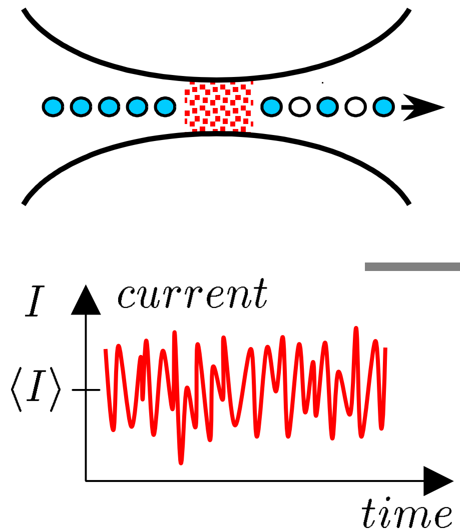
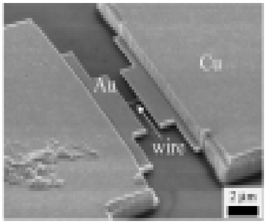
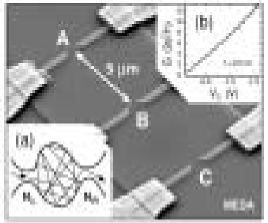
STOCHASTIC PATH INTEGRAL APPROACH TO FCS. LECTURE 1: FORMALISM

EUGENE SUKHORUKOV







1. Why current fluctuates?

2. How strong fluctuations are?

3. Can we use current noise?

References:

Pilgram, Jordan, Sukhorukov, Büttiker, Phys. Rev. Lett.(2003)
Jordan, Sukhorukov, Pilgram, J. Math. Phys. **45**, 4386 (2004)
Jordan, Sukhorukov, Phys. Rev. Lett. **93**, 260604 (2004)

CLASSICAL VS QUANTUM THEORY OF NOISE

QUANTUM THEORY:

- scattering formalism, RMT
- Keldysh technique
- Non-linear σ-model

CLASSICAL THEORY:

- Boltzmann-Langevin equation

$$\mathbf{v}\nabla f_{\mathbf{p}} + \mathcal{I}_{\mathrm{imp}}[f_{\mathbf{p}}] = J^s$$

All theories give the same answers if $N \gg 1$

$\Rightarrow$ quantum phase effects are not important

SYSTEMS:

- Diffusive conductors
- Chaotic cavities
- SN and SNS systems

COMPLETE

CLASSICAL THEORY OF TRANSPORT WAS MISSING

→ **SPI formalism**

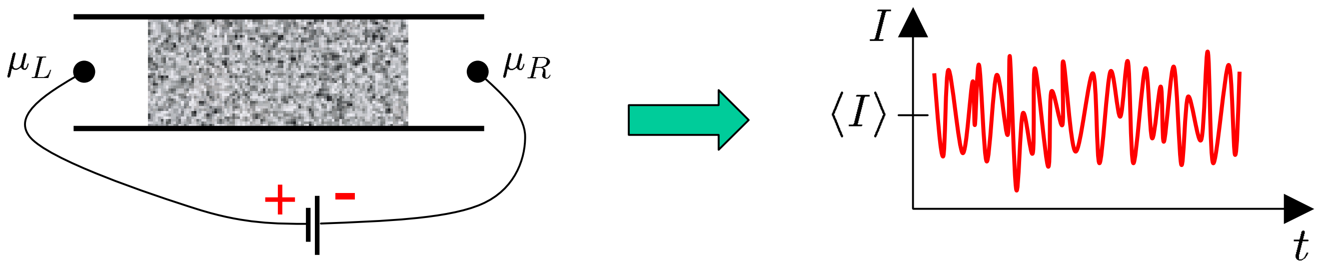
Review: Blanter, Büttiker, Phys. Rep. 336, 1 (2000)

OUTLINE OF LECTURE 1

1. Introduction to noise, motivation.
2. Stochastic network: Basic requirements.
3. Langevin equations.
4. Stochastic Path Integral.
5. Field theory: Diffusive wire.
6. Relation to standard methods.

SIMPLE INTRODUCTION TO NOISE

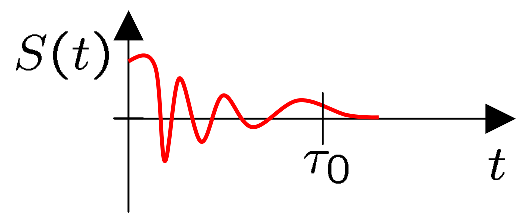
Current fluctuations:



Correlation function:

$$S(t) = \frac{1}{\tau}\int_0^\tau dt' \delta I(t'+t)\delta I(t')$$

$$= \langle \delta I(t) \delta I(0) \rangle,$$



where $\langle \ldots \rangle$ - statistical average.

not easy to measure

Noise power: $S \equiv \int dt\, S(t)$

$$S = \tau^{-1}\int_0^\tau dt \langle \delta I(t) \delta I(0) \rangle = \# \cdot \tau_0 \langle (\delta I(0))^2 \rangle$$

