Quantum Noise

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- Noise and counting statistics for electron transport:

— Background on noise exp/th: (i) electron transport; (ii) optics
— Scattering matrix approach (intro)
— Noise in mesoscopic systems: current partition, binomial statistics
— Generating function for counting statistics
— Passive current detector; Keldysh partition function representation

- Tunneling — Odd vs even moments, nonequilibrium FDT theorem

— Third moment S_3
— Comparison to Glauber theory of photocounting

- Driven many-body systems

— Many particle problem $\to$ one particle problem (the determinant formula)
— Coherent electron pumping
— Phase-sensitive noise, 'Mach-Zender effect', orthogonality catastrophe
— Coherent many-body states — noise-minimizing current pulses
— Comparison to quantum optics

Noise Introduction

Fluctuating current $I(t)$ (electrons or photons)

Correlation function $G_2(\tau) = \overline{I(t)I(t+\tau)}$ (time average, stationary flow)

Temporal correlations due to quantum statistics and/or source

Photons counted individually, destroyed at counting

Electrons counted in a flow, without being pulled out of a many-body system, no single-electron resolution yet (ensemble)

Noise spectrum $S(\omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau} S(\tau) d\tau$,
where $S(\tau) = \langle\langle I(t)I(t+\tau)\rangle\rangle = \overline{I(t)I(t+\tau)} - \overline{I}^2$

Photon correlations

Joint photocount probability of detecting photons at times t and $t+\tau$:

$G_2(\tau) = \langle E^{(-)}(t)E^{(-)}(t+\tau)E^{(+)}(t+\tau)E^{(+)}(t)\rangle = \langle : I(t)I(t+\tau) : \rangle$
where : ... : is normal ordering of quantum fields (no re-emission!)

Normalized second-order correlation function $g_2(\tau) = G_2(\tau)/\overline{I}^2$

For a coherent field $E^{(\pm)}(t) \to \epsilon^{(\pm)}(t)$ (a c-number), thus $g_2 = 1$, while for a generic classical field $g_2(\tau) > 1$ (super-Poissonian noise).

For a field obeying Gaussian statistics with zero mean, $g_2(0) = 2$: $\langle E^{(-)}(t)E^{(-)}(t+\tau)E^{(+)}(t+\tau)E^{(+)}(t)\rangle = \langle E^{(-)}(t)E^{(+)}(t+\tau)\rangle\langle E^{(-)}(t+\tau)E^{(+)}(t)\rangle + \langle E^{(-)}(t)E^{(+)}(t)\rangle\langle E^{(-)}(t+\tau)E^{(+)}(t+\tau)\rangle$

$$g_2(0)_{chaotic} = 2g_2(0)_{coherent}$$

Photon bunching (Hanbury-Brown and Twiss): e.g., $g_2(\tau) = 1 + e^{-\gamma|\tau|}$ (Lorentzian spectrum), with $\gamma^{-1} = \tau_c$ the correlation time ($g_2(\tau \gg \tau_c) \to 1$).

Antibunching for nonclassical light, $g_2(\tau) < 1$, sub-Poissonian noise.

Photon counting, classical theory

Radiation of intensity $I(t)$ makes a counter click with probability $\Delta p(t) = \alpha I(t)dt$, where α is the detector sensitivity.

Assuming independence of photocounts at different times, the probability P_n of n counts between t and t' is

$$P_0(t,t') = \prod_{t<t_i<t'} (1-\Delta p(t_i)) = \exp\left(-\sum_{t<t_i<t'} \Delta p(t_i)\right) = \exp\left(-\alpha \int_t^{t'} I(t_i)dt_i\right)$$

$$P_1(t,t') = \sum_{t''} \Delta p(t'') \prod_{t<t_i<t'} (1-\Delta p(t_i)) = \int_t^{t'} \alpha I(t'')dt'' \exp\left(-\alpha \int_t^{t'} I(t_i)dt_i\right)$$

...

$$P_n(t,t') = \frac{1}{n!}\left(\int_t^{t'} \alpha I(t'')dt''\right)^n \exp\left(-\alpha \int_t^{t'} I(t_i)dt_i\right)$$

Example: constant intensity radiation, $I(t) = \bar{I}$, Poisson distribution $P_n(T) = \frac{\bar{n}^n}{n!}\exp(-\bar{n})$, $\bar{n} = \bar{I}T$

Photon counting, quantum theory

Generalize classical expression (Glauber theory):

$$P_n(T) = \langle : \frac{1}{n!}(\alpha T \hat{I})^n \exp(-\alpha T \hat{I}) : \rangle$$

where $\hat{I} = \frac{1}{T}\int_0^T \hat{I}(t)dt = \frac{1}{T}\int_0^T E^{(-)}(t)E^{(+)}(t)dt$

Note: operator normal ordering $: ... :$ and $\langle ... \rangle = \mathrm{Tr}(...\rho)$

Assumptions: short detector reset time, no re-emission

For a single mode, $\hat{I} = a^+ a$, $\mu(T) = \alpha T$:

$$P_n(T) = \langle : \frac{1}{n!}(\mu a^+ a)^n \exp(-\mu a^+ a) : \rangle = \sum_{m\geq 0} \frac{(-)^m \mu^{n+m}}{n! m!} \langle (a^+)^{m+n} a^{m+n} \rangle$$

Ex I: for a coherent state $a|\psi\rangle = \eta|\psi\rangle$ obtain Poisson distribution $P_n(T) = \frac{\bar{n}^n}{n!}\exp(-\bar{n})$, $\bar{n} = \mu|\eta|^2$

Ex II: for a number state $a^+ a|n\rangle = n|n\rangle$ obtain binomial distribution $P_{m\leq n} = C_n^m \mu^m (1-\mu)^{n-m}$, $C_n^m = \frac{n!}{m!(n-m)!}$

If you are an electron

Electron transport

Coherent elastic scattering (mesoscopic systems, point contacts, etc.) — scattering matrix approach

Interactions (nanotubes, quantum wires, QHE edge states) — Luttinger liquid theory, QHE fractional charge theories

Quantum systems driven out of equilibrium (quantum dots, pumps, turnstiles, qubits)

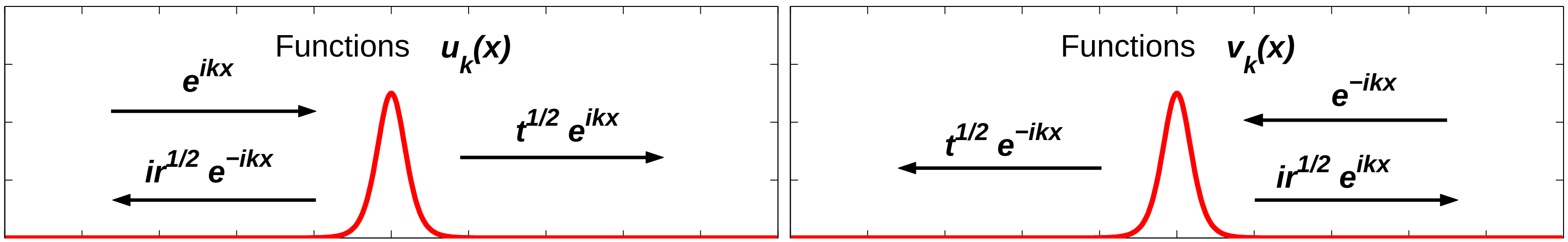
Current autocorrelation function:

$$G_2(\tau) = \langle \frac{1}{2}(I(t)I(t+\tau) + I(t+\tau)I(t)) \rangle$$

(Note: no normal ordering, electrons counted without being destroyed)

Mesoscopic transport (crash course)

A quintessential example (point contact): 1d single channel QM scattering on a barrier, $-\frac{\hbar^2}{2m}\psi'' + U(x)\psi = \epsilon\psi$. Scattering states in asymptotic form:



Express electric current through $\psi(x) = \sum_k \left(\hat{a}_k u_k(x) + \hat{b}_k v_k(x)\right)$:

$\hat{j}(x) = \frac{e}{2m}(-\psi^+(x)\partial_x\psi(x) + \mathrm{h.c.}) = e\sum_{k,k'} e^{i(k-k')x}\frac{k+k'}{2m}\psi^+_{k'}(x)\psi_k(x)$

$$\hat{j}(x) = \sum_{k,k'} e\frac{k+k'}{2m}e^{i(k-k')x}\begin{pmatrix} a^+_{k'} \\ b^+_{k'}\end{pmatrix}\begin{pmatrix} t & i\sqrt{rt} \\ -i\sqrt{rt} & r-1\end{pmatrix}\begin{pmatrix} a_k \\ b_k\end{pmatrix} \qquad (x \gg 0)$$

Time-averaged current (at $eV \ll E_F$ only energies near E_F contribute):
$\langle j(x)\rangle = ev_F\sum_k t\langle a^+_k a_k\rangle + (r-1)\langle b^+_k b_k\rangle = ev_F t\int \frac{dk}{2\pi\hbar}[n_L(\epsilon) - n_R(\epsilon)] = (et/h)\int [f(\epsilon - eV) - f(\epsilon)]\, d\epsilon = \frac{e^2}{h}tV$

Ohm's law: $I = gV$ $(IR = V)$ with conductance $g = 1/R = \frac{e^2}{h}t$ (Landauer)

Multiterminal system, reservoirs, scattering states

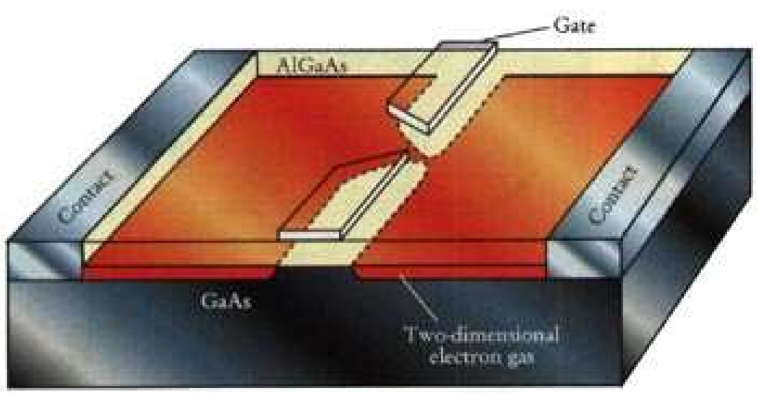
Single channel S-matrix: $S = \begin{pmatrix} \sqrt{t} & i\sqrt{r} \\ i\sqrt{r} & \sqrt{t} \end{pmatrix}$ (optical beam splitter)

