Not Yet

Introduction to Mesoscopics

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ORIGINS

E.P. Wigner, Conference on Neutron Physics by Time of Flight, November 1956

P.W. Anderson, "*Absence of Diffusion in Certain Random Lattices*"; Phys.Rev., 1958, v.109, p.1492

L.D. Landau, "*Fermi-Liquid Theory*" Zh. Exp. Teor. Fiz.,1956, v.30, p.1058

J. Bardeen, L.N. Cooper & J. Schriffer, "*Theory of Superconductivity*"; Phys.Rev., 1957, v.108, p.1175.

Part 1 Without interactions

Random Matrices, Anderson Localization, and Quantum Chaos

RANDOM MATRIX THEORY

$N \times N$ *ensemble of Hermitian matrices with random matrix element* $N \to \infty$

E_α - spectrum (set of eigenvalues)

$\delta_1 \equiv \left\langle E_{\alpha+1} - E_\alpha \right\rangle$ - mean level spacing

$\left\langle \ldots\ldots \right\rangle$ - ensemble averaging

$s \equiv \dfrac{E_{\alpha+1} - E_\alpha}{\delta_1}$ - spacing between nearest neighbors

$P(s)$ - distribution function of nearest neighbors spacing between

Spectral Rigidity

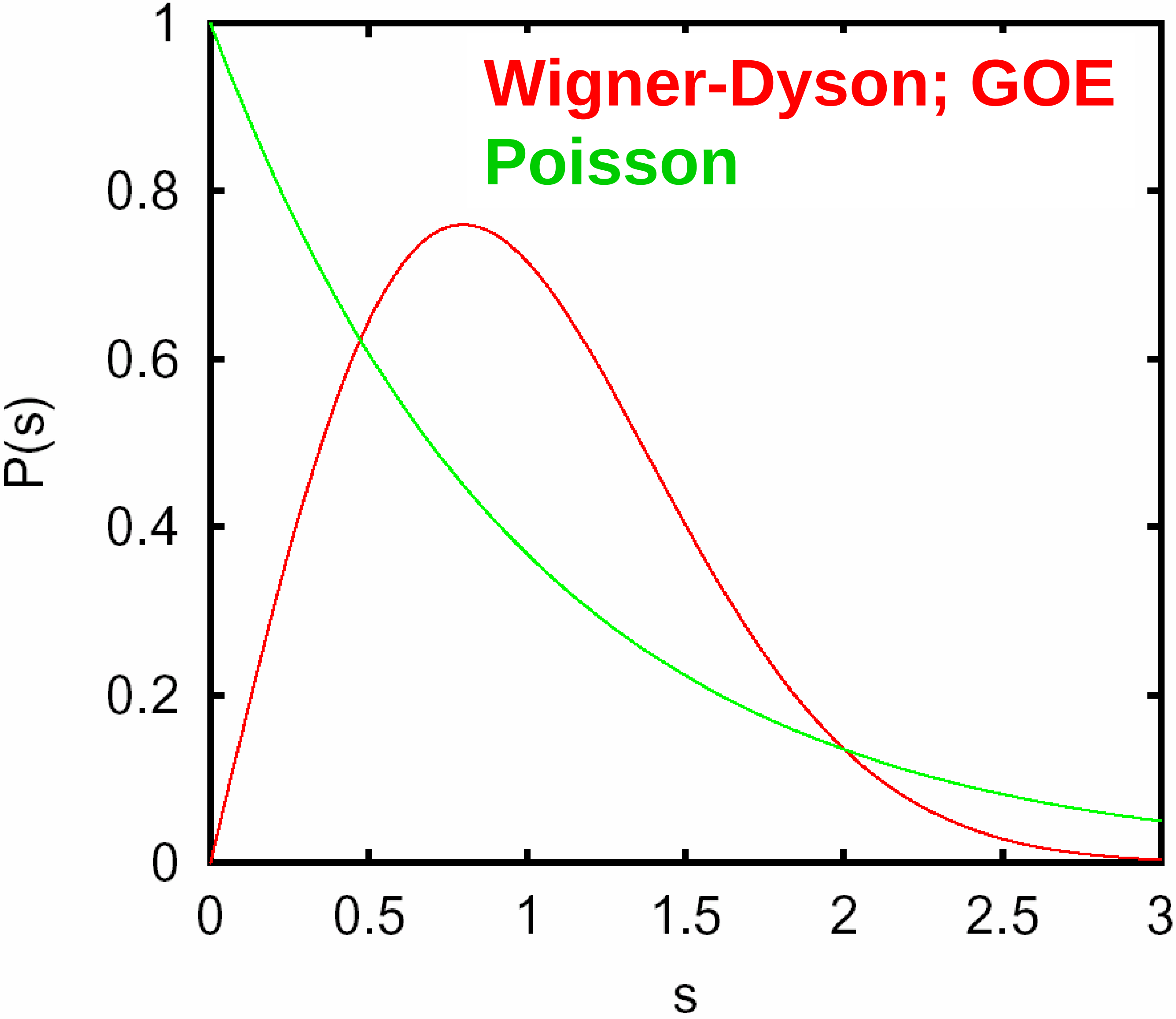
Level repulsion

$$P(s=0)=0$$

$$P(s<<1) \propto s^\beta \quad \beta=1,2,4$$

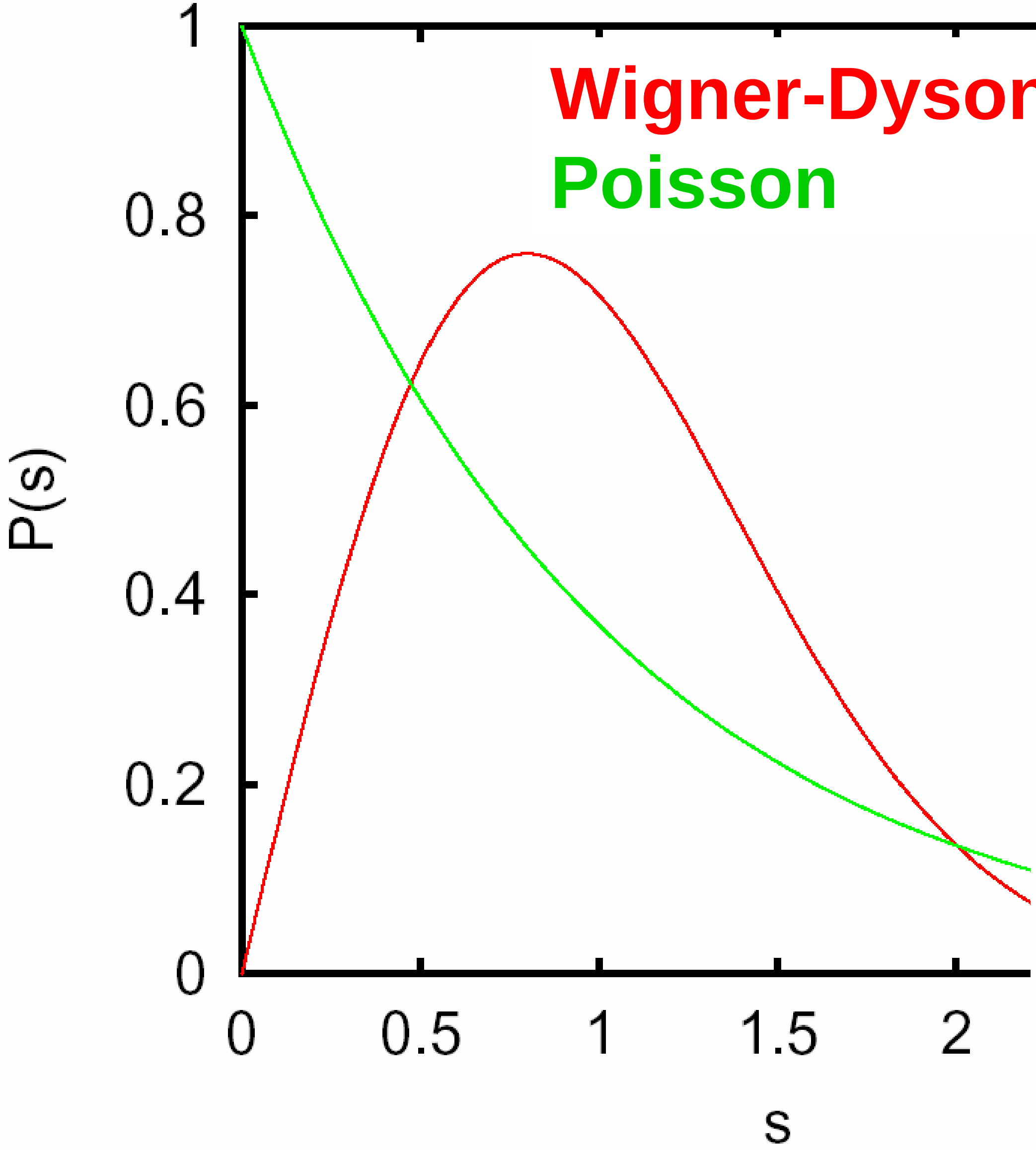
Dyson Ensembles and Hamiltonian systems

Matrix elements	Ensemble	β
real	orthogonal	1
complex	unitary	2
2×2 matrices	simplectic	4



Gaussian
Orthogonal
Ensemble

Poisson – completely uncorrelated levels

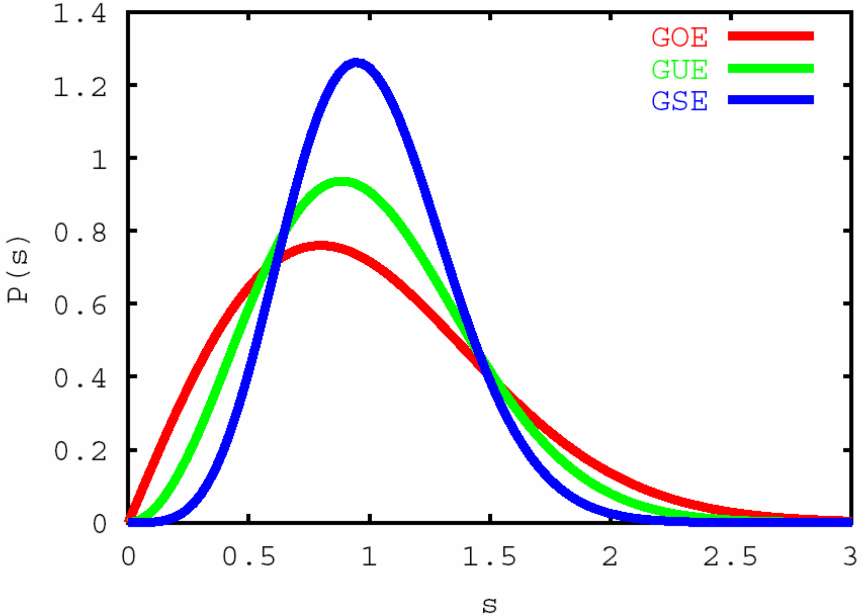


Gaussian Orthogonal Ensemble

Orthogonal
$\beta=1$

Unitary
$\beta=2$

Simplectic
$\beta=4$



Poisson – completely uncorrelated levels