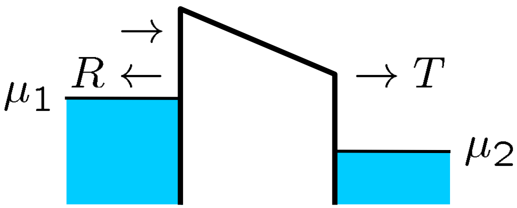
$$S = \tau^{-1}\int\int_0^\tau dt_1 dt_2 \langle \delta I(t_1) \delta I(t_2) \rangle = \langle (\delta Q)^2 \rangle / \tau$$

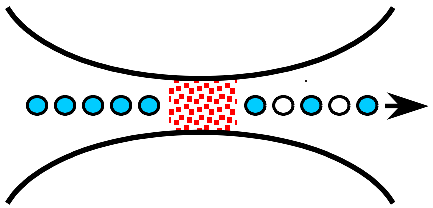
Markovian noise!

$$\langle Q \rangle = \tau \langle I \rangle$$

$$\langle (\delta Q)^2 \rangle = \tau S$$

ELECTRON TRANSPORT – RANDOM PROCESS





Binomial probability game:

$$P_N(n) = \frac{N!}{n!(N-n)!} T^n (1-T)^{N-n}$$

$\rightarrow$ distribution of transmitted charge

Moment generator:

$$M(\lambda) = \sum_n P_N(n) e^{\lambda n}$$

$$\Rightarrow M(0) = 1, \quad M'(0) = \langle n \rangle$$

Binomial statistics:

use $(a+b)^N = \sum_n \frac{N!}{n!(N-n)!} a^n b^{N-n}$

$$\Rightarrow M(\lambda) = \sum_n \frac{N!}{n!(N-n)!} R^n (Te^\lambda)^{N-n}$$

$$\Rightarrow \boxed{M(\lambda) = (R + Te^\lambda)^N.}$$

Normalization:

$$M(0) = (R+T)^N = 1.$$

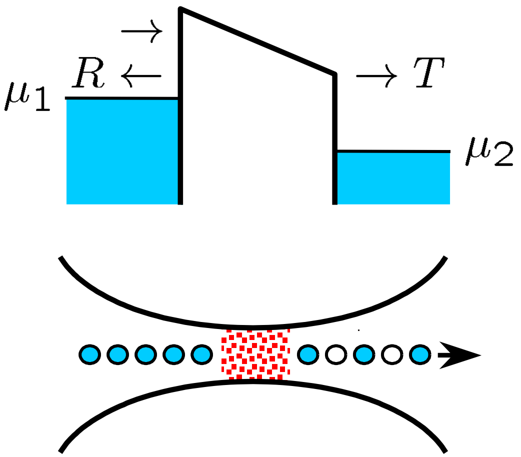
Examples:

$$\langle n \rangle = M'(0) = N\,T,$$

$$\langle n^2 \rangle = M''(0) = N\,T + N(N-1)T^2,$$

$$\Rightarrow \boxed{\langle (\delta n)^2 \rangle = NT(1-T)}$$

CUMULANTS OF CURRENT



In general:

$\langle\langle n^m \rangle\rangle \equiv \partial_\lambda^m \ln[M(\lambda)]|_{\lambda=0} \rightarrow$ cumulants

$\ln[M(\lambda)] = N \ln[1 + T(e^\lambda - 1)]$

Noise correlators:

$\langle I \rangle = \langle n \rangle / \tau = \Omega T,$
where $\Omega = N/\tau$ - attempt frequency

$S = \langle (\delta n)^2 \rangle / \tau = \Omega T(1-T) = \langle I \rangle (1-T)$

$\Rightarrow$ Fano factor: $S/\langle I \rangle = 1 - T,$

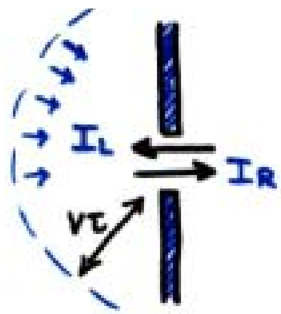
$\Rightarrow$ Low transmission $\rightarrow$ Poissonian limit.

Cumulant generator:

$\langle\langle I^m \rangle\rangle \equiv \langle\langle n^m \rangle\rangle / \tau = \partial_\lambda^m H(\lambda)|_{\lambda=0},$

where $H(\lambda) = (1/\tau) \ln[M(\lambda)] \; = \Omega \ln[1 + T(e^\lambda - 1)]$

NOTES ON BALLISTIC CONDUCTOR 1



Point Contact

$$I = I_R - I_L$$

Reminder:

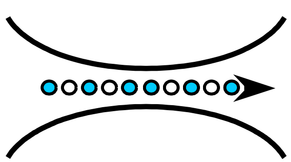
$$I = I_R - I_L = e^{-1} G_{SH} \int d\varepsilon \left(f_L - f_R\right)$$
$$= G_{SH} V$$

Noise power:

$$S = \langle\langle (I_L - I_R)^2 \rangle\rangle = \langle\langle I_L^2 \rangle\rangle + \langle\langle I_R^2 \rangle\rangle$$

Energy is conserved $\Rightarrow$ $\varepsilon \to \{\Delta\varepsilon\}$!

$$\Rightarrow \delta I_L = e^{-1} \Delta\varepsilon\, G_{SH} \sum_{\varepsilon} f_R(\varepsilon)$$



$$\Rightarrow \delta Q_L = \int dt\, \delta I_L = e \mathcal{N} \sum_{\varepsilon} f_R(\varepsilon)$$

where $\mathcal{N} = (G_{SH}/e^2)\, \Delta\varepsilon \cdot \tau$
is total number of sites per $\Delta\varepsilon$.

