Heisenberg operators, $\Omega \equiv \xi \int_t^{t+T} dt' \hat{E}^-(t')\hat{E}^+(t')$.
  – Not correct for long times. 'Open system'. Various generalisations on the market.

5 Correlation functions

Coherence functions

Notation $x = (\mathbf{r}, t)$.

$$G^{(1)}(x, x') \equiv \langle E^{(-)}(x) E^{(+)}(x') \rangle \tag{5.1}$$

$$G^{(2)}(x_1, x_2, x_2', x_1') \equiv \langle E^{(-)}(x_1) E^{(-)}(x_2) E^{(+)}(x_2') E^{(+)}(x_1') \rangle. \tag{5.2}$$

- Based on photon *absorption* $\rightsquigarrow$ intensity $\langle I(x) \rangle = G^{(1)}(x, x)$.
- $G^{(1)}$ describes first order coherence: Mach-Zehnder (Young, Michelson) interference.
- $G^{(2)}$ describes second order coherence: Hanbury Brown, Twiss.

$$G^{(1)}(t, t+\tau) \equiv \langle E^{(-)}(t) E^{(+)}(t+\tau) \rangle \tag{5.3}$$

$$G^{(2)}(t, t+\tau) \equiv \langle E^{(-)}(t) E^{(-)}(t+\tau) E^{(+)}(t+\tau) E^{(+)}(t) \rangle \tag{5.4}$$

$$g^{(2)}(t, t+\tau) \equiv \frac{G^{(2)}(t, t+\tau)}{G^{(1)}(t,t) G^{(1)}(t+\tau, t+\tau)} \tag{5.5}$$

$$\text{number state} \quad \rho(0) = |n\rangle\langle n| \rightsquigarrow g^{(2)}(\tau) = \frac{n(n-1)}{n^2} = 1 - \frac{1}{n}$$

$$\text{coherent state} \quad \rho(0) = |z\rangle\langle z| \rightsquigarrow g^{(2)}(\tau) = \frac{z^* z^* z z}{|z^* z|^2} = 1. \tag{5.6}$$

Definition 2 (bunching, antibunching; sub/super-Poissonian). - Bunching: $g^{(2)}(\tau) < g^{(2)}(0)$, anti-bunching $g^{(2)}(\tau) > g^{(2)}(0)$.
- Super-P. $g^{(2)}(0) > 1$, sub-P. $g^{(2)}(0) < 1$: relation to $p_n(t, t+T)$.

- Mode angular frequency ω, damping κ.
- Master equation.
- Use *quantum regression theorem.*
- Long-time limit, $t \to \infty$, $n_\mathrm{B} = [e^{\beta\omega} - 1]^{-1}$

$$\lim_{t\to\infty} \langle a^\dagger(t) a(t+\tau) \rangle = n_\mathrm{B} e^{-(\kappa + i\omega)\tau} \tag{5.7}$$

$$\lim_{t\to\infty} \langle a^\dagger(t) a^\dagger(t+\tau) a(t+\tau) a(t) \rangle = n_\mathrm{B}^2 \left(1 + e^{-2\kappa\tau}\right). \tag{5.8}$$

- Thus, $g^{(2)}(\tau) = 1 + e^{-2\kappa\tau}$ and $g^{(2)}(\tau) < g^{(2)}(0)$: photon bunching.

(cf. Carmichael book etc.)

6 Source-field dynamics and counting

6.1 Quantum optics basics

Quantization of Maxwell's equations

- Vector potential in Coulomb gauge.
- Fourier expansion into field modes $\mathbf{u}_Q(\mathbf{r})$, mode index Q.

$$(\nabla^2 + \omega_Q^2)\mathbf{u}_Q(\mathbf{r}) = 0.$$

- Quantization, annihilation operator a_Q, creation operator $a_Q^\dagger$.
- Electric field operator

$$\mathbf{E}(\mathbf{r}) = i\sum_Q \left(\frac{\hbar\omega_Q}{2\varepsilon_0}\right)^{1/2} \mathbf{u}_Q(\mathbf{r})a_Q + H.c. = \mathbf{E}^{(+)}(\mathbf{r}) + \mathbf{E}^{(-)}(\mathbf{r}).$$

6.2 Quantum sources of light

Spontaneous emission from a two-level atom

Two-level atom with states $|1\rangle$, $|0\rangle$

$$H = \frac{\omega_0}{2}\sigma_z + \sum_Q \gamma_Q\left(\sigma_+ a_Q + \sigma_- a_Q^\dagger\right) + \sum_Q \omega_Q a_Q^\dagger a_Q. \tag{6.1}$$

Pauli matrices, photon creation operators $a_Q^\dagger$.