RANDOM MATRICES

$N \times N$ matrices with random matrix elements. $N \to \infty$

Dyson Ensembles

Matrix elements	Ensemble	β	realization
real	orthogonal	1	T-inv potential
complex	unitary	2	broken T-invariance (e.g., by magnetic field)
2×2 matrices	simplectic	4	T-inv, but with spin-orbital coupling

Reason for $P(s) \to 0$ when $s \to 0$:

$$\hat{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{12}^{*} & H_{22} \end{pmatrix}$$

$$E_2 - E_1 = \sqrt{\left(H_{22} - H_{11}\right)^2 + \left|H_{12}\right|^2}$$

small small small

1. The assumption is that the matrix elements are statistically independent. Therefore probability of two levels to be degenerate vanishes.

2. If H_{12} is real (orthogonal ensemble), then for s to be small two statistically independent variables ($(H_{22}$- $H_{11})$ and H_{12}) should be small and thus $P(s) \propto s \qquad \beta = 1$

3. Complex H_{12} (unitary ensemble) $\Longrightarrow$ both $Re(H_{12})$ and $Im(H_{12})$ are statistically independent $\Longrightarrow$ three independent random variables should be small $\Longrightarrow$ $P(s) \propto s^2 \qquad \beta = 2$

Dyson Ensembles and Hamiltonian systems

Matrix elements	Ensemble	β	realization
real	orthogonal	1	T-inv potential
complex	unitary	2	broken T-invariance (e.g., by magnetic field)
2×2 matrices	simplectic	4	T-inv, but with spin-orbital coupling

ATOMS

Main goal is to classify the eigenstates in terms of the quantum numbers

NUCLEI

For the nuclear excitations this program does not work

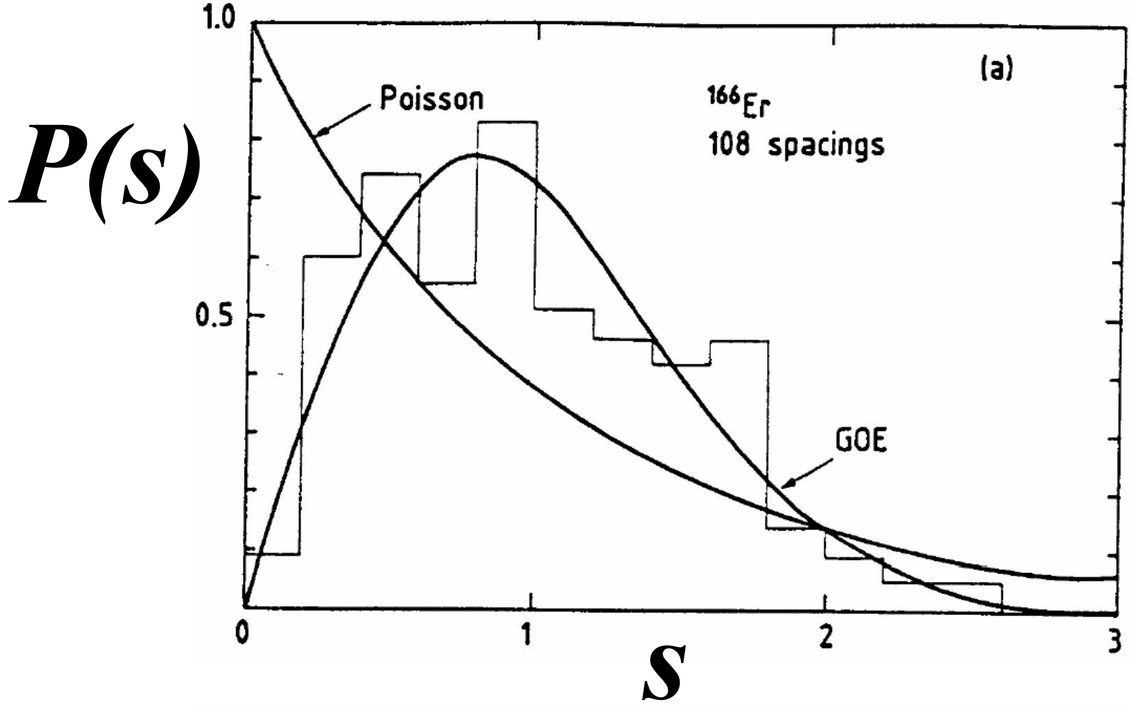
E.P. Wigner:

Study spectral **statistics** of a **particular** quantum system – a given nucleus

Random Matrices	Atomic Nuclei
• *Ensemble*	• ***Particular quantum system***
• *Ensemble averaging*	• ***Spectral averaging (over α)***

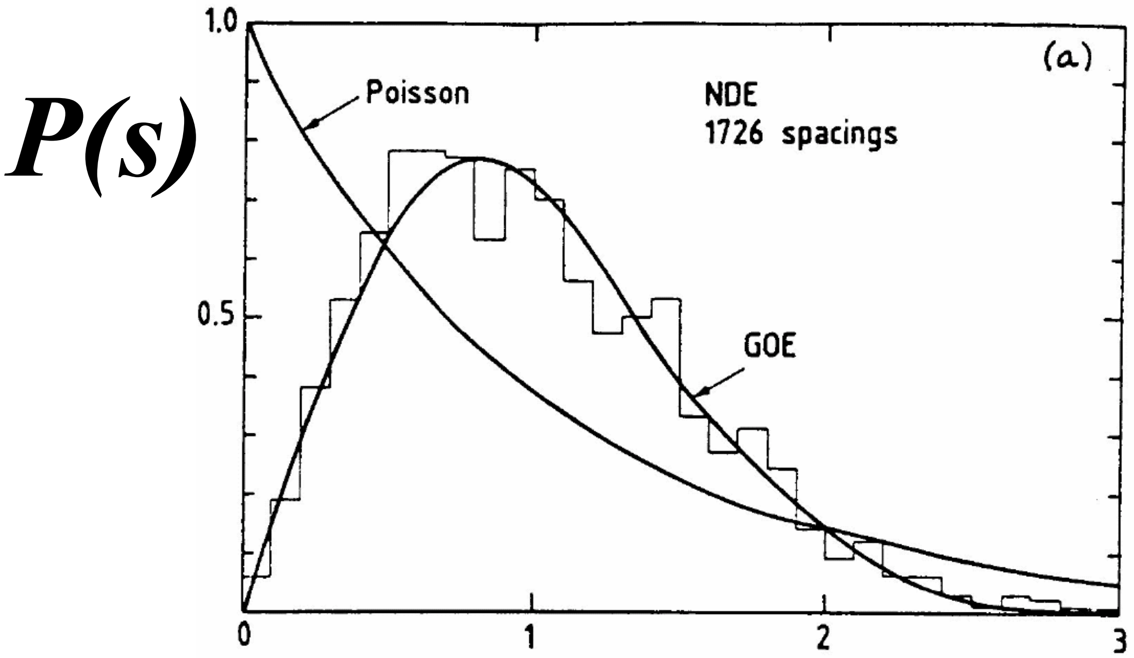
Nevertheless

Statistics of the nuclear spectra are almost exactly the same as the Random Matrix Statistics



Particular nucleus

^{166}Er



Spectra of several nuclei combined (after rescaling by the mean level spacing)

Q: *Why the random matrix theory (RMT) works so well for nuclear spectra* ?

Original answer:

These are systems with a large number of degrees of freedom, and therefore the "complexity" is high

Later it became clear that

there exist very "simple" systems with as many as 2 degrees of freedom (d=2), *which demonstrate RMT - like spectral statistics*

Classical ($\hbar$ =0) Dynamical Systems with *d* degrees of freedom

Integrable Systems

The variables can be separated and the problem reduces to *d* one-dimensional problems

⇒

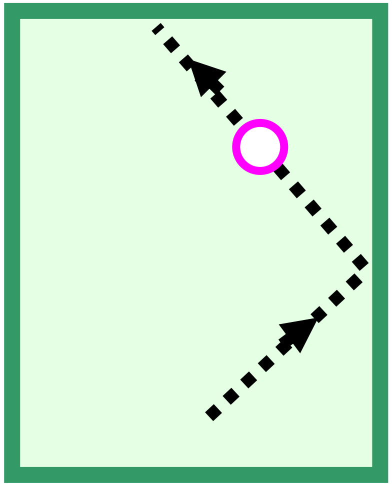
d integrals of motion

Examples

1. A ball inside rectangular billiard; $d=2$

- Vertical motion can be separated from the horizontal one

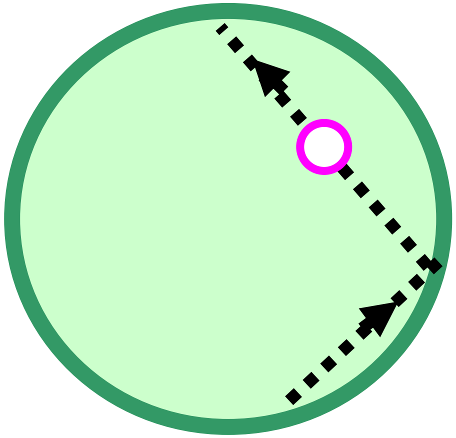
- Vertical and horizontal components of the momentum, are both integrals of motion



2. Circular billiard; $d=2$

- Radial motion can be separated from the angular one

- Angular momentum and energy are the integrals of motion



Classical Dynamical Systems with d degrees of freedom

Integrable Systems

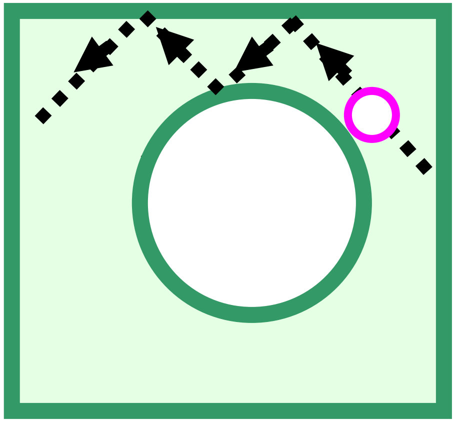
The variables can be separated ⇨ d one-dimensional problems ⇨ d integrals of motion

Rectangular and circular billiard, Kepler problem, . . . ,
1d Hubbard model and other exactly solvable models, . .

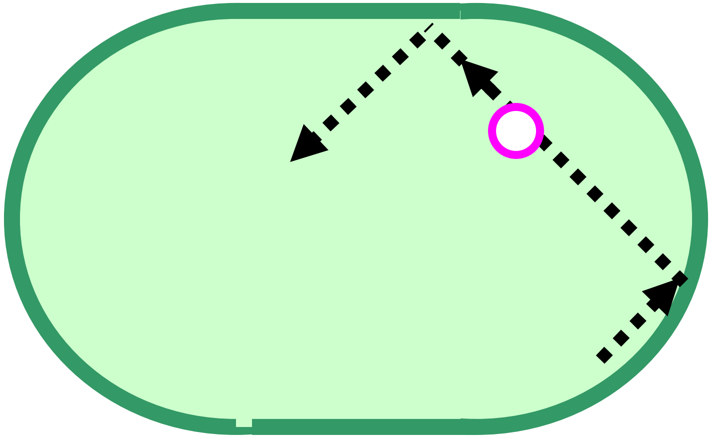
Chaotic Systems

The variables can not be separated ⇨ there is only one integral of motion - energy