$$\Rightarrow \langle \delta Q_L^2 \rangle = e^2 \sum_{\varepsilon} \langle (\mathcal{N} \delta f_R)^2 \rangle = e^2 \mathcal{N} \sum_{\varepsilon} f_R(1 - f_R)$$

$$= \tau\, G_{SH} \int d\varepsilon\, f_R(1 - f_R)$$

Binomial with $T \Rightarrow f_L$

NOTES ON BALLISTIC CONDUCTOR 2

Result:

$$S = G_{SH} \cdot \int d\varepsilon \left[f_L(1-f_L) + f_R(1-f_R) \right]$$

$$= 2 G_{SH} T$$ → thermal (equilibrium) noise

At $T=0$, $V \neq 0$ ⇒ degenerate conductor

⇒ $I = G_{SH} V$

But no noise $S=0$!

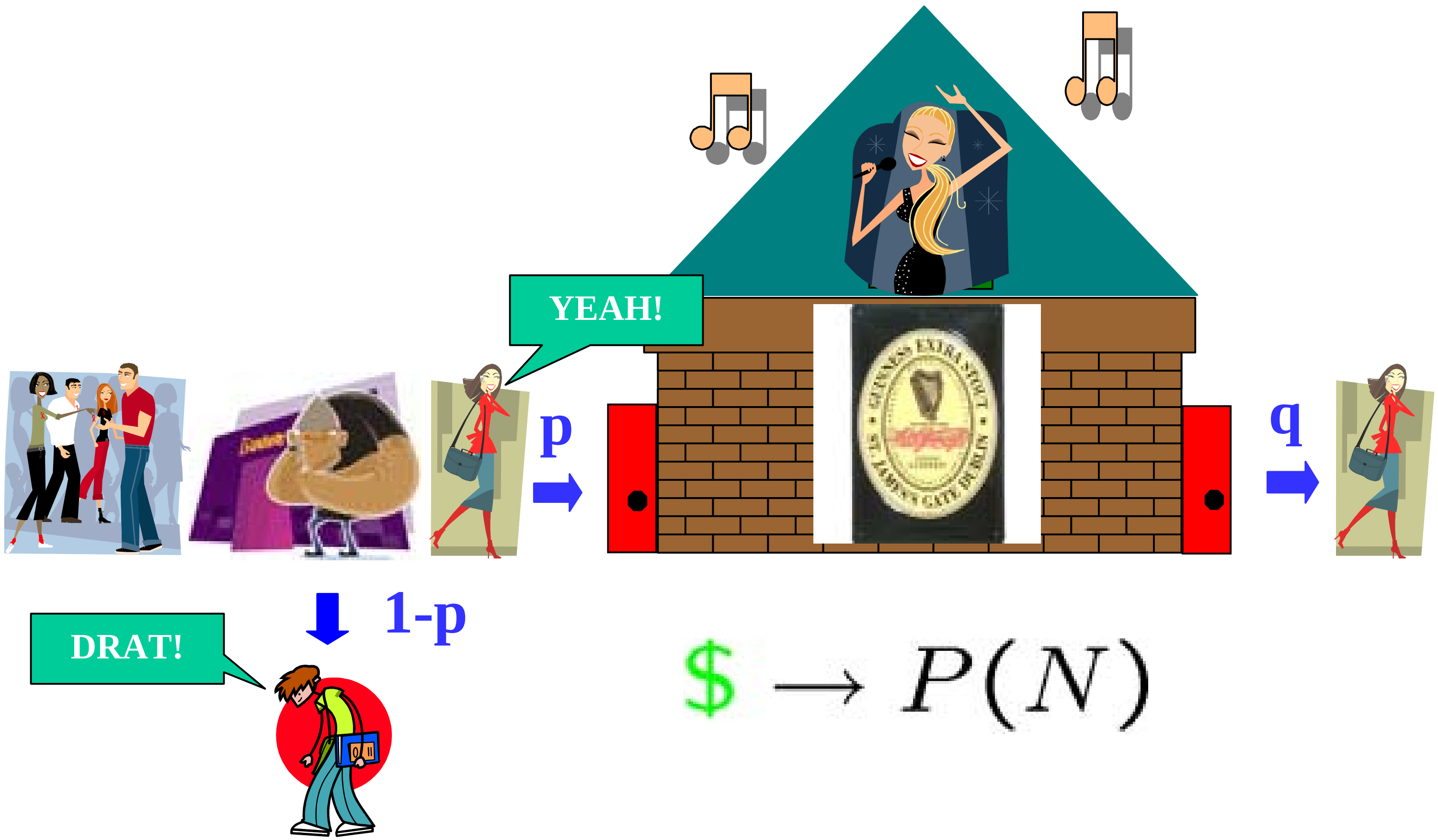
⇒ scattering required; quantum noise?