Algebra of Pauli matrices

$$\begin{aligned}
\sigma_x &\equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
\sigma_- &\equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_+ \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
\sigma_\pm &= \frac{1}{2}(\sigma_x \pm i\sigma_y), \quad \sigma_x = \sigma_+ + \sigma_-, \quad \sigma_y = -i(\sigma_+ - \sigma_-) \\
[\sigma_+, \sigma_-] &= \sigma_z, \quad [\sigma_z, \sigma_\pm] = \pm 2\sigma_\pm.
\end{aligned} \tag{6.2}$$

- Schrödinger equation for total wave function

$$|\Psi(t)\rangle = c(t)|1\rangle|\mathrm{vac}\rangle + \sum_Q b_Q(t)|0\rangle a_Q^\dagger|\mathrm{vac}\rangle, \quad c(0) = 1 \tag{6.3}$$

- Can be solved (Wigner-Weisskopf) within some approximations. In particular, $c(t) = e^{-\Gamma t/2 - i\omega_0 t}$.
- No re-absorption of any emitted photon $\leftrightarrow$ single mode model (only one Q, Jaynes-Cummings Hamiltonian, revivals).
- Electric field $\mathbf{E}^{(+)}(\mathbf{r},t) = \mathbf{E}_f^{(+)}(\mathbf{r},t) + \mathbf{E}_s^{(+)}(\mathbf{r},t)$, source field in terms of *source* operators

- Heisenberg EOM $\dot{a}_Q(t) = -i\omega_Q a_Q(t) - i\gamma_k \sigma_-(t) \rightsquigarrow$

$$a_Q(t) \;=\; a_Q e^{-i\omega_Q t} - i\gamma_Q \int_0^t dt' \sigma_-(t') e^{-i\omega_Q(t-t')}. \tag{6.4}$$

- Field at the detector in terms of atom dipole operator

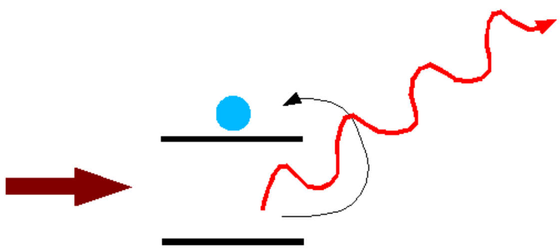
$$\begin{aligned}\underline{\mathbf{E}_s^{(+)}(\mathbf{r},t)} &= \int_0^t dt' \left[\sum_Q \mathbf{f}_Q(\mathbf{r}) e^{-i\omega_Q(t-t')}\right] \sigma_-(t') \qquad (6.5)\\ &\approx \int_0^t dt' \left[\mathcal{E}(\hat{\mathbf{r}})\delta(t-t'-r/c)\right] \sigma_-(t') = \underline{\mathcal{E}(\hat{\mathbf{r}})\sigma_-(t-r/c)}.\end{aligned}$$

- Note dipole form of $\mathcal{E}(\hat{\mathbf{r}})$.

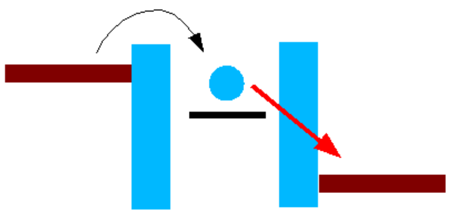
- Not too much can be learned here: transient process, exponentially decaying probability.
- We want to describe stationary processes $\rightsquigarrow$ 'driven spontaneous emission' (resonance fluorescence).
- Analogy to tunneling of a single electron from a single level quantum dot.