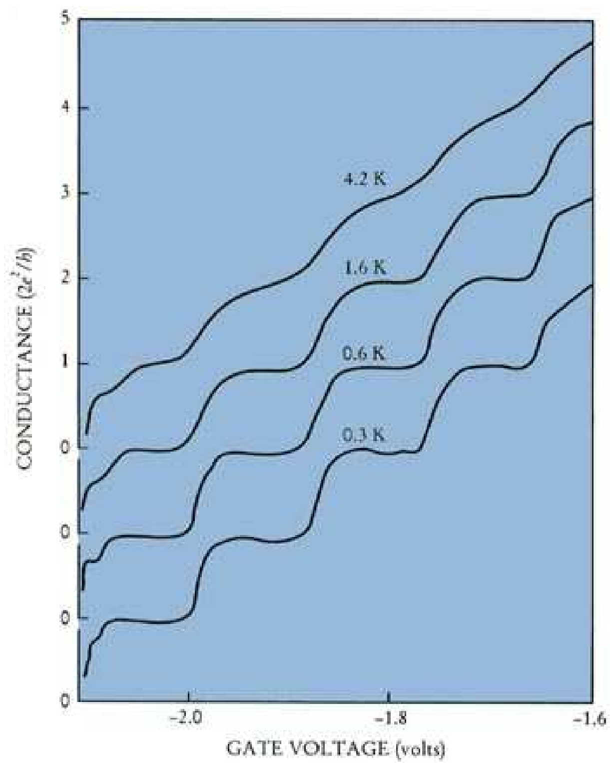
Scattering manifest in transport — quantization of g in point contacts of adjustable width (many parallel channels which open one by one as a function of V_{gate})

$$g = \frac{2e^2}{h} \sum_n t_n$$

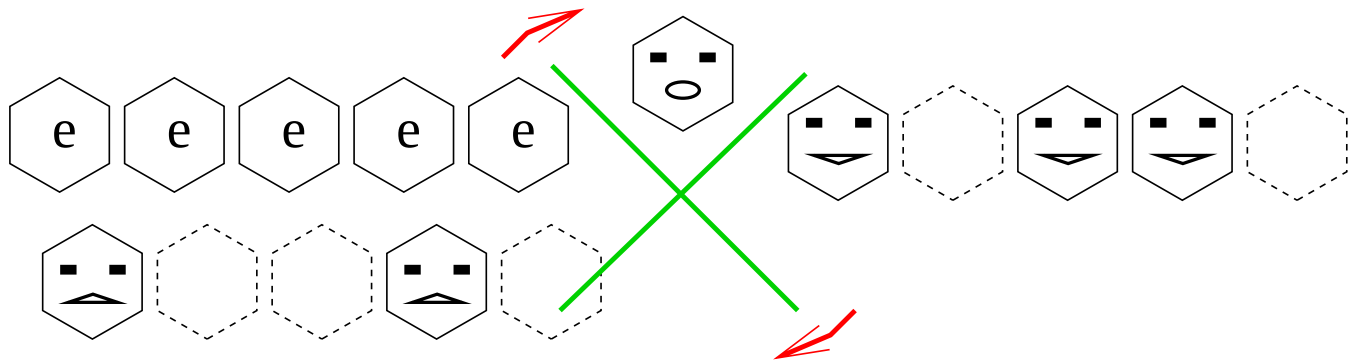
Conductance quantum:
$2e^2/h = 1/13\,\mathrm{k}\Omega^{-1}$

adapted from van Wees et al. (1991)





Electron beam partitioning



Incident electrons transmitted/reflected with probabilities t, $r = 1 - t$;

At $T = 0$, bias voltage $eV = \mu_L - \mu_L$, transport only at $\mu_R < \epsilon < \mu_L$:
(i) Fully filled Fermi at $\epsilon < \mu_L, \mu_R$; (ii) All empty states at $\epsilon > \mu_L, \mu_R$.

Mean current $I = 2eN_{L\to R} = e\int_{\mu_R}^{\mu_L} t\frac{d\epsilon}{2\pi\hbar} = \frac{e^2}{\hbar}tV$ (two spin channels)

Noiseless source (zero temperature) — binomial statistics with the number of attempts during time τ: $N_\tau = (\mu_L - \mu_R)\tau/2\pi\hbar = eV\tau/h$

Transmitted charge $\langle Q\rangle = 2eN_\tau\tau = (2e^2/h)V\tau$ — agrees with microscopic calculation!

Mesoscopic noise

Find noise power spectrum for a single channel conductor (no spin):
$S_\omega = \langle\langle \hat{j}(t)\hat{j}(0) + \hat{j}(0)\hat{j}(t)\rangle\rangle e^{-i\omega t} dt$

$$\hat{j}(t) = \sum_{k,k'} e v_F e^{-i(\epsilon_k - \epsilon_{k'})t} \begin{pmatrix} a^+_{k'} \\ b^+_{k'} \end{pmatrix} \begin{pmatrix} t & i\sqrt{rt} \\ -i\sqrt{rt} & r-1 \end{pmatrix} \begin{pmatrix} a_k \\ b_k \end{pmatrix}$$

$$S_{\omega=0} = \sum_{k,k'} (ev_F)^2 \delta(\epsilon_k - \epsilon_{k'}) \left\langle\left\langle \left[\begin{pmatrix} a^+_{k'} \\ b^+_{k'} \end{pmatrix} \begin{pmatrix} t & i\sqrt{rt} \\ -i\sqrt{rt} & r-1 \end{pmatrix} \begin{pmatrix} a_k \\ b_k \end{pmatrix} \right]^2 \right\rangle\right\rangle$$
$$= \sum_k e^2 v_F \langle\langle [t(a^+_k a_k - b^+_k b_k) + i\sqrt{rt}(a^+_k b_k - b^+_k a_k)] \times [\mathrm{h.c.}]\rangle\rangle$$

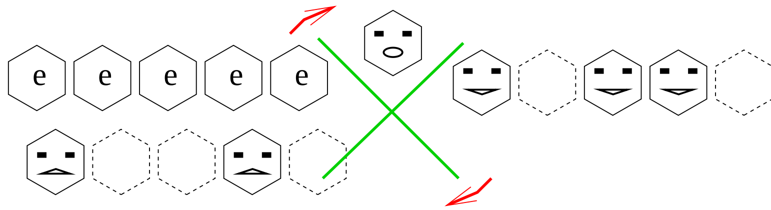
Averaging with the help of Wick's theorem, obtain
$S_0 = \frac{e^2}{h} \int d\epsilon \left[t^2(n_L(1-n_L) + n_R(1-n_R)) + rt(n_L(1-n_R) + n_R(1-n_L)) \right]$

For reservoirs at equilibrium, with $n_{L,R}(\epsilon) = f(\epsilon \mp \frac{1}{2}eV)$, have

$$S_0 = \frac{e^2}{h} \left[t^2 kT + rteV \coth \frac{eV}{2kT} \right] = \begin{cases} gkT & eV \ll kT, \text{ thermal noise;} \\ re^2 geV & eV \gg kT, \text{ shot noise} \end{cases}$$

Note: $S_0(eV \gg kT) = re^2 I$ — Shottky noise, suppressed by $r = 1 - t$ (Lesovik '89)

Noise due to electron beam partitioning



Incident electrons with $\mu_R < \epsilon < \mu_L$ transmitted/reflected with probabilities t, $r = 1 - t$ (No transport at $\epsilon < \mu_L, \mu_R$ and $\epsilon > \mu_L, \mu_R$)

Noiseless source (zero temperature) — binomial statistics with the number of attempts during time τ: $N_\tau = (\mu_L - \mu_R)\tau/2\pi\hbar = eV\tau/h$

Probability of m out of N_τ electrons to be transmitted:
$P_m = C_N^m t^m r^{N-m}$ ($C_N^m = N!/m!(N-m)!$ — binomial coefficients)

Mean value: $\overline{m} = \sum_0^N mP_m = t\partial_t(t+r)^N = tN$
Variance: $\overline{\delta m^2} = \overline{m^2} - \overline{m}^2 = (t\partial_t)^2(t+r)^N - \overline{m}^2 = rtN$

$$\text{Variance} = (1-t) \times \text{Mean}$$

— agrees with microscopic calculation!

Noise in a point contact, experiment

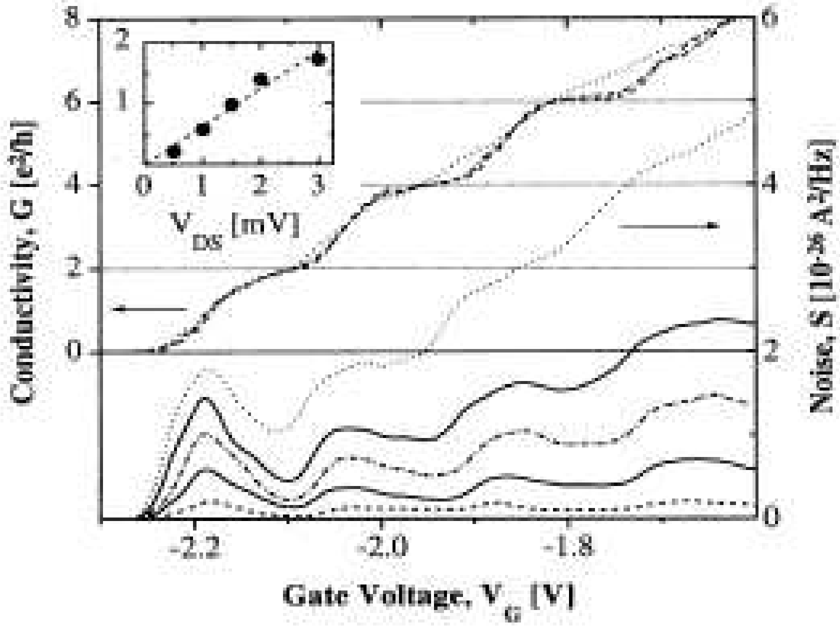


FIG. 2. Noise spectral density $S(\nu)$ and normalized linear conductance G vs gate voltage V_G. The noise is measured for $V_{DS} = 0.5, 1, 1.5, 2$, and 3 mV. Inset: Dependence of the first peak height (same scale as in main figure) on injection voltage V_{DS}. The dashed straight line is the predicted behavior. The conductance is shown for $V_{DS} = 0.5, 1.5$, and 3 mV.