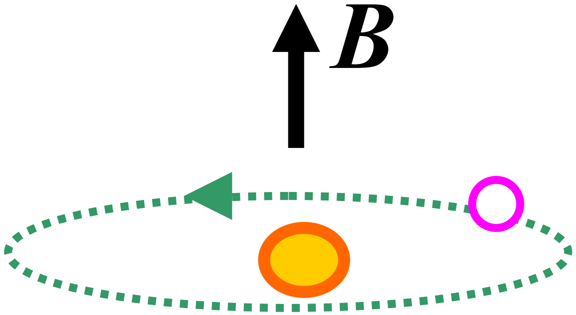
Examples



Sinai billiard



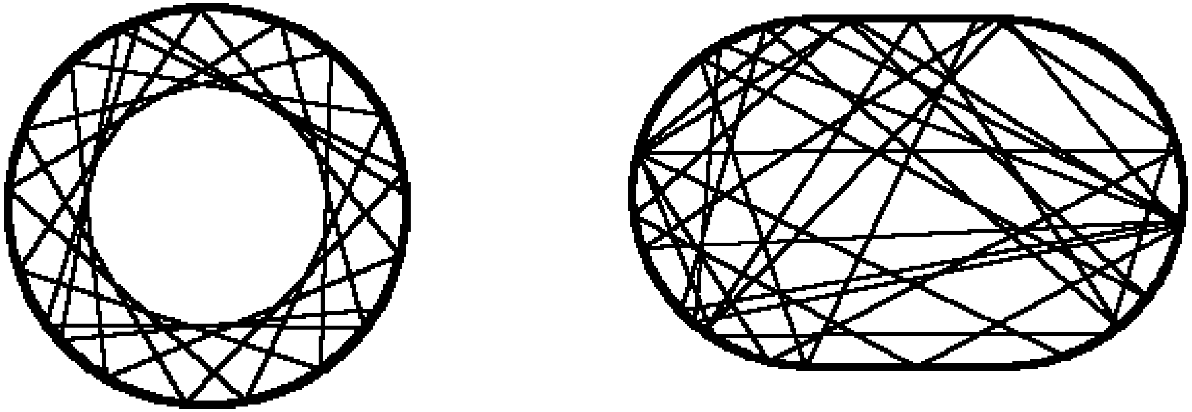
Stadium



Kepler problem in magnetic field

Classical Chaos
$\hbar = 0$

- *Nonlinearities*
- *Lyapunov exponents*
- *Exponential dependence on the original conditions*
- *Ergodicity*



Quantum description of any System with a finite number of the degrees of freedom is a linear problem – Shrodinger equation

Q: What does it mean Quantum Chaos ?

$\hbar \neq 0$ Bohigas – Giannoni – Schmit conjecture

VOLUME 52 2 JANUARY 1984 NUMBER 1

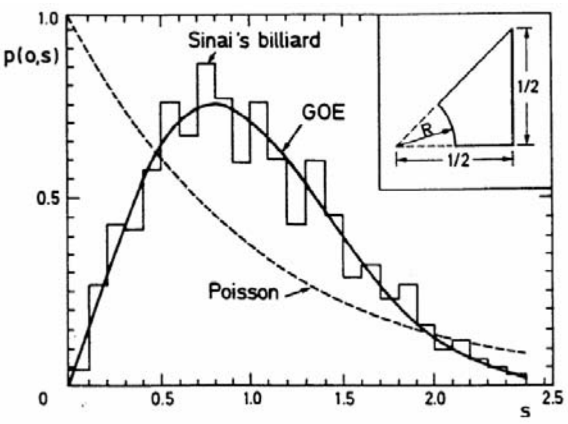
Characterization of Chaotic Quantum Spectra and Universality of Level Fluctuation Laws

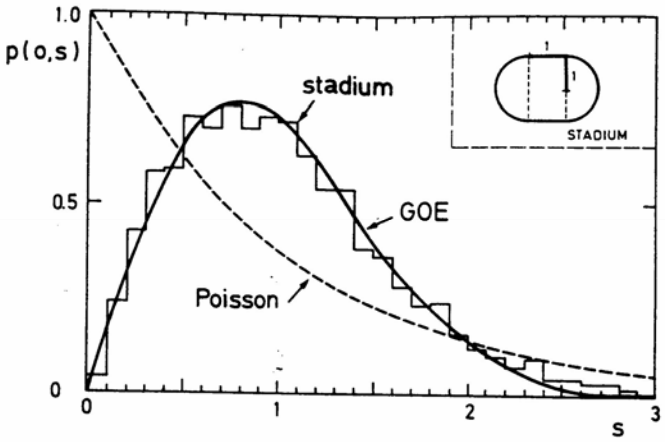
O. Bohigas, M. J. Giannoni, and C. Schmit
Division de Physique Théorique, Institut de Physique Nucléaire, F-91406 Orsay Cedex, France

(Received 2 August 1983)

It is found that the level fluctuations of the quantum Sinai's billiard are consistent with the predictions of the Gaussian orthogonal ensemble of random matrices. This reinforces the belief that level fluctuation laws are universal.

In summary, the question at issue is to prove or disprove the following conjecture: Spectra of time-reversal–invariant systems whose classical analogs are *K* systems show the same fluctuation properties as predicted by GOE





Chaotic classical analog

Wigner- Dyson spectral statistics

No quantum numbers except energy

Q: What does it mean Quantum Chaos ?

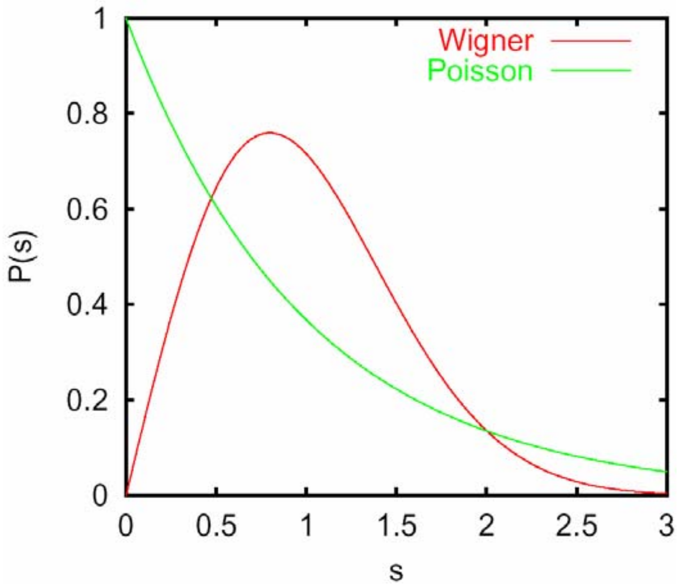
Two possible definitions

Chaotic classical analog	Wigner - Dyson-like spectrum

Classical | Quantum

Integrable ⟷? **Poisson**

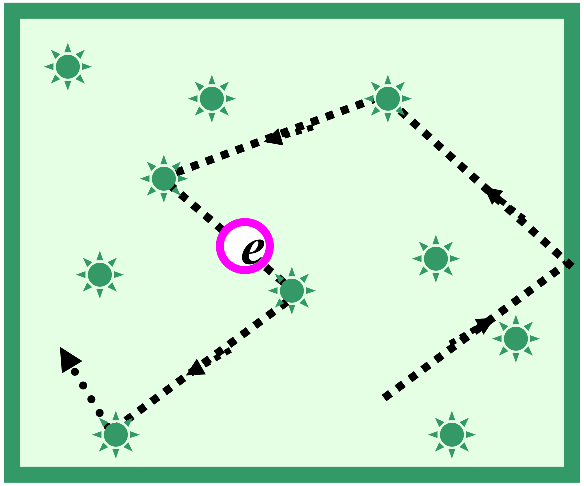
Chaotic ⟷? **Wigner-Dyson**



Poisson to Wigner-Dyson crossover

Important example: quantum particle subject to a random potential – disordered conductor

✹ *Scattering centers, e.g., impurities*



- As well as in the case of Random Matrices (RM) there is a luxury of ensemble averaging.
- The problem is much richer than RM theory
- There is still a lot of universality.