6.3 Resonance fluorescence: driven spontaneous emission

Resonance fluorescence: analogy to single electron tunneling



Resonance fluorescence



CB dot, tunneling

Resonance fluorescence model

- Spontaneous emission from TLS plus driving with classical field $E\cos(\omega_L t)$, *Rabi-frequency* $\Omega \equiv dE/\hbar$, d dipole moment.

$$\mathcal{H}_t \equiv \mathcal{H}_{\rm SE} + \frac{\Omega}{2}\left(e^{-i\omega_L t}\sigma_+ + e^{i\omega t}\sigma_-\right), \quad \text{(RWA).} \tag{6.6}$$

- Time-dependent unitary trafo leaves Liouville-v.Neumann equation invariant

$$\bar{\mathcal{H}}_t \equiv -iU_t^\dagger \frac{\partial U_t}{\partial t} + U_t^\dagger \mathcal{H}_t U_t, \quad \bar{\rho}_t \equiv U_t^\dagger \rho_t U_t. \tag{6.7}$$

- The form $U_t = \exp(-i\hat{N}_{\rm F}\omega_L t)\mathrm{diag}(e^{-i\omega_L t}, 1)$ leads to ($\omega_0 = \omega_L$)

$$\boxed{\bar{\mathcal{H}}_t \equiv \tfrac{\Omega}{2}(\sigma_+ + \sigma_-) + \textstyle\sum_Q \gamma_Q \left(\sigma_+ a_Q + \sigma_- a_Q^\dagger\right) + \sum_Q (\omega_Q - \omega_L) a_Q^\dagger a_Q} \tag{6.8}$$

Master equation for TLS-'source' density operator ρ_t

$\dot{\rho}_t = i\frac{\Omega}{2}[\sigma_+ + \sigma_-, \rho_t] - \beta\left(\sigma_+\sigma_-\rho_t + \rho_t\sigma_+\sigma_- - 2\sigma_-\rho_t\sigma_+\right)$

- Spontaneous emission rate $\beta = \pi\sum_Q \gamma_Q^2 \delta(\omega_L - \omega_Q)$, effect of driving in β neglected ($\leftrightarrow$ 'intra-collisional field effect).
- Compare with our previous detector equation, $\dot{\rho}_t^{(m)} = -i[\mathcal{H}_{\rm F}, \rho_t^{(m)}] - \frac{\gamma}{2}\left(a^\dagger a \rho_t^{(m)} + \rho_t^{(m)} a^\dagger a - 2a\rho_t^{(m-1)} a^\dagger\right)$.
- Remember spontaneous emission: field at the detector in terms of atom dipole operator,

$$\mathbf{E}_s^{(+)}(\mathbf{r}, t) \approx \mathcal{E}(\hat{\mathbf{r}})\sigma_-(t - r/c).$$

- Thus $a \sim \mathbf{E}_s^{(+)} \sim \sigma_-$.
- $\rightsquigarrow$ detector photon absorption $\sim$ electron jumps from up to down, σ_-.