Shot noise summed over channels, $S_0 = \sum_n \frac{2e^2}{h} t_n(1-t_n)$ — minima on QPC conductance plateaus (adapted from Reznikov et al. '95)

Example: photon beam splitter with noiseless source

Consider n identical photons, in a number state $|n\rangle$, incident on a beam splitter.

$$\begin{pmatrix} a_{in} \\ b_{in} \end{pmatrix} = \begin{pmatrix} \sqrt{t} & i\sqrt{r} \\ i\sqrt{r} & \sqrt{t} \end{pmatrix} \begin{pmatrix} a_{out} \\ b_{out} \end{pmatrix}$$

$$|n\rangle = \tfrac{1}{\sqrt{n!}}(a_{in}^{+})^n|0\rangle$$

$$= \tfrac{1}{\sqrt{n!}}(\sqrt{t}a_{out}^{+} + i\sqrt{r}b_{out}^{+})^n|0\rangle$$

$$= \tfrac{1}{\sqrt{n!}}\left(\textstyle\sum_{m=0}^{m=n} i^{n-m} C_n^m t^{m/2} r^{(n-m)/2} (a_{out}^{+})^m (b_{out}^{+})^{n-m}|0\rangle\right)$$

$$= \textstyle\sum_{m=0}^{m=n} i^{n-m} \left(C_n^m\right)^{1/2} t^{m/2} r^{(n-m)/2} |n, n-m\rangle$$

Probability to transmit m out of n photons is $P_m = C_n^m t^m r^{n-m}$ — binomial statistics

Shot noise highlights (for experts)

• Noise suppression relative to Scottky noise $S_2 = eI$ (tunneling current). In a point contact $S_2 = (1-t)eI$ (Lesovik '89, Khlus '87)

• Multiterminal, multichannel generalization; Relation to the random matrix theory; Universal $1/3$ reduction in mesoscopic conductors (Büttiker '90, Beenakker '92)

• Measured in a point contact (Reznikov '95, Glattli '96)

• Measured in a mesoscopic wire (Steinbach, Martinis, Devoret '96, Schoelkopf '97)

• Fractional charge noise in QHE (Kane, Fisher '94, de Picciotto, Reznikov '97, Glattli '97 ($\nu = 1/3$), Reznikov '99($\nu = 2/5$))

• Phase-sensitive (photon-assisted) noise (Schoelkopf '98, Glattli '02)

• Noise in NS structures, charge doubling (Kozhevnikov, Schoelkopf, Prober '00)

• Luttinger liquid, nanotubes (Yamamoto, '03)

• QHE system; Kondo quantum dots; Noise near $0.7e^2/h$ structure in QPC (Weizmann group, recent)

• Third moment S_3 measurement (Reulet, Prober '03, Reznikov '04)

Experimental issues, briefly

- Actually measured is not electric current but EM field. Photons detached from matter, transmitted by $\sim 1\,\mathrm{m}$, amplified, and detected;

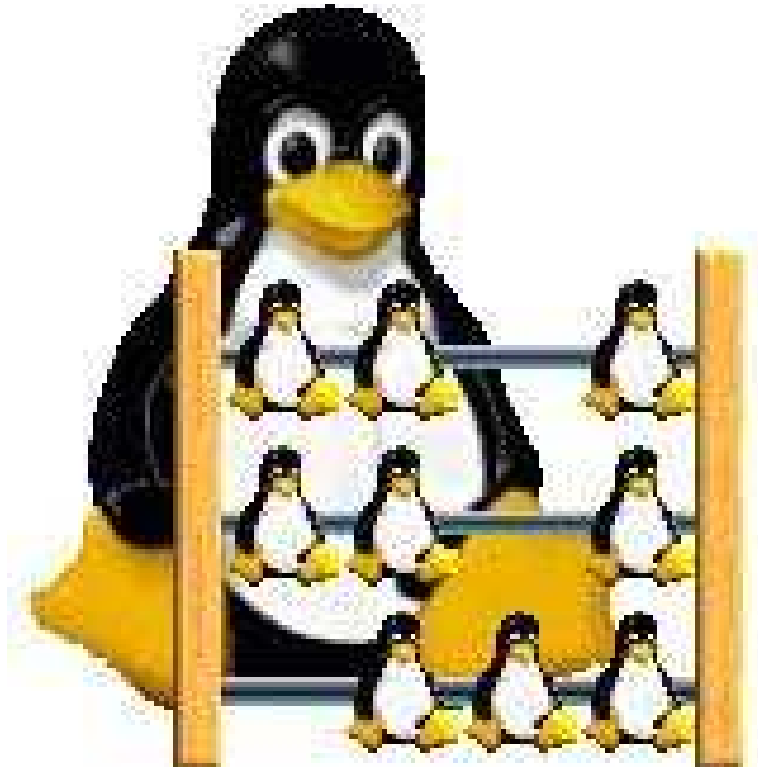
- Matter-to-field conversion harmless if there is no backaction;

- Device $+$ leads $+$ environment. Engineer the circuit so that the interesting noise dominates (e.g. a tunnel junction or point contact of high impedance)

- Detune from $1/f$ noise

- Limitations due to heating and detection sensitivity

Full counting statistics



Counting statistics generating function

Probability distribution $P_n \to$ cumulants $m_k = \langle\langle n^k \rangle\rangle$

$m_1 = \overline{n}$, the mean value;
$m_2 = \overline{\delta n^2} = \overline{n^2} - \overline{n}^2$, the variance;
$m_3 = \overline{\delta n^3} = \overline{(n - \overline{n})^3}$, the skewness;
...

Generating function $\chi(\lambda) = \sum_n e^{i\lambda k} P_k$ (defined by Fourier transform),

$$\ln [\chi(\lambda)] = \sum_{k>0} \frac{m_k}{k!} (i\lambda)^k$$

While P_n is more easy to measure, $\chi(\lambda)$ is more easy to calculate!

The advantage of $\chi(\lambda)$ over P_n similar to partition function in a 'grand canonical ensemble' approach

Ex I: Binomial distribution, $P_n = C_N^n p^n (1-p)^{N-n}$ with N the number of attempts, p the success probability $\to$ $\chi(\lambda) = (pe^{i\lambda} + 1 - p)^N$

Ex II: Poisson distribution, $P_n = \frac{\bar{n}^n}{n!} e^{-\bar{n}}$, $\chi(\lambda) = \exp(\bar{n}(e^{i\lambda} - 1))$

Counting statistics, microscopic formula I

WANTED: A microscopic expression for the generating function $\chi(\lambda) = \sum_q P(q) e^{i\lambda q}$ for a generic many-body system

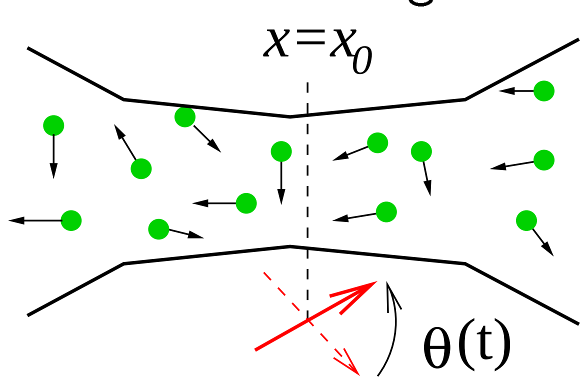
Spin $1/2$ coupled to current: $\mathcal{H}_{el,spin} = \mathcal{H}_{el}(\mathbf{p} - \mathbf{a}\sigma_3, \mathbf{q})$

Counting field $\mathbf{a} = -\frac{\lambda}{2}\hat{\mathbf{x}}\delta(x - x_0)$ measures current through cross-section $x = x_0$

Ex. Coupling to classical current $\mathcal{H} = \frac{\lambda}{2}\sigma_3 I(t)$; time evolution of spin:

$$|\uparrow\rangle = e^{-i\theta(t)/2}|\uparrow\rangle,\ |\downarrow\rangle = e^{i\theta(t)}/2|\downarrow\rangle$$

Spin precesses in the XY plane, precession angle $\theta(t) = \lambda \int_0^t I(t')dt'$ measures time-dependent transmitted charge



Disclaimer: We attempt to clarify microscopic picture of current fluctuations, not to describe realistic measurement

Counting statistics, microscopic formula II

In QM current is an operator and $[I(t), I(t')] \neq 0 \rightarrow$ Need a more careful analysis!