In general:

$$H(\lambda) = G_{SH} \int d\varepsilon \left\{ \ln\left[1 + f_L(e^{\lambda} - 1)\right] + \ln\left[1 + f_R(e^{-\lambda} - 1)\right] \right\}$$

I_R I_L

“NIGHT CLUB” PHYSICS

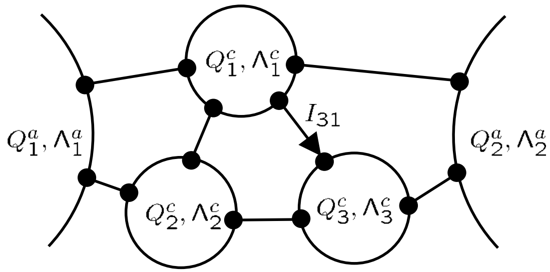


$$\$ \rightarrow P(N)$$

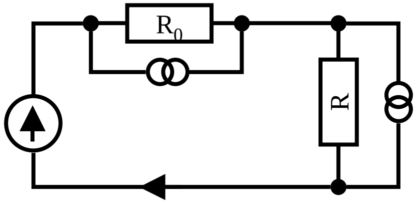
Jordan, *E.S.*, Pilgram, J. Math. Phys. (2004)

BASIC REQUIREMENTS, NETWORK

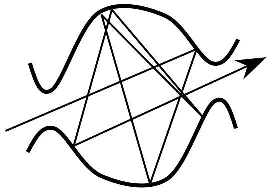
1. **Generalized charges are classical**
2. **Separation of time scales,** $\tau_C \gg \tau_0$
   - Fast fluctuating currents, $\{I^s\}, \; \tau_0$
   - Slow fluctuating charges, $\{Q\}, \; \tau_C$
3. **Continuous charges <=> large parameter**



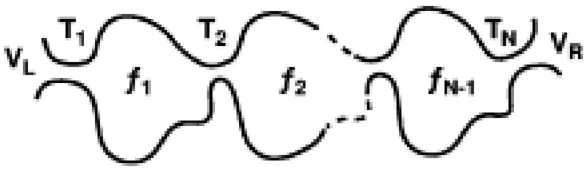
<u>Examples:</u>



Electrical circuit



Mesoscopic cavity



Series of cavities

NOTES ON LANGEVIN EQUATIONS 1

Relaxation dynamics:

$$\dot{Q}_\alpha = \sum_\beta G_{\alpha\beta} Q^c_\beta \quad , \quad \vec{Q} = \{\vec{Q}^a, \vec{Q}^c\}$$

$\vec{Q}^c \to$ conserved charges

$\vec{Q}^a \to$ absorbed charges, reservoirs

$$\hat{G} = \begin{pmatrix} \hat{G}^{cc} & \hat{G}^{ca} \\ \hat{G}^{ac} & \hat{G}^{aa} \end{pmatrix} \to \text{conductances matrix}$$

- Charge conservation,

$$\sum_\alpha \dot{Q}_\alpha = 0 \Rightarrow \sum_\alpha G_{\alpha\beta} = 0$$

$$\Rightarrow G_{\alpha\alpha} < 0$$

- Relaxation, $\vec{Q}^c(t) = \exp\{\hat{G}^{cc} \cdot t\} \vec{Q}^c(0) \Rightarrow 0$
- stationary state, $\vec{Q}^c = 0$, $\dot{\vec{Q}}^a \neq 0$

NOTES ON LANGEVIN EQUATIONS 2

Fluctuations:

$$Q_\alpha \Rightarrow Q_\alpha + \delta Q_\alpha$$

$$I_{\alpha\beta} \Rightarrow I_{\alpha\beta} + \delta I_{\alpha\beta}$$

Separation of time scales

$\Rightarrow$ Langevin Eq:

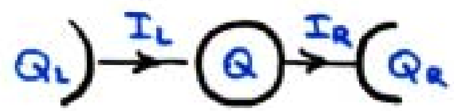
source

$$\delta \dot{Q}_\alpha = \sum_\beta G_{\alpha\beta}\, \delta Q_\beta + \sum_\beta \delta I_{\alpha\beta}$$

$$\iint_{-\tau/2}^{\tau/2} dt_1\, dt_2 \langle \delta I_{\alpha\beta}(t_1)\, \delta I_{\alpha\beta}(t_2) \rangle \equiv \tau \langle\langle I_{\alpha\beta} \rangle\rangle$$

NOTES ON LANGEVIN EQUATIONS 3

Example (night club):



Langevin equations:

Feedback (slow)

$$\delta \dot{Q}_R = G_R\, \delta Q + \delta I_R$$

$$-\delta \dot{Q}_L = G_L\, \delta Q + \delta I_L$$

fluctuations at entrance or exit (fast)

$$\delta \dot{Q} = -(G_L + G_R)\,\delta Q - I_R + I_L$$

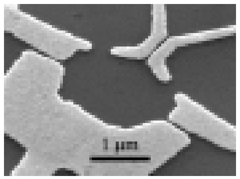
$\Downarrow$ $\tau \gg \tau_c \rightarrow \delta \dot{Q} = 0$

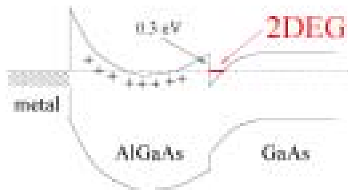
$$\delta Q = \frac{\delta I_L - \delta I_R}{G_L + G_R}\,, \quad \delta \dot{Q}_R = \frac{\delta I_L\, G_R + \delta I_R \cdot G_L}{G_L + G_R}$$