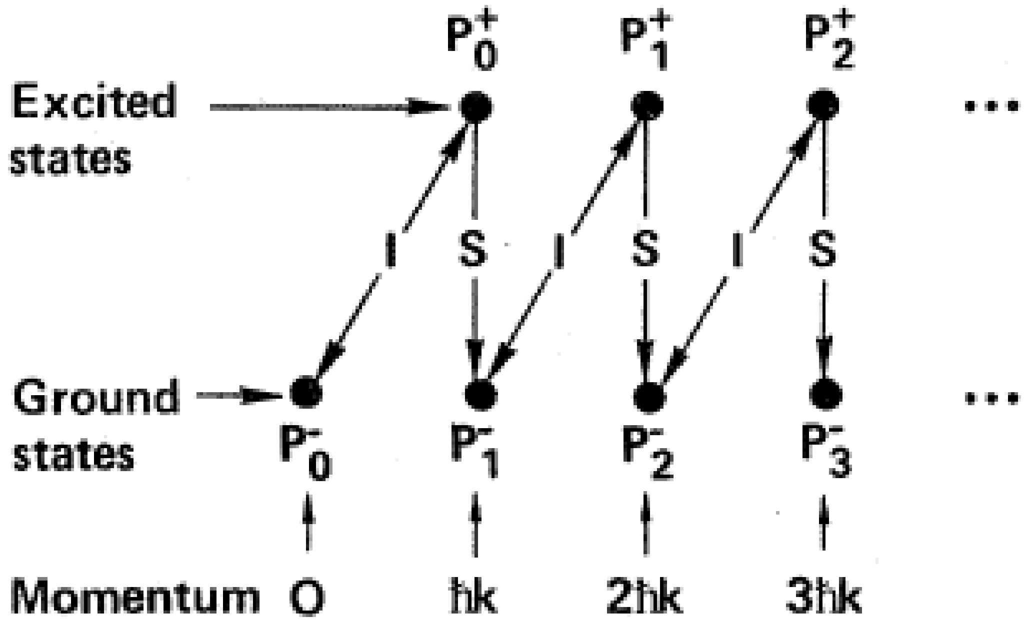
Cook's 'counting at the source'

R. J. Cook PRA **23**, 1243 (1981)

***n*-resolved master equation for resonance fluorescence of driven TLS**

$\dot{\rho}_t^{(n)} = i\frac{\Omega}{2}[\sigma_+ + \sigma_-, \rho_t^{(n)}] - \beta\left(\sigma_+\sigma_-\rho_t^{(n)} + \rho_t^{(n)}\sigma_+\sigma_- - 2\sigma_-\rho_t^{(n-1)}\sigma_+\right)$

- Splitting up $\rho_t = \sum_{n=0}^\infty \rho_t^{(n)}$, n photon emissions.



- Cook's original idea: momentum transfers between atom and driving field.
- Count number of discrete displacements $n\hbar k$.
- Alternatively, count number of spontaneous emission events.
- Jump super-operator J with $J\rho = 2\beta\sigma_-\rho\sigma_+ = 2\beta|-\rangle\langle+|\rho|+\rangle\langle-\rangle$.
- Generating operator as usual, $G(s,t) \equiv \sum_{n=0}^{\infty} s^n \rho_t^{(n)}$; *counting variable* s.
- Counting statistics as $p_n(0,t) = \mathrm{Tr}\rho_t^{(n)}$.
- Photons are integrated out: just 4 by 4 equation

$\partial_t G = i\frac{\Omega}{2}[\sigma_+ + \sigma_-, G] - \beta\left(\sigma_+\sigma_- G + G\sigma_+\sigma_- - 2s\sigma_- G\sigma_+\right)$.

- Solution $G = \exp\{(\mathcal{L}_0 + sJ)t\}\rho(0)$, needs diagonalisation.
- In Laplace space, $\hat{G}(s,z) = [z - \mathcal{L}_0 - sJ]^{-1}\rho(0)$, needs Laplace inversion.
- $\hat{G}$ as vector, resolvent matrix

$$[z - \mathcal{L}_0 - sJ]^{-1} = \begin{pmatrix} z+2\beta & 0 & 0 & -\Omega \\ -2\beta s & z & 0 & \Omega \\ 0 & 0 & z+\beta & 0 \\ \frac{\Omega}{2} & -\frac{\Omega}{2} & 0 & z+\beta \end{pmatrix}.$$

Result in Laplace space

$$
\begin{aligned}
&\mathrm{Tr}\hat{G}(s,z) = \\
&\frac{(z+\beta)(z+2\beta)+\Omega^2+(s-1)2\beta\left[(z+\beta)\rho_0^{++}+\Omega\mathrm{Im}\rho_0^{+-}\right]}{z(z+\beta)(z+2\beta)+\Omega^2[z+\beta(1-s)]}.
\end{aligned}
\tag{6.9}
$$

We now have a closer look a this formula. Let us assume initial conditions $\rho_0^{++}=\rho_0^{+-}=0$, $\rho_0^{--}=1$ for simplicity in the following.