Anderson localization (1958)

At strong enough disorder all eigenstates are localized in space

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P.W. Anderson, “*Absence of Diffusion in Certain Random Lattices*”; Phys.Rev., 1958, v.109, p.1492

L.D. Landau, ”*Fermi-Liquid Theory*” Zh. Exp. Teor. Fiz.,1956, v.30, p.1058

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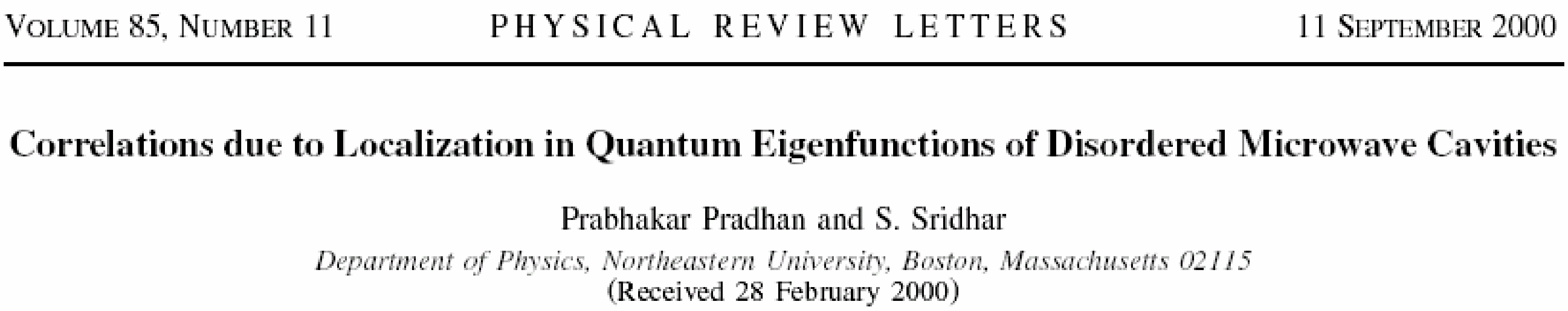

VOLUME 85, NUMBER 11 PHYSICAL REVIEW LETTERS 11 SEPTEMBER 2000

Correlations due to Localization in Quantum Eigenfunctions of Disordered Microwave Cavities

Prabhakar Pradhan and S. Sridhar
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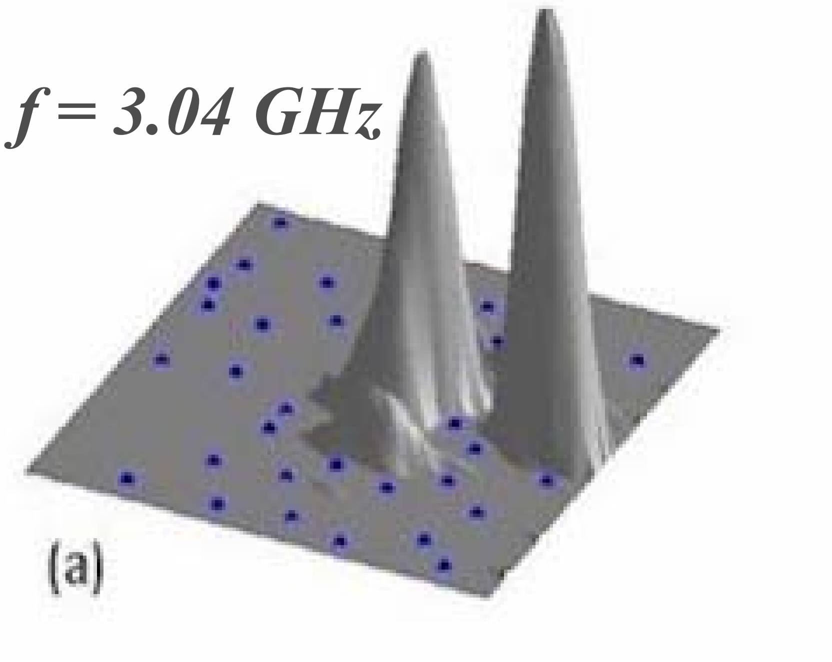
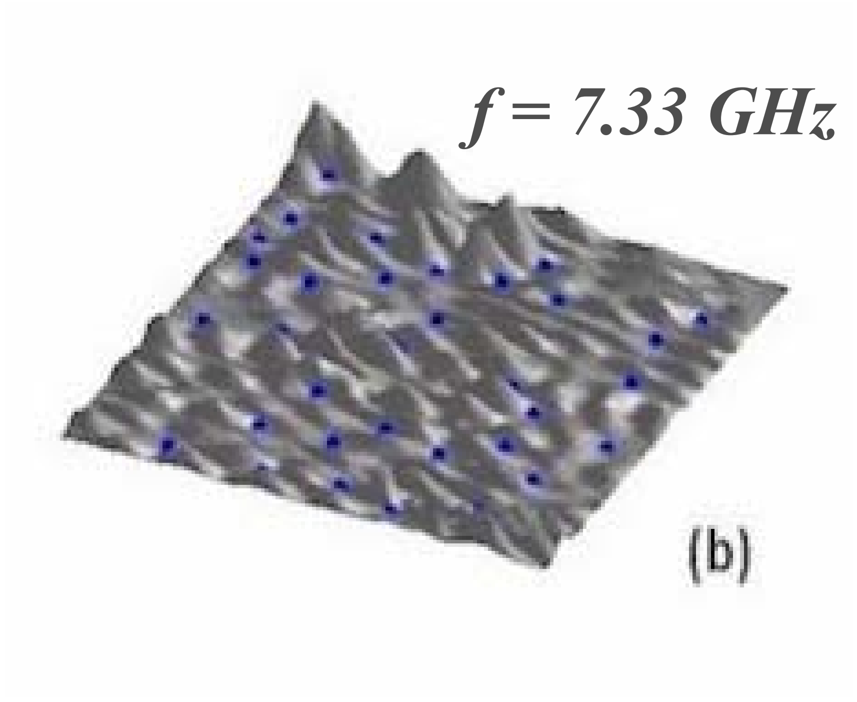
(Received 28 February 2000)



f = 3.04 GHz



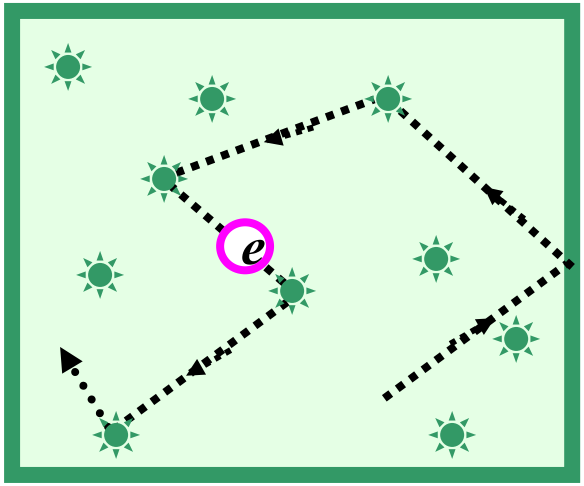
f = 7.33 GHz

Anderson Insulator

Anderson Metal

Poisson to Wigner-Dyson crossover

Important example: quantum particle subject to a random potential – disordered conductor

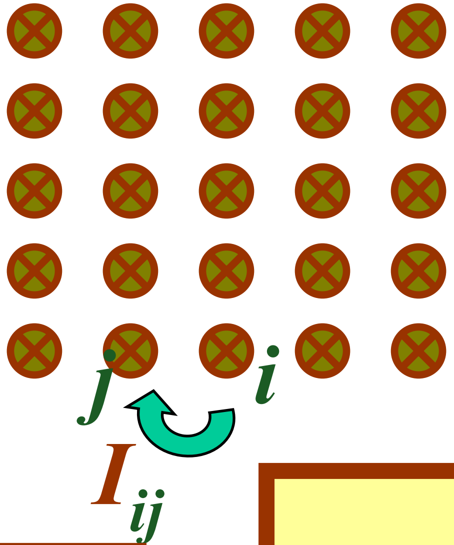


✹ *Scattering centers, e.g., impurities*

Models of disorder:

Randomly located impurities
White noise potential
Lattice models
  Anderson model
  Lifshits model

Anderson Model



- Lattice - tight binding model
- Onsite energies ε_i - random
- Hopping matrix elements I_{ij}

$$-W < \varepsilon_i < W$$

uniformly distributed

$$I_{ij} = \begin{cases} I & i \text{ and } j \text{ are nearest neighbors} \\ 0 & \text{otherwise} \end{cases}$$

Anderson Transition

$I < I_c$

Insulator

All eigenstates are localized

Localization length ξ

$I > I_c$

Metal

There appear states extended all over the whole system

Anderson Transition

$I < I_c$

Insulator

All eigenstates are localized

Localization length ξ

The eigenstates, which are localized at different places will not repel each other

Poisson spectral statistics

$I > I_c$

Metal

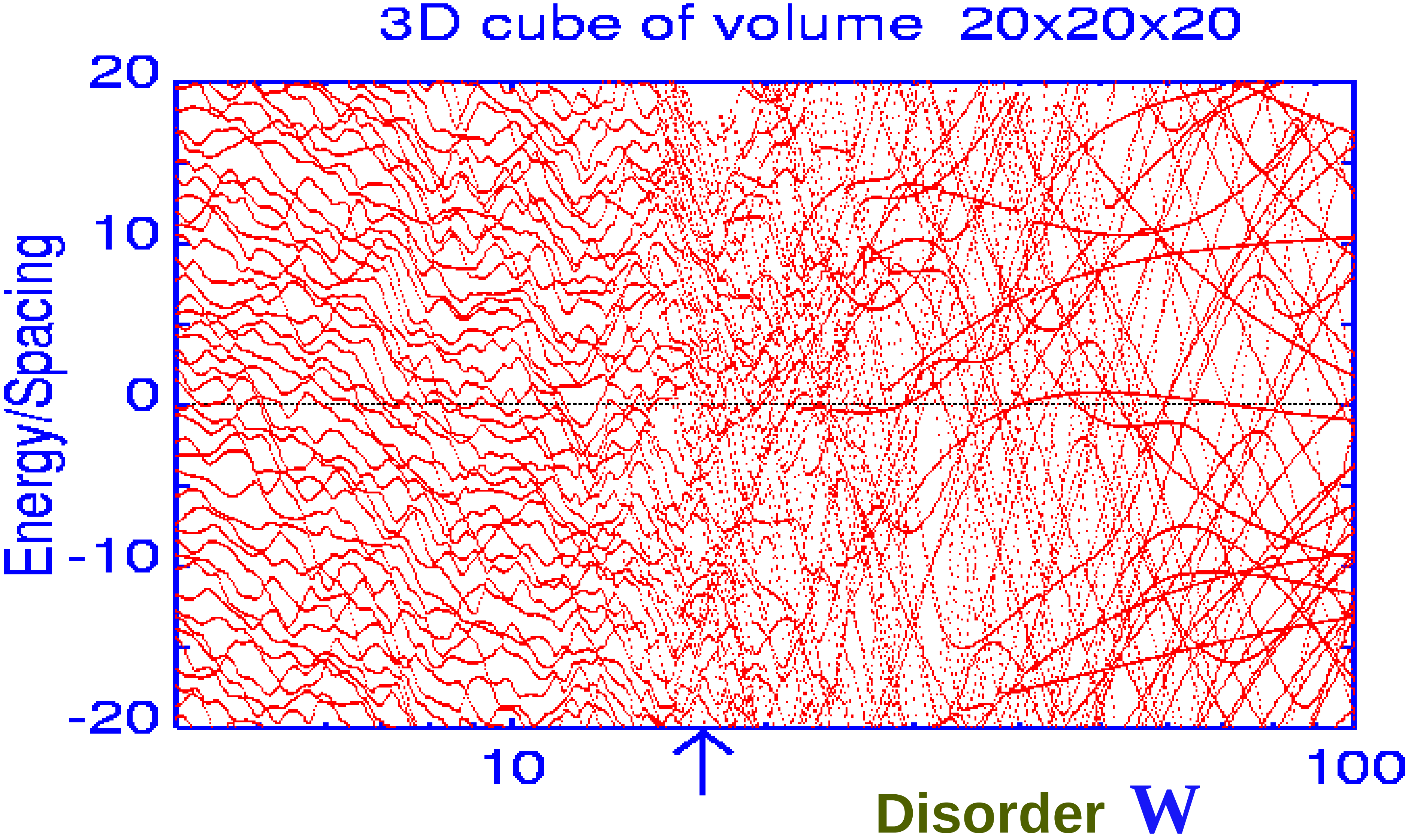
There appear states extended all over the whole system

Any two extended eigenstates repel each other

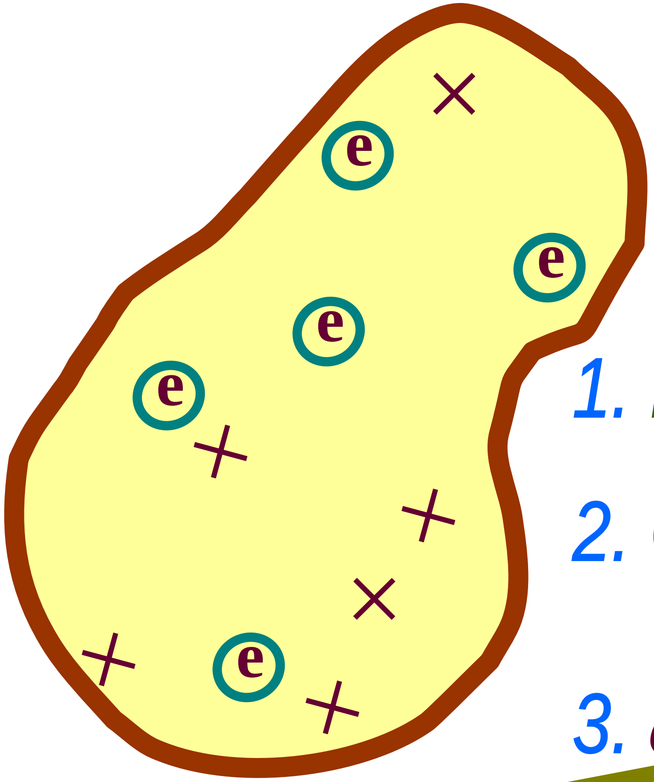
Wigner – Dyson spectral statistics

Zharekeschev & Kramer.

Exact diagonalization of the Anderson model



Quantum Dot



1. *Disorder (× – impurities)*
2. *Complex geometry*
3. ~~*e-e interactions*~~ for a while

Realizations:

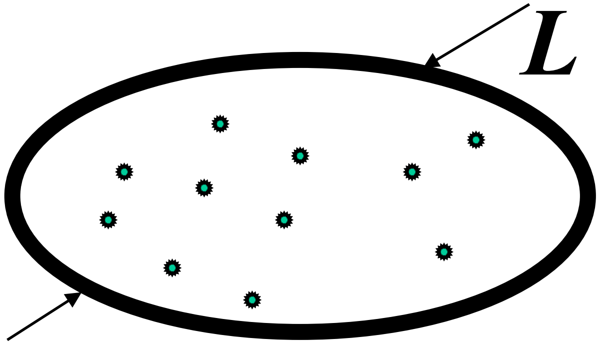
- Metallic clusters
- Gate determined confinement in 2D gases (e.g. ***GaAs/AlGaAs***)
- Carbon nanotubes