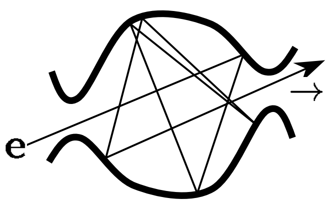
$\Downarrow$ using $\langle \delta I_L\, \delta I_R \rangle = 0$

$$S = \frac{1}{\tau}\langle \delta Q_R^2 \rangle = \frac{G_R^2 \langle\langle I_L^2 \rangle\rangle + G_L^2 \langle\langle I_R^2 \rangle\rangle}{(G_L + G_R)^2}$$

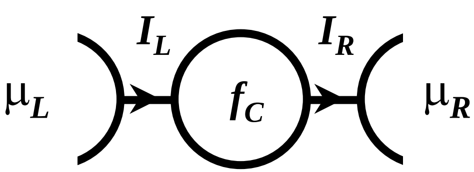
NOISE OF MESOSCOPIC BILLIARD







Mesoscopic billiard



Network representation

Noise sources:

$\langle\langle I_L^2 \rangle\rangle = G_L T_C, \quad \langle\langle I_R^2 \rangle\rangle = G_R T_C,$

where $T_C = \Delta\mu\, f_C(1 - f_C)$

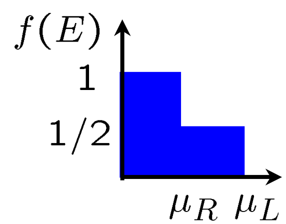
Noise power:

$\langle\langle I^2 \rangle\rangle = G T_C$, where $G = \frac{G_L G_R}{G_L + G_R}$.

Fano factor: $F = f_C(1 - f_C)$.

Symmetric billiard: $G_L = G_R$,

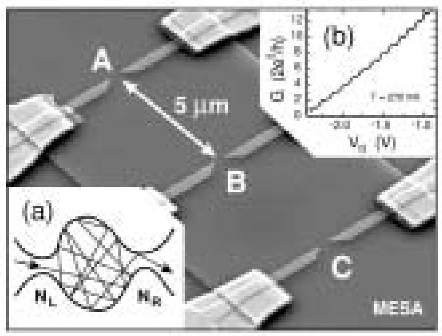
$f_C = 1/2 \Rightarrow F = 1/4$.

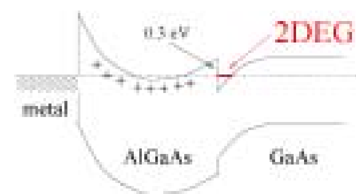


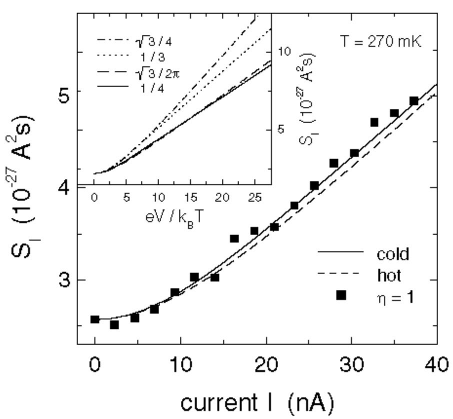
Blanter & ES, PRL 84, 1280 (2000)

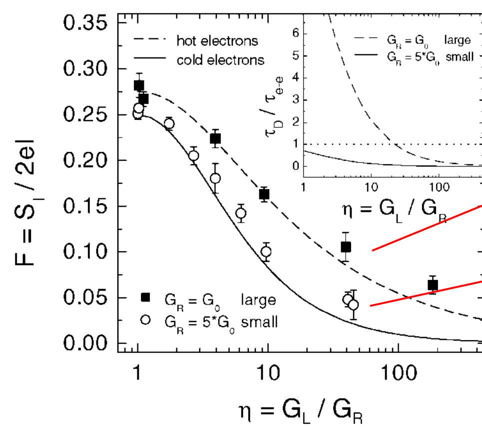
MESOSCOPIC BILLIARD: EXPERIMENT

Chaotic billiard in 2D electron system:









$$F(\eta) = \frac{\sqrt{3 N_L N_R}}{\pi(N_L + N_R)} = \frac{\sqrt{3\eta}}{\pi(1+\eta)}$$

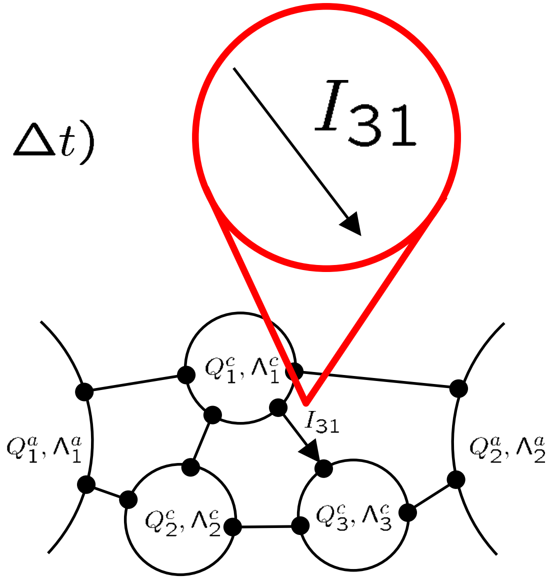
$$F(\eta) = \frac{N_L N_R}{(N_L + N_R)^2} = \frac{\eta}{(1+\eta)^2}$$