Spin density matrix evolution (ensemble-averaged):

$$\rho(t) = \langle e^{-i\mathcal{H}t}\rho_0 e^{i\mathcal{H}t}\rangle_{el} = \begin{pmatrix} \rho^{(0)}_{\uparrow\uparrow} & \langle e^{i\mathcal{H}_{-\lambda}t}e^{-i\mathcal{H}_{\lambda}t}\rangle_{el}\rho^{(0)}_{\uparrow\downarrow} \\ \langle e^{i\mathcal{H}_{\lambda}t}e^{-i\mathcal{H}_{-\lambda}t}\rangle_{el}\rho^{(0)}_{\downarrow\uparrow} & \rho^{(0)}_{\downarrow\downarrow} \end{pmatrix}$$

$$\langle ...\rangle_{el} = \mathrm{Tr}_{el}(...\rho_{el})$$

Examine the classical current case: spin precesses by $\theta_n = \lambda n$ for n transmitted particles.

$$\rho(t) = \sum_n P_n \begin{pmatrix} \rho^{(0)}_{\uparrow\uparrow} & e^{-i\theta_n}\rho^{(0)}_{\uparrow\downarrow} \\ e^{i\theta_n}\rho^{(0)}_{\downarrow\uparrow} & \rho^{(0)}_{\downarrow\downarrow} \end{pmatrix} = \begin{pmatrix} \rho^{(0)}_{\uparrow\uparrow} & \langle e^{i\lambda n}\rangle\rho^{(0)}_{\uparrow\downarrow} \\ \langle e^{-i\lambda n}\rangle\rho^{(0)}_{\downarrow\uparrow} & \rho^{(0)}_{\downarrow\downarrow} \end{pmatrix}$$

Identify $\langle e^{i\theta(t)}\rangle = \langle e^{i\lambda q}\rangle$ with $\chi(\lambda)$, the generating function

Main result:

$\chi(\lambda)$ is given by Keldysh partition function

$$\chi(\lambda) = \left\langle \mathrm{T}_{\mathrm{K}} \exp\left(-i \oint_{C_{0,t}} \hat{\mathcal{H}}_{\lambda}(t')dt' \right) \right\rangle$$

with the counting field $\lambda(t) = \pm\lambda$ antisymmetric on the forward and backward parts of the Keldysh contour $C_{0,t} \equiv [0 \to t \to 0]$

Properties:

1. Normalization: $\sum_q P_q = 1$, since $\chi(\lambda = 0) = 1$
2. $P_q = \int e^{-iq\lambda}\chi(\lambda)\frac{d\lambda}{2\pi} \geq 0$
3. Charge quantization: $\chi(\lambda)$ is 2π-periodic in λ (for noninteracting particles)

Features:

— Describes not just spin $1/2$ but a wide class of passive charge detectors, such as heavy particle $\mathcal{H} = p^2/2M - \lambda f(t)q$ at large M (no recoil);
— Minimal backaction, measurement affects only forward scattering: $[\mathcal{H}, \sigma_z] = 0$;
— Good for generic many-body system

Variety of topics

• Tunneling problem. S_2, S_3 Nonequilibrium FDT theorem. Relation with Glauber theory of photocounting.

• Driven many-body systems. For noninteracting particles (fermions or bosons) $\chi(\lambda)$ can be expressed through time-dependent one-particle S-matrix. Pumps, coherent current pulses, photon-assisted noise $\rightarrow$ next lecture

• Mesoscopic noise in normal and superconducting systems (Nazarov, Nagaev)

• Mesoscopic photon sources (Beenakker)

• Entangled EPR states, counting statistics (Fazio)

• Spin current noise (Lamacraft)

• Backaction of spin $1/2$ counter (Muzykantsky)

• Role of environment (Kindermann)

• Quantum information/entropy (Callan & Wilczek, Vidal, Kitaev)

• Orthogonality catastrophe, Fermi-edge singularity

Counting statistics of tunneling current

Focus on the tunneling problem (generic interacting system). Tunneling Hamiltonian

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_1 + \hat{\mathcal{H}}_2 + \hat{V}$$

where $\hat{\mathcal{H}}_{1,2}$ and $\hat{V} = \hat{J}_{12} + \hat{J}_{21}$) describe leads and tunneling coupling). The counting field $\lambda(t)$ is added to the phase of the tunneling operators $\hat{J}_{12}$, $\hat{J}_{21}$ as

$$\hat{V}_\lambda = e^{\frac{i}{2}\lambda(t)} \hat{J}_{12}(t) + e^{-\frac{i}{2}\lambda(t)} \hat{J}_{21}(t)$$

with $\lambda_{0<t<\tau} = \lambda$. (Justified using one-particle tunneling problem.)

Transform the bias voltage into a phase factor, $\hat{J}_{12} \to \hat{J}_{12} e^{-ieVt}$, $\hat{J}_{21} \to \hat{J}_{21} e^{ieVt}$. In the interaction representation, write

$$\chi(\lambda) = \left\langle \mathrm{T_K} \exp\left(-i \oint_{C_{0,t}} \hat{V}_{\lambda(t')}(t') dt' \right) \right\rangle$$

using cumulant expansion, as a sum of linked cluster diagrams.

The lowest order in the tunneling coupling $\hat{J}_{12}$, $\hat{J}_{21}$ is given by linked clusters of order two. Obtain $\chi(\lambda) = e^{W(\lambda)}$, where

$$W(\lambda) = -\frac{1}{2}\oint\oint_{C_{0,t}} \left\langle \mathrm{T_K} \hat{V}_{\lambda(t')}(t') \hat{V}_{\lambda(t'')}(t'') \right\rangle dt' dt''$$

More explicitly,

$$W(\lambda) = (e^{i\lambda} - 1) N_{1\to 2}(t) + (e^{-i\lambda} - 1) N_{2\to 1}(t)$$

$$N_{1\to 2} = \int_0^t \int_0^t \langle \hat{J}_{21}(t') \hat{J}_{12}(t'') \rangle \, dt' dt'' \quad N_{2\to 1} = \int_0^t \int_0^t \langle \hat{J}_{12}(t') \hat{J}_{21}(t'') \rangle \, dt' dt''$$

with $N_{j\to k}(t) = n_{jk} t$ the mean particle number transmitted between the contacts in a time t (cf. Kubo formula).

Resulting counting statistics is bi-directional Poissonian:

$$\chi(\lambda) = \exp\left[(e^{i\lambda} - 1) N_{1\to 2}(t) + (e^{-i\lambda} - 1) N_{2\to 1}(t) \right]$$

True in any interacting system, in the tunneling regime.

Nonequilibrium Fluctuation-Dissipation theorem

The cumulants are generated as $\ln\chi(\lambda) = \sum_{k=1}^{\infty} \frac{(i\lambda)^k}{k!} \frac{\langle\langle \delta q^k \rangle\rangle}{q_0^k}$ with q_0 the tunneling charge. Obtain

$$\langle\langle \delta q^k \rangle\rangle = q_0^k \begin{cases} (n_{12} - n_{21})t, & k \text{ odd} \\ (n_{12} + n_{21})t, & k \text{ even} \end{cases}$$

Setting $k = 1, 2$, relate $n_{12} \pm n_{21}$ with the time-averaged current and the low frequency noise power:

$$n_{12} - n_{21} = I/q_0, \quad n_{12} + n_{21} = S_2/q_0^2.$$

Relate the second and the first correlator:

$$S_2 = \langle\langle \delta q^2 \rangle\rangle / t = (N_{1\to 2} + N_{2\to 1})/(N_{1\to 2} - N_{2\to 1}) q_0 I = \coth(eV/2k_{\mathrm{B}}T) q_0 I$$

Nyquist for $eV < k_{\mathrm{B}}T$, Schottky for $eV > k_{\mathrm{B}}T$.

A universal relation — holds for any $I - V$ characteristic (linear responce not required, *cf.* FDT in equilibrium)

Application in metrology

Primary Electronic Thermometry Using the Shot Noise of a Tunnel Junction

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We present a thermometer based on the electrical noise from a tunnel junction. In this thermometer, temperature is related to the voltage across the junction by a relative noise measurement with only the use of the electron charge, Boltzmann's constant, and assumption that electrons in a metal obey Fermi-Dirac statistics. We demonstrate proof-of-concept operation of this primary thermometer over four orders of magnitude in temperature, with as high as 0.1% accuracy and 0.02% precision in the range near 1 kelvin. The self-calibrating nature of this sensor allows for a much faster and simpler measurement than traditional Johnson noise thermometry, making it potentially attractive for metrology and for general use in cryogenic systems.

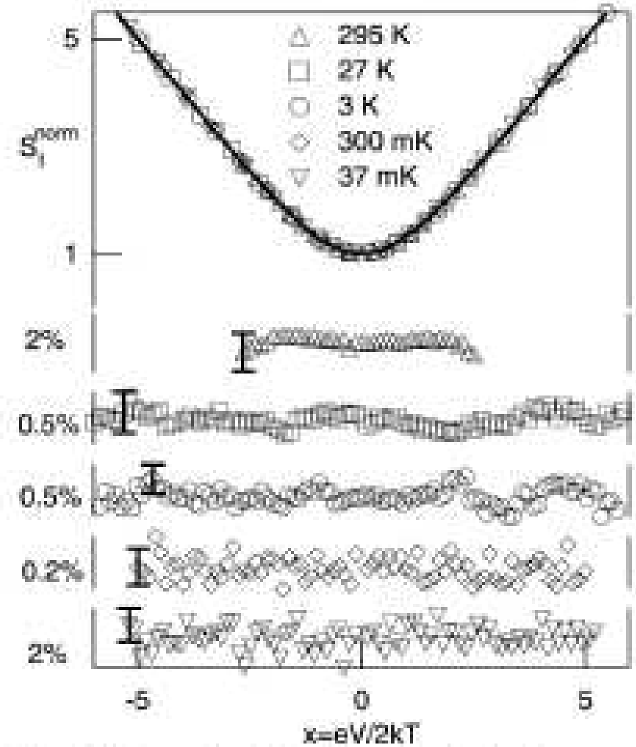


Fig. 3. Normalized junction noise plotted versus normalized voltage at various temperatures.