Resonance fluorescence: sub-Poissonian counting statistics

Information contained in

$$
\mathrm{Tr}\hat{G}(s,z) = \frac{f(z)}{zf(z)+\beta\Omega^2(1-s)}, \quad f(z) \equiv (z+\beta)(z+2\beta)+\Omega^2.
$$

- Need to transform back into time-domain.

$$
p_n(0,t) = \frac{\partial^n}{\partial s^n}\,\mathrm{Tr}G(s,t)|_{s=0}\,. \tag{6.10}
$$

$$
\langle n\rangle_t = \frac{\partial}{\partial s}\,\mathrm{Tr}G(s,t)|_{s=1} \quad \text{1st moment.} \tag{6.11}
$$

$$
\langle n(n-1)_t\rangle = \frac{\partial^2}{\partial s^2}\,\mathrm{Tr}G(s,t)|_{s=1} \quad \text{2nd factorial moment.} \tag{6.12}
$$

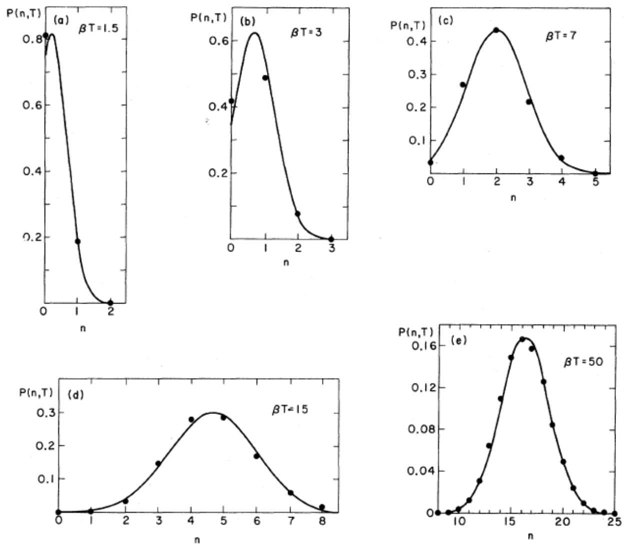
- Large t: pole z_0 closest to $z=0$.
- Expand $z_0=\sum_{m=1}^{\infty} c_m(s-1)^m$

$$
\begin{aligned}
\rightsquigarrow \langle n\rangle_{t\to\infty} &= \frac{\beta\Omega^2}{2\beta^2+\Omega^2}t \\
\rightsquigarrow \sigma_t^2 \equiv \langle \Delta n^2\rangle_{t\to\infty} &= \langle n\rangle_{t\to\infty}\left[1-\frac{6\beta^2\Omega^2}{(2\beta^2+\Omega^2)^2}\right].
\end{aligned}
\tag{6.13}
$$

- Negative Mandel Q-parameter $Q\equiv F-1$, Fano factor $F\equiv\langle\Delta n^2\rangle/\langle n\rangle<1$.
- Large $t\gg\beta^{-1}$: counting statistics $p_n(t)$ becomes a Gaussian!

$$
\lim_{t\to\infty} p_n(t) = \frac{1}{\sqrt{2\pi\sigma_t^2}}e^{-(n-\bar{n}_t)^2/2\sigma_t^2}. \tag{6.14}
$$

(D. Lenstra, PRA **26**, 3369 (1982)).



7 Master equations and quantum jumps

7.1 Counting the jumps

Counting in quantum optics: towards a counting formula...

- Direct 'counting at the source': the savest option...
  - n-resolved master equations with 'jumpers' $J \to sJ$, generating operators. Cook 81 (Lesovik 89, Gurvitz 99, Bagrets/Nazarov 03 ...)
- Mandel (Poissonian) $p_n(t, t+T) = \frac{\bar{n}^n}{n!} e^{-\bar{n}}$, $\quad \bar{n} = \eta I(\mathbf{r})T$.
  - Classical field with intensity $I(\mathbf{r})$.
  - Golden rule (photo-electric effect) plus Markov.
- Mollow, Scully-Lamb single mode $p_n(0,t) = \mathrm{Tr}\rho(0) : \frac{1}{n!}\left(a^\dagger a \eta_t\right)^n \exp(-a^\dagger a \eta_t) :$, $\quad \eta_t \equiv 1 - e^{-\gamma t}$.
  - Correctly describes detector backaction. 'Closed system'.
  - Free cavity fields only, no sources.
- 'Quantum Mandel', Kelley-Kleiner $p_n(t, t+T) = \langle : \frac{\hat{\Omega}^n}{n!} e^{-\hat{\Omega}} : \rangle$.
  - Heisenberg operators, $\Omega \equiv \xi \int_t^{t+T} dt' \hat{E}^-(t')\hat{E}^+(t')$.
  - Not correct for long times. 'Open system'. Various generalisations on the market.