Oberholzer et al., PRL 86, 2114 (2001)

GENERALIZATION: STATISTICS OF SOURCES

Statistics of sources:

$$P(Q, \Delta t + \Delta t') = \int dQ' P(Q - Q', \Delta t') P(Q', \Delta t)$$



Generators:

$$P(Q) = \int \frac{d\lambda}{2\pi} \exp[-i\lambda Q + S(\lambda, \Delta t)]$$

$$S(\Delta t + \Delta t', \lambda) = S(\Delta t, \lambda) + S(\Delta t', \lambda)$$

$$\to S = \Delta t\, H \to \langle\langle Q^n \rangle\rangle = \Delta t \langle\langle I^n \rangle\rangle$$

$$H(\lambda) = \sum_n \frac{1}{n!} \langle\langle I^n \rangle\rangle (i\lambda)^n$$

Examples:

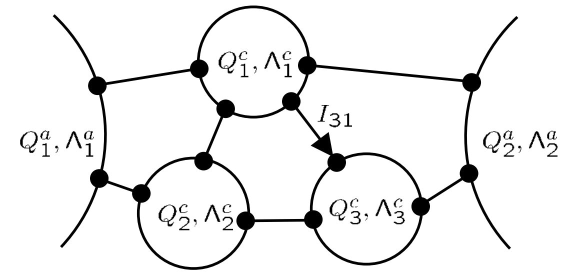
Poissonian noise: $H = \langle I \rangle [e^{\lambda} - 1]$

Binomial noise: $H = (\langle I \rangle / T) \ln[1 + T(e^{\lambda} - 1)]$

GENERALIZATION: NETWORK EVOLUTION

<u>**Langevin equation:**</u>

$$\delta \dot{Q}_\alpha = \sum_\beta G_{\alpha\beta}\delta Q^c_\beta + \sum_\beta I^s_{\alpha\beta}$$



<u>**Solution:**</u>

$$\Gamma(Q,t) = \int dQ' U(QQ',t)\Gamma(Q',0)$$

$$U(QQ',t) = \int \mathcal{D}Q\mathcal{D}\Lambda \exp\{S(Q,\Lambda\}$$

$$S(Q,\Lambda) = \int_0^t dt'[-i\Lambda\cdot\dot{Q} + (1/2)\sum_{\alpha\beta} H_{\alpha\beta}(Q^c, \lambda_\alpha - \lambda_\beta)]$$

$$H_{\alpha\beta} = \sum_n \frac{i^n}{n!}\langle\langle(I_{\alpha\beta})^n\rangle\rangle(\lambda_\alpha - \lambda_\beta)^n$$

A node is called "absorbing" if H is independent of its charge.

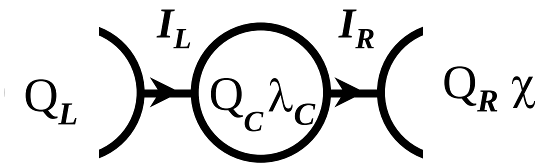
$$i\int dt' \Lambda^a \dot{Q}^a = -i\int dt' Q^a \dot{\Lambda}^a + i\Lambda^a[Q^a(t) - Q^a(0)]$$

$$\rightarrow \Lambda^a = const$$

$\rightarrow$ ln$[U(\Lambda^a)]$ – Cumulant generator

Pilgram, Jordan, *E.S.*, Büttiker, PRL (2003) Jordan, *E.S.*, Pilgram, J. Math. Phys. (2004)

DERIVATION OF THE PATH INTEGRAL IN 4 STEPS



1. **Time discretization,** $Q_{L,R} = \int_0^{\Delta t} dt' I_{L,R}$

   $P_{L,R}(Q_{L,R}) = FT[\exp(S_{L,R})]$

2. **Independent Markovian currents => ** $S_{L,R} = H_{L,R}\Delta t$

   $$P(\mathcal{Q}) = \Pi_n \int \{dQ_{L,R}\} P_L[Q_L(t_n)] P_R[Q_R(t_n)] \delta[\mathcal{Q} - \sum_n Q_R(t_n)]$$

3. **Conservation of charge,**

   $\delta[Q_C(t_{n+1}) - Q_C(t_n) - Q_L(t_n) + Q_R(t_n)] \to \lambda_C, \chi$

4. **Integrate out QL and QR, continuum limit.**

$$\exp[S(\chi,t)] = \int \mathcal{D}Q_C \mathcal{D}\lambda_C \ \exp\left\{\int_0^t dt' [-i\lambda_C \dot{Q}_C + H(\chi, \lambda_C, Q_C)]\right\}$$

$$H(\chi, \lambda_C, Q_C) = H_L(\lambda_C, Q_C) + H_R(-\lambda_C + \chi, Q_C)$$

= "Hamiltonian"

Pilgram, Jordan, *E.S.*, Büttiker, PRL (2003)

Jordan, *E.S.*, Pilgram, J. Math. Phys. (2004)