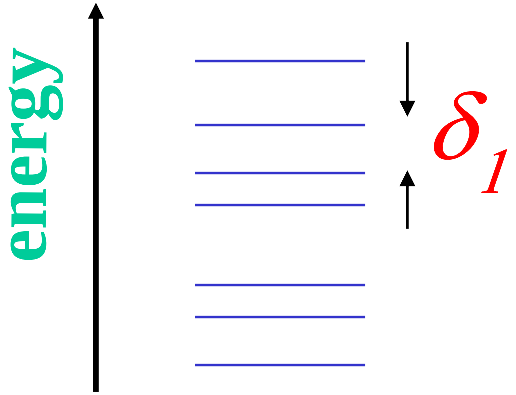
One-particle problem (*Thouless, 1972*)

Energy scales

1. Mean level spacing

$$\delta_1 = 1/\nu \times L^d$$





L *is the system size;*

d *is the number of dimensions*

2. Thouless energy

$$E_T = hD/L^2$$

D *is the diffusion const*

E_T *has a meaning of the* *inverse diffusion time* *of the traveling through the system or the* *escape rate* *(for open systems)*

$$g = E_T/\delta_1$$

dimensionless
Thouless
conductance

$$g = Gh/e^2$$

Thouless Conductance and One-particle Spectral Statistics

Localized states
Insulator

Extended states
Metal

Poisson spectral statistics

Wigner-Dyson spectral statistics

$N \times N$
Random Matrices

Quantum Dots with Thouless conductance g

$N \to \infty$

$g \to \infty$

The same statistics of the random spectra and one-particle wave functions (eigenvectors)

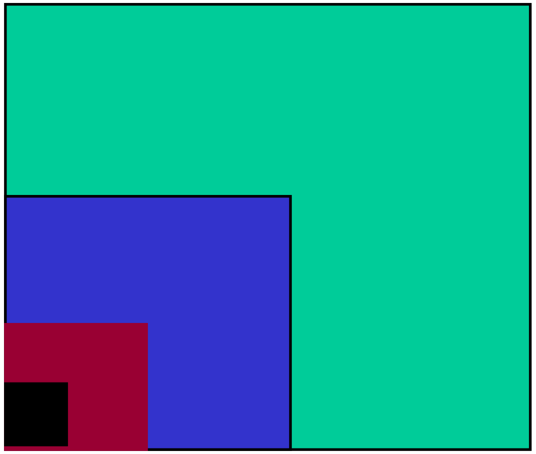
Scaling theory of Localization

(Abrahams, Anderson, Licciardello and Ramakrishnan 1979)

$$g = E_T / \delta_1$$

Dimensionless Thouless conductance

$$g = Gh/e^2$$



$$L = 2L = 4L = 8L \ldots.$$

without quantum corrections

$$E_T \propto L^{-2} \qquad \delta_1 \propto L^{-d}$$

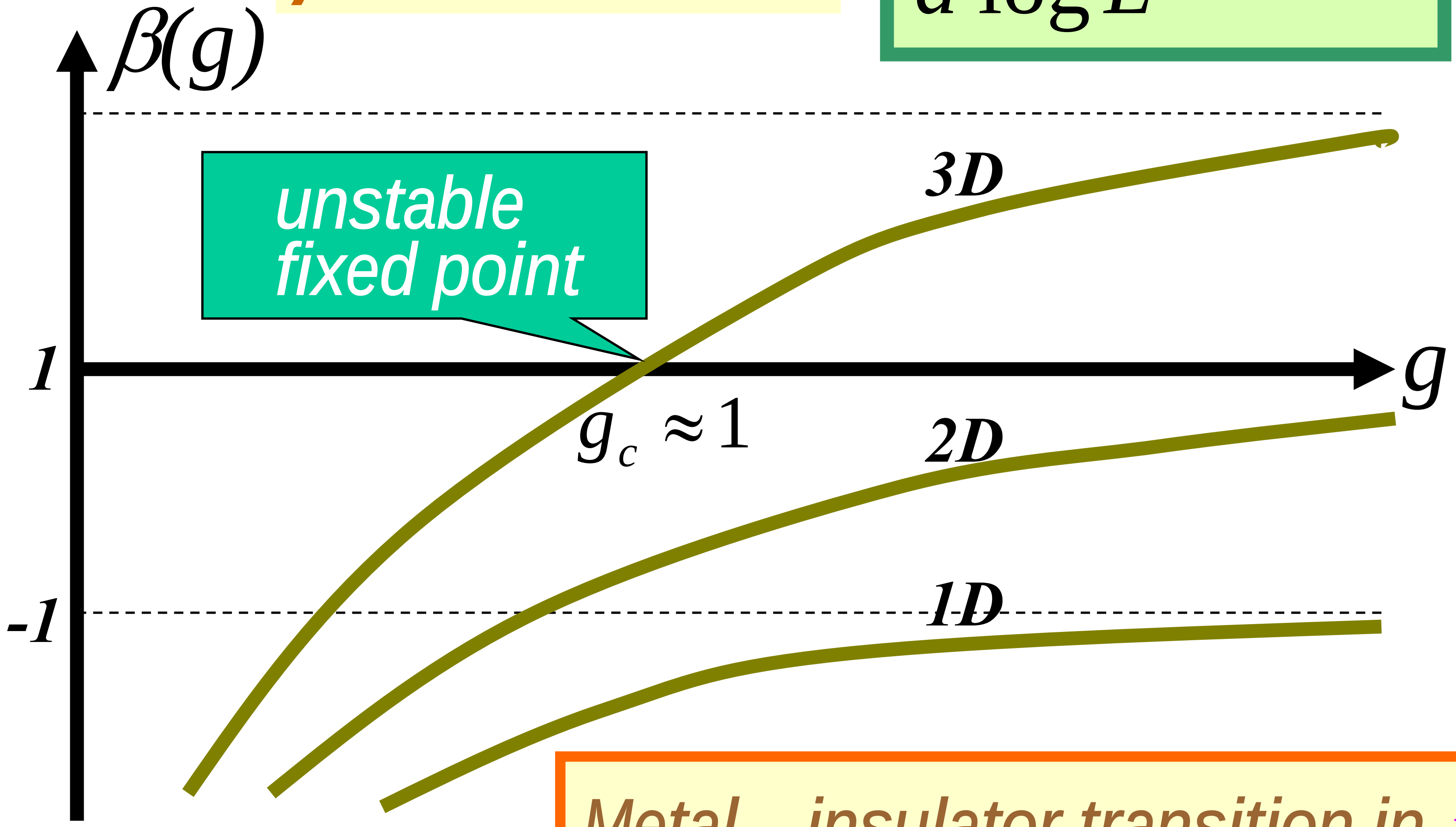
$$\begin{matrix} E_T & \rightarrow & E_T & \rightarrow & E_T & \rightarrow & E_T \\ \delta_1 & & \delta_1 & & \delta_1 & & \delta_1 \end{matrix}$$