Generalized Schottky formula for S_3

Relate the third cumulant $\langle\langle \delta q^3 \rangle\rangle \equiv \overline{\left(\delta q - \overline{\delta q}\right)^3} = S_3 t$ with $\langle \delta q \rangle = It$. Obtain a Schottky-like relation for the third correlator spectral power S_3:

$$S_3 \equiv \langle\langle \delta q^3 \rangle\rangle / t = q_0^2 I$$

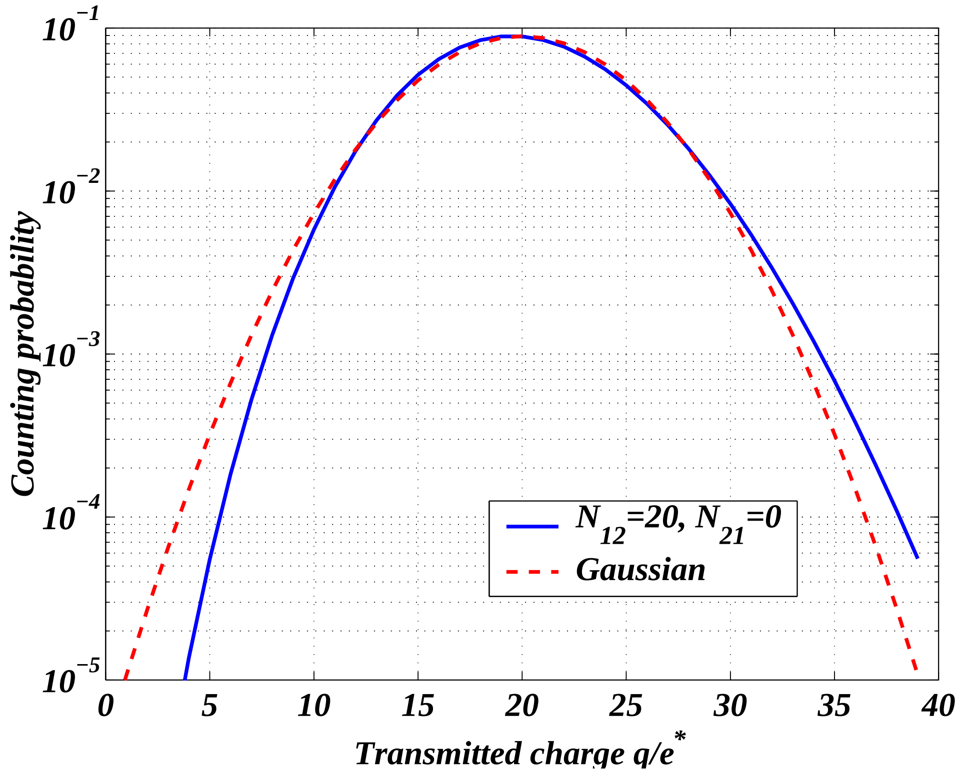
— independent of the mean/variance ratio $(n_{12} - n_{21})/(n_{12} + n_{21})$.

Since $N_{1\to 2}/N_{2\to 1} \equiv n_{12}/n_{21} = \exp(eV/k_{\rm B}T)$ (detailed balance), the relation $S_3 = q_0^2 I$ holds at any voltage/temperature ratio.

Good for using shot noise to determine particle charge in Luttinger liquids and fractional QHE (heating limitation: $S_2 = q_0 I$ requires $eV > k_{\rm B}T$).

A possibility to measure tunneling quasiparticle charge at temperatures $k_{\rm B}T \geq eV$

The 3-rd correlator in terms of the counting distribution profile:
$\langle q \rangle = It =$ mean; $\langle\langle \delta q^2 \rangle\rangle = S_2 t =$ variance; $\langle\langle \delta q^3 \rangle\rangle = S_3 t =$ **skewness**



The third moment determines *skewness* of the distribution $P(q)$ profile. This is illustrated by a bi-directional Poissonian distribution and a Gaussian with the same mean and variance. For $S_3 > 0$ the peak tails are stretched more to the right than to the left.

Measurement of S_3:
Tunnel junction (B. Reulet, J. Senzier, and D.E. Prober, Phys. Rev. Lett. 91, 196601 (2003)); point contact (M. Reznikov, unpublished)

Tunneling summary:

1) The counting statistics of tunneling current is bi-directional Poissonian, universally and independently of the character of interactions and thus of the form of $I - V$ dependence.

2) Nonequilibrium FDT relation $S_2 = \coth(eV/2k_\mathrm{B}T)q_0 I$

3) Shottky-like relation for the third correlator $S_3 = q_0^2 I$ at both large and small $eV/k_\mathrm{B}T$.

Photocounting as 'tunneling of photons'

A system of photons interacting with atoms in a photodetector:

$$\mathcal{H} = \mathcal{H}_{\rm p} + \mathcal{H}_{\rm a} + V$$

with the free EM field and atoms in the detector described by $\mathcal{H}_{\rm p}$ and $\mathcal{H}_{\rm a}$,

$$\hat{V} = \sum_{j,k} \left(u_{j,k} e^{\frac{i}{2}\lambda} b_j^{\dagger} a_k + u_{j,k}^{*} e^{-\frac{i}{2}\lambda} a_k^{\dagger} b_j \right)$$

a "tunneling operator" which describes photon-atom interaction (photon absorption and atom excitation). Use canonical Bose operators of photon modes a_k and the operators b_j describing atom excitation.

<u>Note:</u> (i) Bose statistics OK; (ii) 'Tunnel current' unidirectional (no photon re-emission); (iii) Counting field inserted in $\hat{V}$.

Finding $\chi(\lambda)$ for photodetection

$$\chi_t(\lambda) = \left\langle \mathcal{U}_{-\lambda}^{-1}(t)\,\mathcal{U}_\lambda(t) \right\rangle, \qquad \mathcal{U}_\lambda(t) = \mathrm{Texp}\left(-i\int_0^t \hat{V}_\lambda(t')dt' \right) ,$$

The task simplified by the weakness of the photon-atom interaction $\rightarrow$ 'Markov apprx' analogous to our 2-nd order cumulant apprx in the tunneling problem. Only pairwise averages of atom operators are needed: $\langle b_j^\dagger(t) b_{j'}(t')\rangle = 0$, $\langle b_j(t) b_{j'}^\dagger(t')\rangle = \tau_j \delta(t-t')\delta_{jj'}$ with τ_j a microscopic 'click' time (the δ-function has width τ_j).

Main difference from tunneling: photon coherence time can be much longer than the measurement time t. Need to account for the long coherence times.

<u>Method:</u> average over atoms in the partition function $\chi(\lambda)$, while keeping the photon variables free.

Consider $\chi_{t+\Delta}(\lambda) - \chi_t(\lambda)$ with $\tau_j \ll \Delta \ll t, \tau_{coh}$. Expand:

$$\mathcal{U}_\lambda(t+\Delta) = \left(1 - i\int_t^{t+\Delta} \hat{V}_\lambda(t')dt' - \frac{1}{2}\int_t^{t+\Delta}\!\!\int_t^{t'} \hat{V}_\lambda(t')\hat{V}_\lambda(t'')dt'dt'' \right) \mathcal{U}_\lambda(t)$$

(similar for $\mathcal{U}_{-\lambda}^{-1}(t+\Delta)$).

Comparison to Glauber theory

$$\partial_t \chi_t(\lambda) = (\chi_{t+\Delta}(\lambda) - \chi_t(\lambda))/\Delta = \sum_k \eta_k (e^{i\lambda} - 1) \left\langle \mathcal{U}_{-\lambda}^{-1}(t) a_k^\dagger a_k \mathcal{U}_\lambda(t) \right\rangle$$

with $\eta_k = \sum_j \tau_j |u_{j,k}|^2$ the detector efficiency parameters.

Glauber formula is the solution of this Eqn:

$$\chi_t(\lambda) = \prod_k \chi_t^{(k)}(\lambda), \quad \chi_t^{(k)}(\lambda) = \left\langle \, : \exp\left(\eta_k t (e^{i\lambda} - 1) a_k^\dagger a_k\right) : \right\rangle_k ,$$

with $:\ldots:$ the normal ordering symbol and $\langle \ldots \rangle_k = \mathrm{Tr}(\ldots \rho_k)$ (different photon modes are independent).

Recall: normal ordering $\equiv$ no re-emission

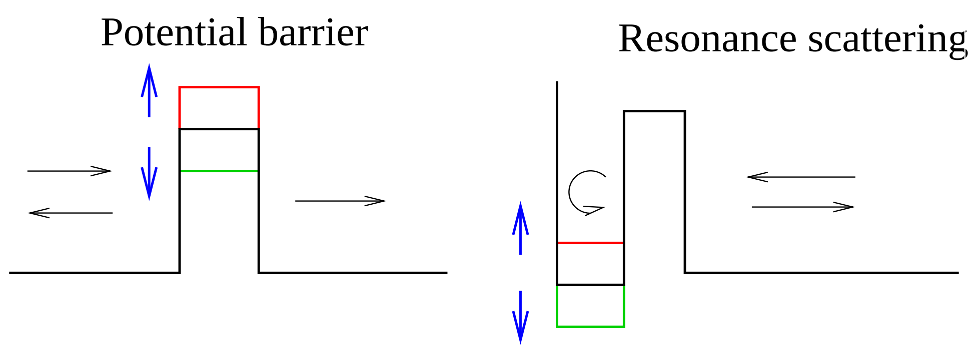
Counting probability of m photons in one mode:

$$p_m^{(k)} = \frac{(\eta_k t)^m}{m!} \langle \, : (a_k^\dagger a_k)^m e^{-\eta_k t a_k^\dagger a_k} : \rangle_k \, .$$

Larmor clock for QM collision

Nuclear reaction, tunneling, resonance scattering, etc. — How long does it take?

Add a fictitious spin $1/2$ to particle: $U(x) \to U_{eff} = U(x) + \frac{1}{2}\omega(x)\sigma_z$ with fictitious field $\omega(x)$ nonzero in the spatial region of interest.



Spin precesses about the Z axis during collision: precession angle measures time. Analysis similar to passive detector (one particle!) yields

$$\chi(\omega) = \mathrm{Tr}(S_{-\omega}^{-1} S_\omega \rho) = \int e^{-i\omega\tau} P(\tau) d\tau$$

with $P(\tau)$ interpreted as probability to spend time τ in the region of interest

Resonance scattering: $S(\epsilon) = \frac{\epsilon - \epsilon_0 + i\gamma/2}{\epsilon - \epsilon_0 - i\gamma/2}$ gives $\chi(\omega) = \frac{\omega - \Delta - i\gamma}{\omega - \Delta + i\gamma} \times \frac{\omega + \Delta - i\gamma}{\omega + \Delta + i\gamma}$ with detuning $\Delta = 2(\epsilon - \epsilon_0)$. Obtain positive or negative probabilities!