Ueda's photodetector theory

M. Ueda PRA **41**, 3875 (1990). (Relatively) consistent attempt to put everything together ?

- Source-field interaction.
- Detector-field backaction.

Three parties (source, field, receiver/detector).

Multi-mode photodetector

$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\rm D} + \mathcal{H}_{\rm FD}, \mathcal{H}_0 = \mathcal{H}_{\rm S} + \mathcal{H}_{\rm FS} + \mathcal{H}_{\rm F}$

$$\mathcal{H}_{\rm FD} = \sum_{Qkj} \left(V_k^Q c_{kj}^\dagger |0\rangle_j \langle 1| a_Q + H.c. \right), \quad \text{field-detector interaction} \tag{7.1}$$

- Neglect $\mathcal{H}_{\rm FS}$ in deriving non-unitary part of master equation for χ_t (field-source density operator).

$$\begin{aligned} \frac{d}{dt}\chi_t^{(m)} &= -i[\mathcal{H}_0, \chi_t^{(m)}] \\ &- \frac{1}{2}\sum_{QQ'} \gamma_{QQ'} \left(a_Q^\dagger a_{Q'} \chi_t^{(m)} + \chi_t^{(m)} a_Q^\dagger a_{Q'} - 2 a_{Q'} \chi_t^{(m-1)} a_Q^\dagger \right). \end{aligned} \tag{7.2}$$

- Assumes 'broadband detection', $\gamma_{QQ'} = 2\pi N \sum_k V_k^Q \bar{V}_k^{Q'} \delta(\varepsilon - \varepsilon_{kj})$, $N \gg m$ detector atoms.

Formal solution

Generating operator G, 'damper' $\mathcal{L}_0$, 'jumper' J.

- Write $\partial_t G = \mathcal{L}_0 G + sJG$, $G(s,t) \equiv \sum_{m=0}^\infty s^m \chi_t^{(m)}$.
- $\mathcal{L}_0 X \equiv YX + XY^\dagger$, $Y \equiv -i\mathcal{H}_0 - \frac{1}{2}\sum_{QQ'} \gamma_{QQ'} a_Q^\dagger a_{Q'}$.
- $JX \equiv \sum_{QQ'} \gamma_{QQ'} a_{Q'} X a_Q^\dagger$.

- Interaction picture $G(s,t) \equiv S_t \tilde{G}(s,t)$, $S_t \equiv e^{\mathcal{L}_0 t}$.
- Here, $S_t X \equiv e^{\mathcal{L}_0 t} X = e^{Yt} X e^{Y^\dagger t}$.
- Counting and jumping in interaction picture,

$$\partial_t \tilde{G}(s,t) = s e^{-\mathcal{L}_0 t} J e^{\mathcal{L}_0 t} \tilde{G}(s,t). \tag{7.3}$$

Solution of $\partial_t \tilde{G}(s,t) = s e^{-\mathcal{L}_0 t} J e^{\mathcal{L}_0 t} \tilde{G}(s,t)$ as formal power series,

$$\begin{aligned} \tilde{G}(s,t) &= \tilde{G}(s,0) + \int_0^t dt' s e^{-\mathcal{L}_0 t'} J e^{\mathcal{L}_0 t'} \left\{ \tilde{G}(s,0) + \int_0^{t'} dt'' s ... \right\} \\ &= \sum_{m=0}^\infty s^m \int_0^t dt_m ... \int_0^{t_2} dt_1 S_{-t_m} J S_{t_m - t_{m-1}} J ... J S_{t_m} \chi(0) \\ G(s,t) &= \sum_{m=0}^\infty s^m \int_0^t dt_m ... \int_0^{t_2} dt_1 S_{t-t_m} J S_{t_m - t_{m-1}} J ... J S_{t_m} \chi(0). \end{aligned} \tag{7.4}$$