SADDLE-POINT APPROXIMATION

Large parameter: $\tau_C/\tau_0 \gg 1$

$$S_{\rm tot}(\Lambda^a) = S_{\rm sp} + S_{\rm fluc}$$

Dominates

Equations of motion:

$$\dot{Q}^c = \frac{\partial H}{\partial \Lambda^c}\,, \qquad \dot{\Lambda}^c = -\frac{\partial H}{\partial Q^c}$$

Stationary limit: $t \gg \tau_C \quad \Rightarrow \quad \dot{\Lambda}^c = \dot{Q}^c = 0$

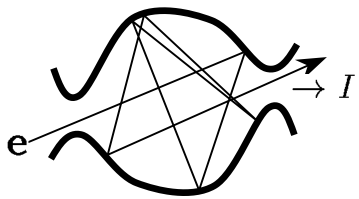
Cumulant generator: $S_{\rm sp} = t \cdot H(\bar{Q}^c, \bar{\Lambda}^c, \Lambda^a)$,

where $\partial_{Q^c} H = \partial_{\Lambda^c} H = 0 \Rightarrow \{\bar{Q}^c, \bar{\Lambda}^c\}$.

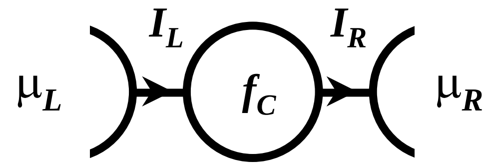
Normalization:

If $\Lambda^a = 0 \quad \Rightarrow \quad \bar{\Lambda}^c = 0 \quad \Rightarrow \quad H = 0$

FULL COUNTING STATISTICS OF A BILLIARD



Mesoscopic billiard



Network representation

The "Hamiltonian":

$$H(\lambda_C) = H_L(\lambda_C) + H_R(\lambda_C + \lambda),$$

$$H_{L,R}(\lambda_C) = G_{L,R}\Delta\mu \left\{ \ln[1 + f_C(e^{\lambda_C} - 1)] - \lambda_C f_{L,R} \right\}$$

Reminder on FCS of point contact:

$$H(\lambda) = G_{SH} \int d\varepsilon \left\{ \ln\left[1 + f_L(e^{\lambda} - 1)\right] + \ln\left[1 + f_R(e^{-\lambda} - 1)\right] \right\}$$

Symmetric billiard:

Saddle point: $\partial_{\lambda_C} H = \partial_{f_C} H = 0$

$\Rightarrow \bar{f}_C = 1/2,\ \bar{\lambda}_C = 0$

$\Rightarrow \boxed{H_{\mathrm{SP}} = 4\langle I \rangle \left[\ln(1 + e^{\lambda/2}) - \ln 2 \right]}$

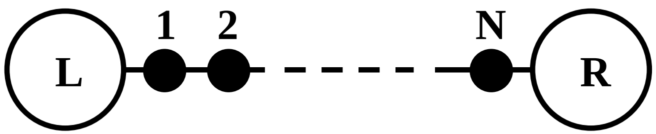
$\Rightarrow$ RMT result with $F = 1/4$

Pilgram, Jordan, *E.S.*, Büttiker, PRL (2003)

FIELD THEORY

Action: $$S = \int dt \sum_{\alpha} \{-\lambda_\alpha \dot{Q}_\alpha + H(Q_\alpha, Q_{\alpha-1}; \lambda_\alpha - \lambda_{\alpha-1})\}$$

Continuum limit:



$H \to h(\rho, \lambda)\Delta z$

$Q_\alpha \to \rho(z)\Delta z$

$G_\alpha (\Delta z)^2 \to D(\rho)$

$S_\alpha \Delta z \to F(\rho)$

Result:

$$S = -\int dt \int dz \left[\lambda\dot{\rho} + D\,\rho'\lambda' - \tfrac{1}{2}F\,(\lambda')^2\right]$$

Conservation law:

gauge invariance

1. Conditional current: $\mathcal{J} = -D(\rho)\rho' + F(\rho)\lambda'$
2. Hamiltonian density: $h = -D(\rho)\rho'\lambda' + (1/2)F(\rho)(\lambda')^2$

invariance under translations

Jordan, *E.S.*, Pilgram, J. Math. Phys. (2004) Bodineau, Derrida, PRL (2004)

TWO TERMINALS: UNIVERSALITY

Boundary conditions: at $z = 0$ and $z = L$,

$\rho(0) = \rho_L,\ \rho(L) = \rho_R,\ \lambda(0) = 0,\ \lambda(L) = \chi$

Solution:

$h = const \Rightarrow \boxed{S = tL \cdot h(\chi).}$

Legendre transform $\Rightarrow \ln[P(I)] = S(\chi) - tI \cdot \chi$.