Counting statistics of a driven many-body system

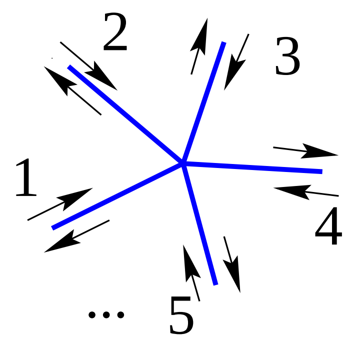
Time-dependent external field and scattering (noninteracting fermions)

We obtain the generating function in the form of a functional determinant in the single-particle Hilbert space:

$$\chi(\lambda) = \det\left(\hat{\mathbf{1}} + n(t,t')\left(\hat{T}_\lambda(t) - \hat{\mathbf{1}}\right)\right)$$

$$\hat{T}_\lambda(t) = S^\dagger_{-\lambda}(t) S_\lambda(t), \qquad S_\lambda(t)_{ij} = e^{i\frac{\lambda_i}{4}} S(t)_{ij} e^{-i\frac{\lambda_j}{4}}$$

with reservoirs density matrix $n(t,t')_{T=0} = \frac{i}{2\pi(t-t'+i\delta)} = \sum_{\epsilon<0} e^{-i\epsilon(t-t')}$, a time-dependent S-matrix $S(t)$ and separate λ_i for each channel



Time-dependent field (voltage $V(t)$, etc.) included in $S_\lambda(t)$. No time delay: $S(t,t') \simeq S(t)\delta(t-t')$ (instant scattering apprx. — nonessential)

Simple and not-so-simple facts from matrix alebra

Useful relations between 2-nd quantized and single-particle operators:

$$A \longrightarrow \Gamma(A) = \sum_{i,j=1}^{N} A_{ij} a_i^+ a_j$$

(mapping of matrices $N \times N \longrightarrow 2^N \times 2^N$).

$$\mathrm{Tr}\, e^{\Gamma(A)} = \det\left(1 + e^{A}\right)$$

— fermion partition function $Z = \mathrm{Tr}\, e^{-\beta\mathcal{H}}$ with $-\beta\mathcal{H} = \Gamma(A)$)

Note: For $A = 0$ obtain $2^N = 2^N$

$$\mathrm{Tr}\left(e^{\Gamma(A)} e^{\Gamma(B)}\right) = \det\left(1 + e^{A} e^{B}\right)$$

$$\mathrm{Tr}\left(e^{\Gamma(A)} e^{\Gamma(B)} e^{\Gamma(C)}\right) = \det\left(1 + e^{A} e^{B} e^{C}\right)$$

$$\ldots$$

Proven using Baker-Hausdorff series for $\ln(e^X e^Y)$ (commutator algebra for X, Y the same as for $\Gamma(X)$, $\Gamma(Y)$)

Deriving the determinant formula I

Write c.s. generating function as

$$\chi(\lambda) = \mathrm{Tr}\,\left(\rho_{el} e^{i\mathcal{H}_{-\lambda}t} e^{-i\mathcal{H}_{\lambda}t}\right)$$

with $\mathcal{H}_{\pm\lambda} = \Gamma(h_{\pm\lambda})$, $\rho = \frac{1}{Z}e^{-\beta\mathcal{H}_0}$. Obtain

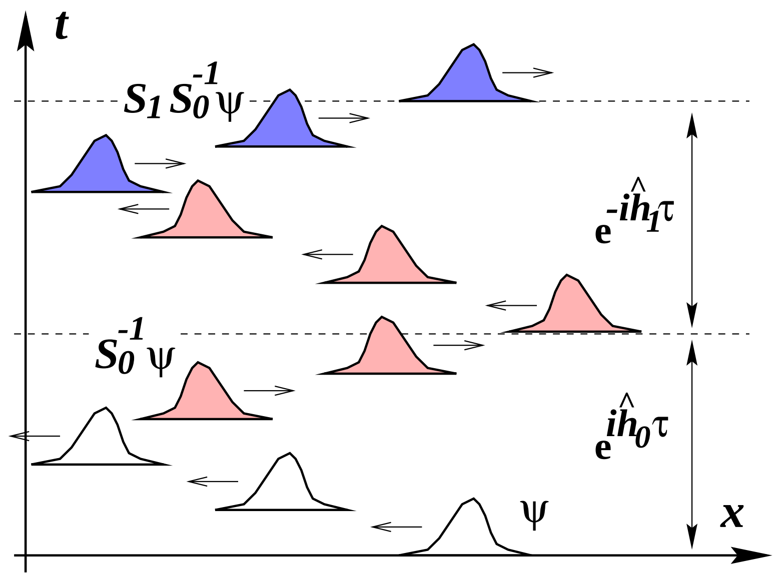
$$\chi = Z^{-1}\det\,\left(1 + e^{-\beta h_0} e^{ih_{-\lambda}t} e^{-ih_{\lambda}t}\right) = \det\,\left[\left(1 + e^{-\beta h_0}\right)^{-1}\left(1 + e^{-\beta h_0} e^{ih_{-\lambda}t} e^{-ih_{\lambda}t}\right)\right]$$

Finally, with $\hat{n} = \left(1 + e^{\beta h_0}\right)^{-1}$, have

$$\chi(\lambda) = \det\,\left(1 - \hat{n} + \hat{n} e^{ih_{-\lambda}t} e^{-ih_{\lambda}t}\right)$$

Deriving the determinant formula II

Relate forward-and-backward evolution in time with scattering operator:



Thus $\langle t'|e^{ih_{-\lambda}t}e^{-ih_\lambda t}|t\rangle = S^{-1}_{-\lambda}(t)S_\lambda(t)\delta(t-t')$
with $|t\rangle$ a wavepacket arriving at scatterer at time t.

$$\chi(\lambda) = \det\left(1 - \hat{n} + \hat{n}S^{-1}_{-\lambda}S_\lambda\right)$$

A single-particle quantity — Fermi-statistics accounted for by det!

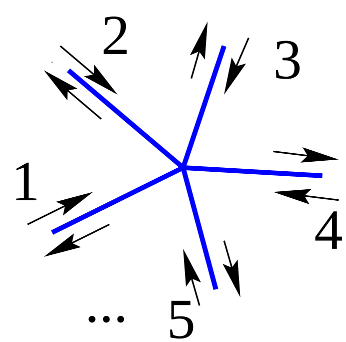
(Generalize to bosons: $\det(1+\ldots) \longrightarrow \det(1-\ldots)^{-1}$)

A more intuitive approach: DC transport

For elastic scattering different energies contribute independently:

$$\chi(\lambda) = \prod_{\epsilon} \chi_\epsilon(\lambda)\,, \quad \text{i.e.} \quad \chi(\lambda) = \exp\left(t\int \ln \chi_\epsilon(\lambda)\frac{d\epsilon}{2\pi\hbar}\right),$$

(quasiclassical $d\mathcal{V} = d\epsilon dt/2\pi\hbar$).



Sum over all multiparticle processes:

$$\chi_\epsilon(\lambda) = \sum_{i_1,\ldots,i_k,j_1,\ldots,j_k} e^{\frac{i}{2}(\lambda_{i_1}+\ldots+\lambda_{i_k}-\lambda_{j_1}-\ldots-\lambda_{j_k})} P_{i_1,\ldots,i_k\,|\,j_1,\ldots,j_k}\,,$$

with the rate of k particles transit $i_1, ..., i_k \longrightarrow j_1, ..., j_k$ is given by

$$P_{i_1,\ldots,i_k \,|\, j_1,\ldots,j_k} = \left| S^{j_1,\ldots,j_k}_{i_1,\ldots,i_k} \right|^2 \prod_{i \neq i_\alpha} (1 - n_i(\epsilon)) \prod_{i = i_\alpha} n_i(\epsilon) \,.$$

with $S^{j_1,\ldots,j_k}_{i_1,\ldots,i_k}$ antisymmetrized product of k single particle amplitudes.

This is equal to our determinant (Proven by reverse engineering)

Positive probabilities!

Point contact (beam splitter): 2×2 matrices

$$\chi_\epsilon(\lambda) = (1-n_1)(1-n_2) + (|S_{11}|^2 + e^{\frac{i}{2}(\lambda_2-\lambda_1)}|S_{21}|^2)n_1(1-n_2)$$
$$+ (|S_{22}|^2 + e^{\frac{i}{2}(\lambda_1-\lambda_2)}|S_{12}|^2)n_2(1-n_1) + |\det S|^2 n_1 n_2\,, \quad (1)$$

with $n_{1,2}(\epsilon) = f(\epsilon \mp \frac{1}{2}eV)$ and S_{ij} is unitary: $|S_{1i}|^2 + |S_{2i}|^2 = 1$, $|\det S| = 1$.

Simplify: $\chi_\epsilon(\lambda) = 1 + t(e^{i\lambda}-1)n_1(1-n_2) + t(e^{-i\lambda}-1)n_2(1-n_1)$ Here $t = |S_{21}|^2 = |S_{12}|^2$ is the transmission coefficient and $\lambda = \lambda_2 - \lambda_1$.

At $T=0$, since $n_F(\epsilon) = 0$ or 1, for $V>0$ have

$$\chi_\epsilon(\lambda) = \begin{cases} e^{i\lambda}t + 1 - t\,, & |\epsilon| < \frac{1}{2}eV; \\ 1\,, & |\epsilon| > \frac{1}{2}eV \end{cases}$$

Full counting statistics: $\chi_\tau(\lambda) = (e^{i\lambda}p + 1 - p)^{N(\tau)}$ — binomial, with the number of attempts $N(\tau) = (eV/h)\tau$.

Similar at $V<0$, with $e^{i\lambda} \longrightarrow e^{-i\lambda}$ (DC current sign reversal).