Single-mode case first for simplicity ($A(t) \equiv e^{-Yt} a e^{Yt}$):

$$\begin{aligned} \tilde{\rho}_t^{(m)} &= \gamma^m \int_0^t dt_m ... \int_0^{t_2} dt_1 A(t_m) ... A(t_1) \chi(0) A^\dagger(t_1) ... A^\dagger(t_m) \\ \rho_t^{(m)} &= \gamma^m \int_0^t dt_m ... \int_0^{t_2} dt_1 e^{Yt} A(t_m) ... A(t_1) \chi(0) A^\dagger(t_1) ... A^\dagger(t_m) e^{Y^\dagger t}. \end{aligned}$$

Single mode case, taking traces:

$$\begin{aligned}
\mathrm{Tr}\tilde{\rho}_t^{(m)} &= \gamma^m \int_0^t dt_m ... \int_0^{t_2} dt_1 \langle A^\dagger(t_1)...A^\dagger(t_m)A(t_m)...A(t_1)\rangle \\
\mathrm{Tr}\rho_t^{(m)} &= \gamma^m \int_0^t dt_m ... \int_0^{t_2} dt_1 \langle A^\dagger(t_1)...A^\dagger(t_m)e^{Y^\dagger t}e^{Yt}A(t_m)...A(t_1)\rangle .
\end{aligned}$$

Relation with Kelley-Kleiner formula

Ueda vs Kelley-Kleiner

$$\begin{aligned}
p_m^{\mathrm{U}}(t) &= \gamma^m \int_0^t dt_m ... \int_0^{t_2} dt_1 \langle A^\dagger(t_1)...A^\dagger(t_m)e^{Y^\dagger t}e^{Yt}A(t_m)...A(t_1)\rangle \\
p_m^{\mathrm{KK}}(t) &= \langle : \frac{\hat{\Omega}^m}{m!} e^{-\hat{\Omega}} : \rangle, \quad \hat{\Omega} \equiv \xi \int_0^t dt' a^\dagger(t')a(t')
\end{aligned} \tag{7.5}$$

- No detector backaction in KK.
- Replace damped time-evolution $A(t) \equiv e^{-Yt}ae^{Yt}$ by free time-evolution $a(t) \equiv e^{i\mathcal{H}_0 t}ae^{-i\mathcal{H}_0 t}$.
- Remember single mode case (Mollow, Scully-Lamb) $p_m(t) = \mathrm{Tr}\left\{\rho(0) : \frac{1}{m!}\left(a^\dagger a \eta_t\right)^m \exp(-a^\dagger a \eta_t) :\right\}$, $\eta_t \equiv 1 - e^{-\gamma t}$.
- KK is short-time limit $\gamma t \ll 1 \rightsquigarrow \eta_t = \gamma t$.

Up to first order in γ

$$\begin{aligned}
e^{Y^\dagger t}e^{Yt} &= \left(1 + \frac{\gamma}{2}\int_0^t dt' a^\dagger(t')a(t')...\right)\left(1 + \frac{\gamma}{2}\int_0^t dt' a^\dagger(t')a(t')...\right) \\
&= \left(1 + \gamma \int_0^t dt' a^\dagger(t')a(t')...\right) \\
&= : \exp\left(\gamma \int_0^t dt' a^\dagger(t')a(t')\right) :
\end{aligned} \tag{7.6}$$

- Sum-rule $\sum_{m=0}^\infty p_m(0,t) = 0$ fulfilled for

$$\begin{aligned}
& p_m(0,t) \equiv \mathrm{Tr}\rho_t^{(m)} = \\
&= \gamma^m \int_0^t dt_m ... \int_0^{t_2} dt_1 \langle : a^\dagger(t_1)a(t_1)...a^\dagger(t_m)a(t_m) e^{\gamma \int_0^t dt' a^\dagger(t')a(t')} : \rangle \\
&= \langle : \frac{1}{m!}\left[\gamma \int_0^t dt' a^\dagger(t')a(t')\right]^m e^{\gamma \int_0^t dt' a^\dagger(t')a(t')} : \rangle .
\end{aligned} \tag{7.7}$$