Universality:

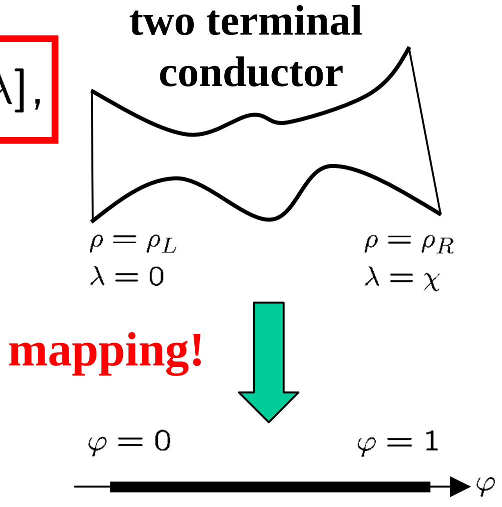
Field theory $\boxed{S = t \int d\mathbf{r}\, [-\nabla\lambda\hat{D}\nabla\rho + (1/2)\nabla\lambda\hat{F}\nabla\lambda],}$

where $\hat{D} = D(\rho)\,\hat{T},\ \hat{F} = F(\rho)\,\hat{T},\ \forall\ \hat{T}$

is solved by $\rho(\mathbf{r}) = \rho\,[\varphi],\ \ \lambda(\mathbf{r}) = \lambda[\varphi]$

where $\nabla \cdot [\hat{T}\nabla\varphi(\mathbf{r})] = 0$

$\Rightarrow \boxed{S = t\mathcal{G} \cdot \mathcal{H},}$ where $\mathcal{G} = \int d\mathbf{r}\, \nabla\varphi\hat{T}\nabla\varphi$.



Jordan, *E.S.*, Pilgram, J. Math. Phys. (2004)

DIFFUSIVE WIRE

Reminder:

$S_\alpha = 2G \int d\varepsilon f_\alpha(1-f_\alpha)$

Action:

$$S = \langle I \rangle t \int_{-1/2}^{1/2} dz \, [-\rho' \lambda' + \underbrace{\rho(1-\rho)}_{F[\rho]/2}(\lambda')^2]$$

Boundary conditions:

$$\rho_L = 1, \ \rho_R = 0, \ \lambda_L = 0, \ \lambda_R = \chi$$

Fano factor

$$\Rightarrow \langle\langle I^2 \rangle\rangle / \langle I \rangle = S''/S' = 1/3$$

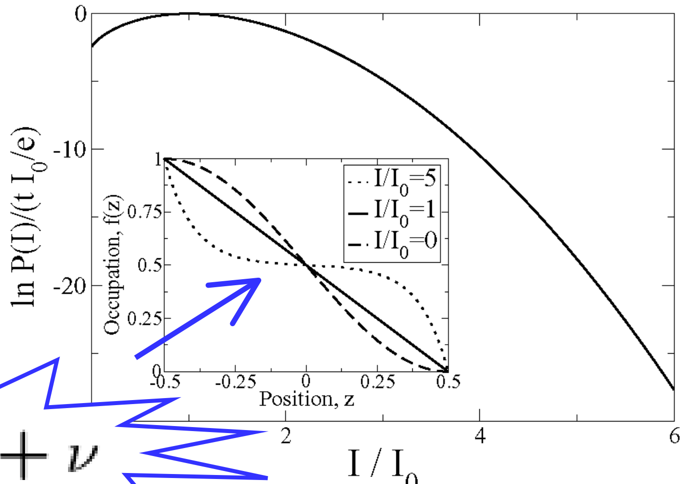
Cumulant generator:

$$S_{\text{sp}}(\chi) = \langle I \rangle t \, \text{arcsinh}^2 \left[\sqrt{\exp(\chi) - 1} \right]$$

Occupation function:

$$\rho(z, I) = \frac{1}{2} \left[1 - \frac{\sinh(2\alpha z)}{\sinh \alpha} \right]$$

$$\alpha \coth(\alpha) = I / \langle I \rangle$$



$$I = -\rho' + \nu$$

Jordan, *E.S.*, Pilgram, J. Math. Phys. (2004)

SPI AND STANDARD METHODS

Master equation:

weak transitions, $W\tau_0$ ¿ 1

$$\dot{\Gamma}(\mathbf{Q},t) = \int d\mathbf{Q}' \left[W(\mathbf{Q},\mathbf{Q}')\Gamma(\mathbf{Q}',t) - W(\mathbf{Q}',\mathbf{Q})\Gamma(\mathbf{Q},t)\right]$$

$$\Rightarrow \quad \boxed{\int d\mathbf{Q} \left[e^{i(\mathbf{Q}-\mathbf{Q}')\cdot\mathbf{\Lambda}} - 1\right] W(\mathbf{Q},\mathbf{Q}') = H(\mathbf{Q}',\mathbf{\Lambda})}$$

Fokker-Planck equation:

Gaussian noise

$$\dot{\Gamma}(\mathbf{Q},t) = -\nabla D(\mathbf{Q})\Gamma(\mathbf{Q},t) + (1/2)\nabla^2 F(\mathbf{Q})\Gamma(\mathbf{Q},t)$$

$$\Rightarrow \quad \boxed{H(\mathbf{Q},\mathbf{\Lambda}) = D(\mathbf{Q})\Lambda + (1/2)F(\mathbf{Q})\Lambda^2}$$

Langevin equation:

$$\mathbf{j} = -\hat{D}(\rho)\nabla\rho + \nu, \qquad \langle \nu(\mathbf{r},t)\nu(0)\rangle = \delta(t)\delta(\mathbf{r})\hat{F}(\rho)$$

$$\Rightarrow \quad \boxed{h = -\hat{D}(\rho)\nabla\rho\nabla\lambda + (1/2)\hat{F}(\rho)\nabla\lambda\nabla\lambda}$$

SPI $\vee$ non-perturbative in Λ and *W*

Jordan, *E.S.*, Pilgram, J. Math. Phys. (2004)