Note: the noninteger number of attempts is an artifact of a quasiclassical calculation. More careful analysis gives a narrow distribution P_N of the number of attempts peaked at $\overline{N} = N(\tau)$, and the generating function as a weighted sum $\sum_N P_N \chi_N(\lambda)$. The peak width is a sublinear function of the measurement time τ (in fact, $\overline{\delta N^2}_{T=0} \propto \ln \tau$), the statistics still binomial, to leading order in t.

Strategies for handling the determinant

Two strategies:

1) For periodic $S(t)$, in the frequency representation $\hat{n}$ is diagonal, $n(\omega) = f(\omega)$, while $S(t)$ has matrix elements $S_{\omega',\omega}$ with discrete frequency change $\omega' - \omega = n\Omega$, with Ω the pumping frequency. In this method the energy axis is divided into intervals $n\Omega < \omega < (n+1)\Omega$, and each interval is treated as a separate conduction channel with time-independent S-matrix $S_{\omega',\omega}$.

2) The determinant can also be analyzed directly in the time domain:

$$\partial_\lambda \ln \chi(\lambda) = \mathrm{Tr}\left[\left(\mathbf{1} + \mathbf{n}(\mathbf{T}_\lambda - \mathbf{1}) \right)^{-1} \partial_\lambda T_\lambda \right]$$

with $T_\lambda(t,t') = S_{-\lambda}^{-1}(t) S_\lambda(t) \delta(t-t')$, $n(t,t') = \frac{i}{2\pi}(t - t' + i\delta)^{-1}$.

The problem of inverting the integral operator $\mathbf{R} = \mathbf{1} + \mathbf{n}(\mathbf{T}_\lambda - \mathbf{1})$ is the so-called Riemann-Hilbert problem (matrix generalization of Wiener-Hopf, well-studied, exact and approximate solutions can be constructed)

Phase-sensitive noise

Charge flow induced by voltage pulse $V(t)$ in a point contact

A pulse $V(t)$corresponds to a step $\Delta\theta$ in the forward scattering phase:

$$S(t) = \begin{pmatrix} e^{-i\theta(t)}\sqrt{t} & i\sqrt{r} \\ i\sqrt{r} & e^{i\theta(t)}\sqrt{t} \end{pmatrix} \quad \theta(t) = \frac{e}{\hbar}\int_{-\infty}^{t} V(t')dt'$$

The mean transmitted charge $\overline{q} = te\Delta\theta/2\pi$ — independent of pulse shape
In contrast, the variance $\overline{\delta q^2}$ exhibits complex dependence:

$$\overline{\delta q^2} = 2t(1-t)e^2 \int\int \frac{1-e^{i\theta_{12}}}{(t_1-t_2)^2} dt_1 dt_2 \quad \theta_{12} = \frac{e}{\hbar}\int_{t_1}^{t_2} V(t')dt'$$

Gives $\overline{\delta q^2} \propto (1-\cos\Delta\theta)\ln(t_{max}/t_0) + \mathrm{const}$ — periodic in the pulse area and log-divergent ($t_{max} \sim \hbar/k_{\rm B}T$)

Interpret the log as orthogonality catastrophe: long-lasting change of scatterer at $\Delta\theta \neq 2\pi n$ causes infinite number of soft particle-hole excitations

Shot noise is phase-sensitive — Mach-Zender effect in electron noise

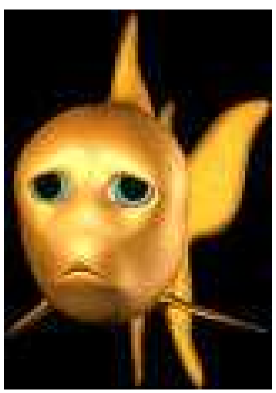
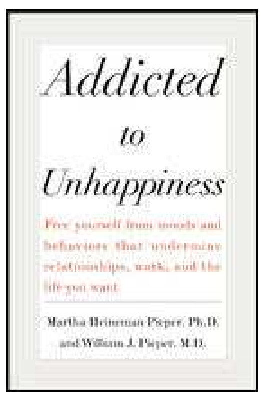
Minimizing unhappiness

Find noise-minimizing pulse shapes: $\iint \frac{1-\exp\left(i\int_{t_1}^{t_2}\frac{e}{\hbar}V(t')dt'\right)}{(t_1-t_2)^2}dt_1dt_2 \;\to\; \min$
— an interesting variational problem, solved by pulses of integer area $2\pi n$:

$$V(t)=\frac{\hbar}{e}\sum_{i=1\ldots n}\frac{2\tau_i}{(t-t_i)^2+\tau_i^2}\qquad(\tau_i>0)$$

Lorentzian pulses (overlapping or nonoverlapping)

Degeneracy: $\overline{\delta q^2}=e^2t(1-t)n$, the same for all t_i, τ_i


Coherent time-dependent many-body states

Variance of transmitted charge $\overline{\delta q^2} = e^2 t(1-t)n$ — independent of pulse parameters t_i, τ_i, same value as for binomial distribution with the number of attempts n

Binomial counting statistics from the functional determinant (exactly solvable Riemann-Hilbert problem):

$$\chi(\lambda) = \left(te^{i\lambda} + 1 - t\right)^n$$

Interpretation: pulses $\simeq$ independent attempts to transmit charge

Coherent current pulses:

Noise reduced as much as the beam splitter partition noise permits

Similarity to coherent states (QM uncertainty minimized)

Analogs in quantum optics?

Measurement of phase sensitive noise

Instead of a train of pulses (which is difficult to realize) used a combination of DC and AC voltage, $V = V_{DC} + V_{AC}\cos\Omega t$ — oscillations in noise power (Bessel functions), while DC current is ohmic:

$$\frac{\partial S_0}{\partial V_{DC}} = \frac{2}{\pi}e^3\left(\sum_m t_m(1-t_m)\right)\sum_n J_n^2(V_{AC}/\hbar\Omega)\theta(eV_{DC}-n\hbar\Omega)$$

Observed in mesoscopic wires (Schoelkopf '98), point contacts (Glattli '02)

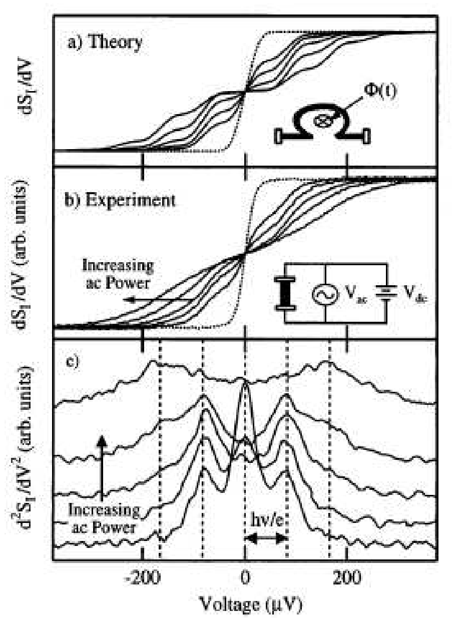


FIG. 1. Theoretical and experimental variation of differential shot noise versus voltage, with 20 GHz ac excitation.

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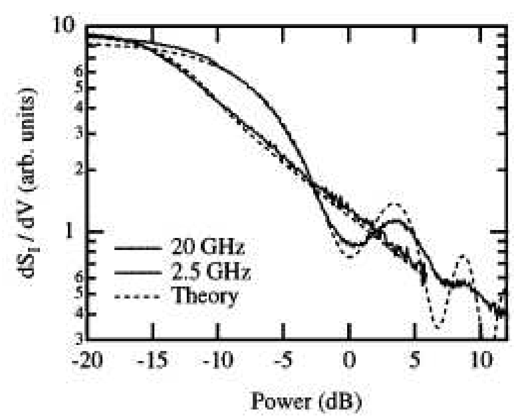


FIG. 2. Differential noise (dS_I/dV) at a voltage of ~40 μV as a function of the applied microwave power. Solid curves show the measured noise for microwave frequencies of 20 and 2.5 GHz, while the dashed lines are calculated according to Eq. (1). For high applied frequencies, the data show a damped oscillatory behavior consistent with the expected Bessel functions.

Case studies

(i) Voltage pulses of different signs (D Ivanov, H-W Lee, and LL, PRB **56**, 6839 (1997)):

$$V(t) = \frac{h}{e}\left(\frac{2\tau_1}{(t-t_1)^2+\tau_1^2} - \frac{2\tau_2}{(t-t_2)^2+\tau_2^2}\right)$$

give rise to the counting distribution

$$\chi(\lambda) = 1 - 2F + F(e^{i\lambda} + e^{-i\lambda})\,, \quad F = t(1-t)\left|\frac{z_1^* - z_2}{z_1 - z_2}\right|^2$$

with $z_{1,2} = t_{1,2} + i\tau_{1,2}$. The quantity $A = |...|^2$ is a measure of pulses' overlap in time: $A = 0$ (full overlap), $A = 1$ (no overlap).
Note: $\chi(\lambda)$ factorizes for nonoverlapping pulses

(ii) Two-channel model of electron pump ((D Ivanov and LL, JETP Lett **58**, 461 (1993)):

$$S(\tau) \equiv \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} = \begin{pmatrix} B + be^{-i\Omega\tau} & \bar{A} + \bar{a}e^{i\Omega\tau} \\ A + ae^{-i\Omega\tau} & -\bar{B} - \bar{b}e^{i\Omega\tau} \end{pmatrix},$$

which is unitary provided $|A|^2 + |a|^2 + |B|^2 + |b|^2 = 1$, $A\bar{a} + B\bar{b} = 0$ (time-independent parameters).