Multi-mode form

$$p_m(0,t) \equiv \mathrm{Tr}\rho_t^{(m)} = \sum_{Q_1Q_1'...Q_mQ_m'} \gamma_{Q_1Q_1'}...\gamma_{Q_mQ_m'} \times$$
$$\times \int_0^t dt_m ... \int_0^{t_1} dt_1 \mathrm{Tr}\left(\chi_0 a_{Q_1}^\dagger(t_1)...a_{Q_m}^\dagger(t_m) e^{Y^\dagger t} e^{Yt} a_{Q_m'}(t_m)...a_{Q_1'}(t_1)\right).$$

- Somewhat impractical ...
- Counting-at-source method much simpler.
- Alternative: integrate out fields in $\partial_t G = \mathcal{L}_0 G + sJG$ (?)

7.2 Quantum trajectories

Quantum jump method, Monte-Carlo for master equation

Example: spontaneous emission from TLS (rotating frame)

$\dot{\rho}_t = -\beta\left(\sigma_+\sigma_-\rho_t + \rho_t\sigma_+\sigma_- - 2\sigma_-\rho_t\sigma_+\right)$

- Jump super-operator J with $J\rho = 2\beta\sigma_-\rho\sigma_+$
- Solve $\partial_t\rho_t = (\mathcal{L}_0 + J)\rho_t$.
- Interaction picture with respect to $\mathcal{L}_0$: $\rho_t \equiv S_t\tilde{\rho}_t$, $S_t \equiv e^{\mathcal{L}_0 t}$.
- Solution of $\partial_t\tilde{\rho}(t) = e^{-\mathcal{L}_0 t} J e^{\mathcal{L}_0 t}\tilde{\rho}(t)$ as formal power series,

$$\rho(t) = \sum_{m=0}^{\infty} \int_0^t dt_m ... \int_0^{t_2} dt_1 \underline{S_{t-t_m} J S_{t_m - t_{m-1}} J ... J S_{t_1} \rho(0)}. \quad (7.8)$$

- m quantum jumps occuring at times $t_1, ..., t_m$.
- Sum over all 'trajectories' with $m = 0, 1, ..., \infty$ jumps between 'free' (but damped) time-evolution.

Monte-Carlo procedure. Fixed time step Δt.

- Step 1: start with pure *wave function* $|\Psi\rangle$.
- Step 2: calculate collaps probability, $P_{\mathrm{col}} = \beta\Delta t\langle\Psi|\sigma_+\sigma_-|\Psi\rangle$
- Step 3: compare P_{col} with random number $0 \le r \le 1$.
  - If $P_{\mathrm{col}} > r$, replace $|\Psi\rangle \to \sigma_-|\Psi\rangle/\|\sigma_-|\Psi\rangle\|$.
  - If $P_{\mathrm{col}} \le r$, no emission but time-evolution $|\Psi\rangle \to (1 - i\Delta t H_{\mathrm{eff}}|\Psi\rangle/\mathcal{N}$, where $H_{\mathrm{eff}}) = -i\beta\sigma_+\sigma_-$.
- Go back to Step 2.
- Repeat procedure in order to obtain average.
- Widely used in quantum optics community.
- Note: splitting $\mathcal{L} = \mathcal{L}_0 + J$ is not unique.
- Literature: Carmichael (book); Plenio,Knight (review).

Summary

- Multi-mode quantum optics: field as 'bath'.
- Correlation (coherence) functions.
- Resonance fluorescence: 'counting at the source', sub-Poissonian, anti-bunched.
- Multi-mode photo-detector theory.
- Quantum trajectories.

Still to do

- Microscopic models for source-field-detector.
- Further understanding of counting statistics $p_n(t)$.
- More complex quantities, e.g. time-resolved probabilities $P_n(t_1, ..., t_n; [t, t+T])$.