For $T = 0$ and $\mu_L = \mu_R$ the charge distribution for m pumping cycles is described by

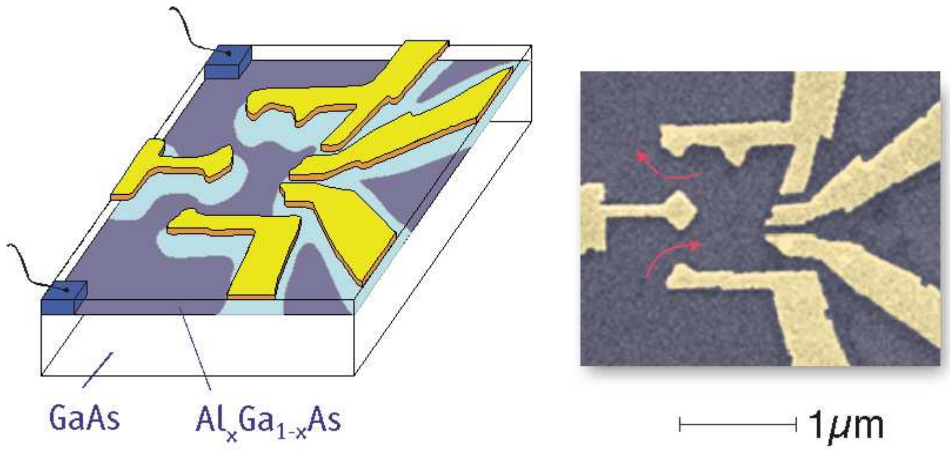
$$\chi(\lambda) = \left(1 + p_1(e^{i\lambda} - 1) + p_2(e^{-i\lambda} - 1)\right)^m$$

with $p_1 = |a|^4/(|a|^2 + |b|^2)$ and $p_2 = |b|^4/(|a|^2 + |b|^2)$: at each pumping cycle an electron is pumped in one direction with probability p_1, or in the opposite direction with probability p_2, or no charge is pumped with probability $1 - p_1 - p_2$.

Also can be solved at $\mu_L \neq \mu_R$: more complicated statistics.

Counting statistics of a charge pump

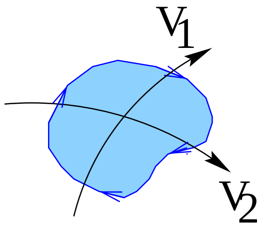
DC current from an AC-driven open quantum dot. (Exp: Marcus group '99)



The time-averaged pumped current is a purely geometric property of the path in the S-matrix parameter space, insensitive to path parameterization (Brouwer '98, Buttiker'94).

Noise dependence on the pumping cycle? Bounds on the ratio noise/current?

Here, consider generic but small path $V_1(t)$, $V_2(t)$:



Counting statistics for a generic pumping cycle

Focus on the weak pumping regime, a small loop in the matrix parameter space: $S(t) = e^{A(t)} S^{(0)}$ with perturbation $A(t)$ (antihermitian, $\mathrm{tr} A^\dagger A \ll 1$) and $S^{(0)}$ is the S-matrix in the absence of pumping.

Expand $\ln \chi(\lambda) = \det \left(\hat{\mathbf{1}} + n(t,t') \left(S^\dagger_{-\lambda}(t) S_\lambda(t) - \hat{\mathbf{1}} \right) \right)$ in $A(t)$:

$$\ln \chi = \frac{1}{2} \mathrm{tr} \left(\hat{n} \left(A^2_{-\lambda} + A^2_\lambda - 2 A_{-\lambda} A_\lambda \right) \right) - \frac{1}{2} \mathrm{tr} (\hat{n} B_\lambda)^2$$

with $A_\lambda(t) = e^{i \frac{\lambda}{4} \sigma_3} A(t) e^{-i \frac{\lambda}{4} \sigma_3}, \quad B_\lambda(t) = A_\lambda(t) - A_{-\lambda}(t)$
Using $\hat{n}^2_{T=0} = \hat{n}_{T=0}$, separate a commutator:

$$\ln \chi(\lambda) = \frac{1}{2} \mathrm{tr} \left(\hat{n} [A_\lambda, A_{-\lambda}] \right) + \frac{1}{2} \left(\mathrm{tr} \left(\hat{n}^2 B^2_\lambda \right) - \mathrm{tr} (\hat{n} B_\lambda)^2 \right)$$

The commutator is regularized as the Schwinger anomaly (splitting points, $t', t'' = t \pm \epsilon/2$), which gives

$$\frac{1}{2} \oint n(t', t'') \mathrm{tr} \left(A_{-\lambda}(t'') A_\lambda(t') - A_\lambda(t'') A_{-\lambda}(t') \right) dt$$

Average over small ϵ (insert additional integrals over t', t'', or just replace $A_\lambda(t) \to \frac{1}{2}(A_\lambda(t) + A_\lambda(t'))$, etc.) In the limit $\epsilon \to 0$, obtain

$$(\ln \chi)_1 = \frac{i}{8\pi} \oint \mathrm{tr}\left(A_{-\lambda}\partial_t A_\lambda - A_\lambda \partial_t A_{-\lambda}\right) dt$$

The second term of $\ln \chi$ is rewritten as

$$(\ln \chi)_2 = \frac{1}{4(2\pi)^2} \oint\oint \frac{\mathrm{tr}\left(B_\lambda(t) - B_\lambda(t')\right)^2}{(t-t')^2} dt dt'$$

Now, combine $(\ln \chi)_1$ with $(\ln \chi)_2$, and simplify

Bi-directional Poissonian statistics

Convenient decomposition $A = a_0 + z + z^\dagger$, such that $[\sigma_3, a_0] = 0$, $[\sigma_3, z] = -2z$, $\left[\sigma_3, z^\dagger\right] = 2z^\dagger$, gives

$$A_\lambda \equiv e^{-i\frac{\lambda}{4}\sigma_3} A e^{i\frac{\lambda}{4}\sigma_3} = a_0 + e^{i\frac{\lambda}{2}} z^\dagger + e^{-i\frac{\lambda}{2}} z$$

$$B_\lambda = \left(e^{i\frac{\lambda}{2}} - e^{-i\frac{\lambda}{2}}\right) W, \quad W \equiv z^\dagger - z$$

In this representation,

$$\ln \chi = \frac{\sin\lambda}{8\pi} \oint \mathrm{tr}\left(\left[\sigma_3, W\right] \partial_t W\right) dt + \frac{(1-\cos\lambda)}{2(2\pi)^2} \oint\oint \frac{\mathrm{tr}\left(W(t) - W(t')\right)^2}{(t-t')^2} dt dt'$$

The first term is identical to the Brouwer result (invariant under reparameterization and has a purely geometric character), the second term describes noise.

Note the λ-dependence: $u(e^{i\lambda} - 1) + v(e^{-i\lambda} - 1) \to$ 2P statistics

$$\chi(\lambda) = \exp\left(u(e^{i\lambda} - 1) + v(e^{-i\lambda} - 1)\right)$$

Prove that

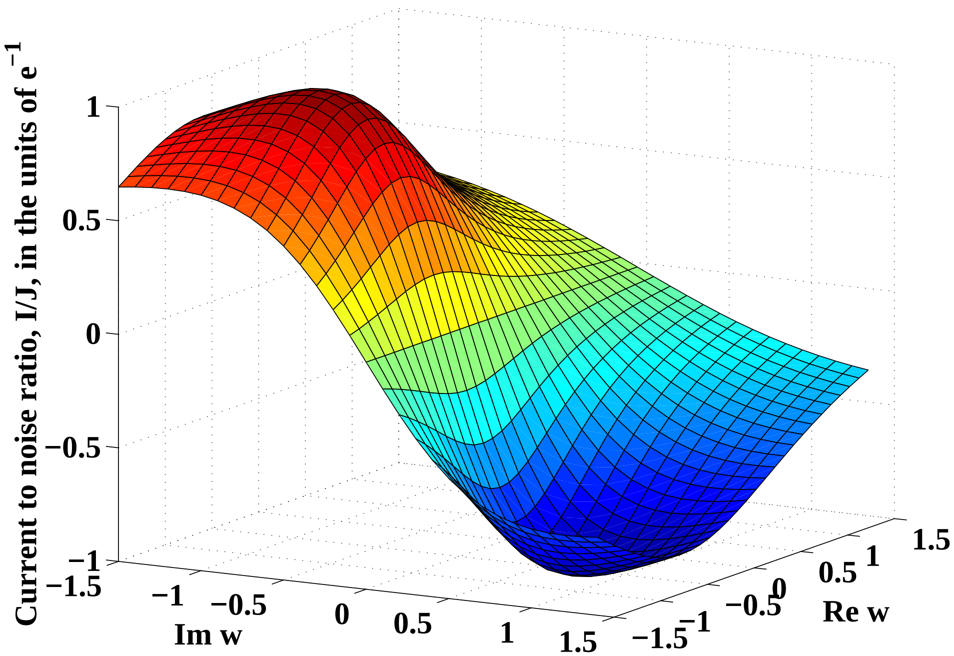
1) $u, v \geq 0$ for generic pumping cycle $W(t)$;

2) $u = 0$ or $v = 0$ for special paths $W(t)$ holomorphic in the upper/lower half-plane of complex time t;

Then the ratio $\mathrm{current/noise} = (u - v)(u + v)$ is maximal or minimal (equal $\pm q_0^{-1}$ per cycle), when $u = 0$ or $v = 0$. The counting statistics in this case is pure poissonian.

Example of a single channel system (two leads) driven by $V_1(t) = a_1 \cos(\Omega t + \theta)$, $V_2(t) = a_2 \cos(\Omega t)$. The pumping cycle:

$$W(t) = \begin{pmatrix} 0 & z(t) \\ z^*(t) & 0 \end{pmatrix}, \quad z(t) = z_1 V_1(t) + z_2 V_2(t), \quad (z_{1,2} \text{ system parameters})$$



Current to noise ratio, $I/J = q_0^{-1}(u-v)/(u+v)$, as a function of the driving signal parameters for a single channel pump. The two harmonic signals driving the system are characterized by relative amplitude and phase, $w = (V_1/V_2)e^{i\theta}$. Maximum and minimum, as a function of w, are $I/J = \pm q_0^{-1}$.

Pump noise summary:

1) Pumping noise is super-poissonian; counting statistics is double-poissonian;

2) The current/noise ratio can be maximized by varying the pumping cycle (relative amplitude or phase of the driving signals);

3) Extremal cycles correspond to poissonian counting statistics with a universal ratio

$$\text{current}/\text{noise} = \pm q_0^{-1}$$

(another generalization of the Schottky formula).