$$g \rightarrow g \rightarrow g \rightarrow g$$

$$\boxed{\frac{d(\log g)}{d(\log L)} = \beta(g)}$$

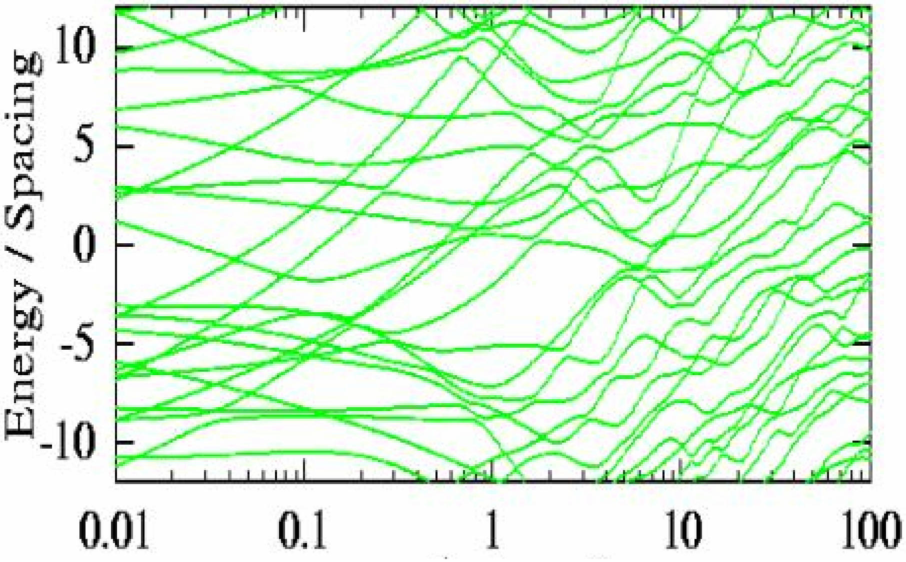
β - function

$$\frac{d \log g}{d \log L} = \beta(g)$$

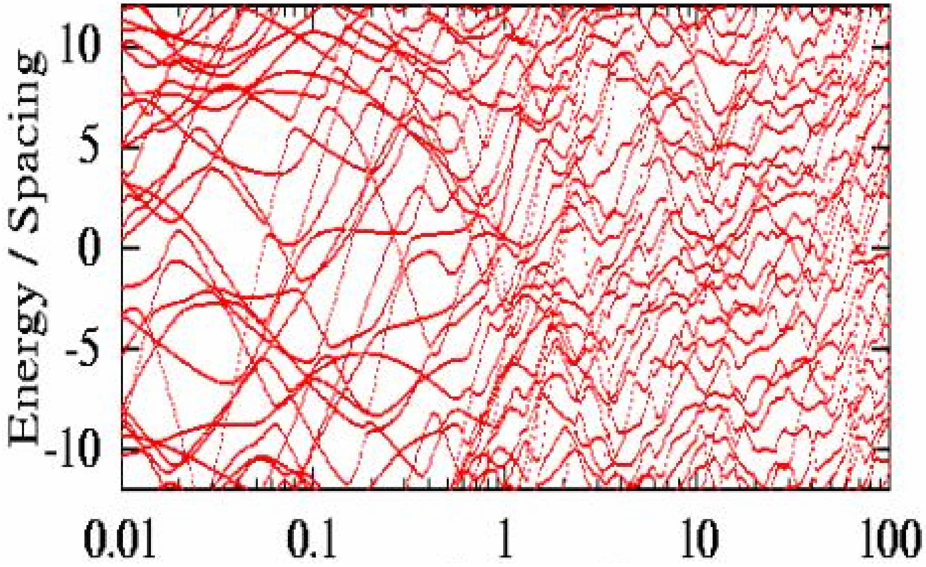


Metal – insulator transition in ***3D***

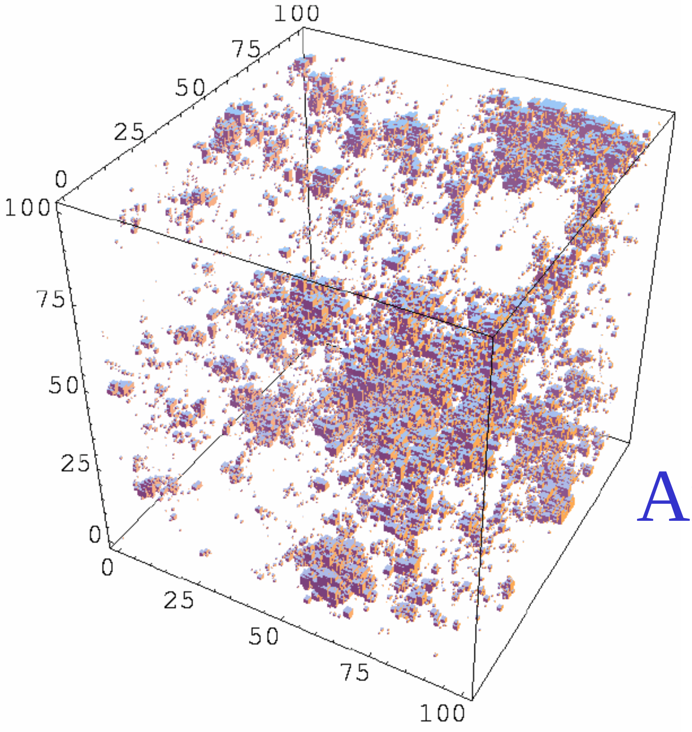
All states are localized for ***d=1,2*** volume = 8 x 8 x 8



volume = 20 x 20 x 20



Critical electron eigenstate at the Anderson transition



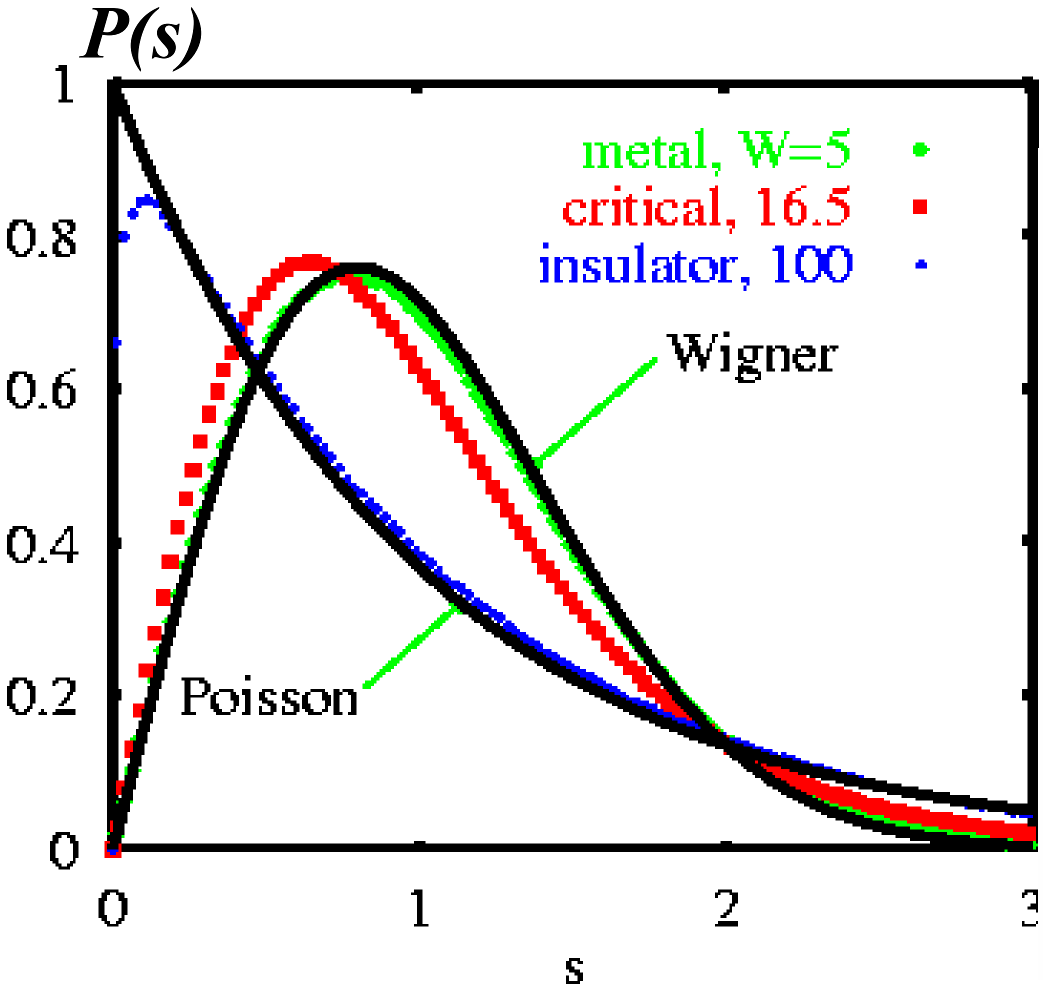
Zharekeshev, Computer Phys. Commun. 1999

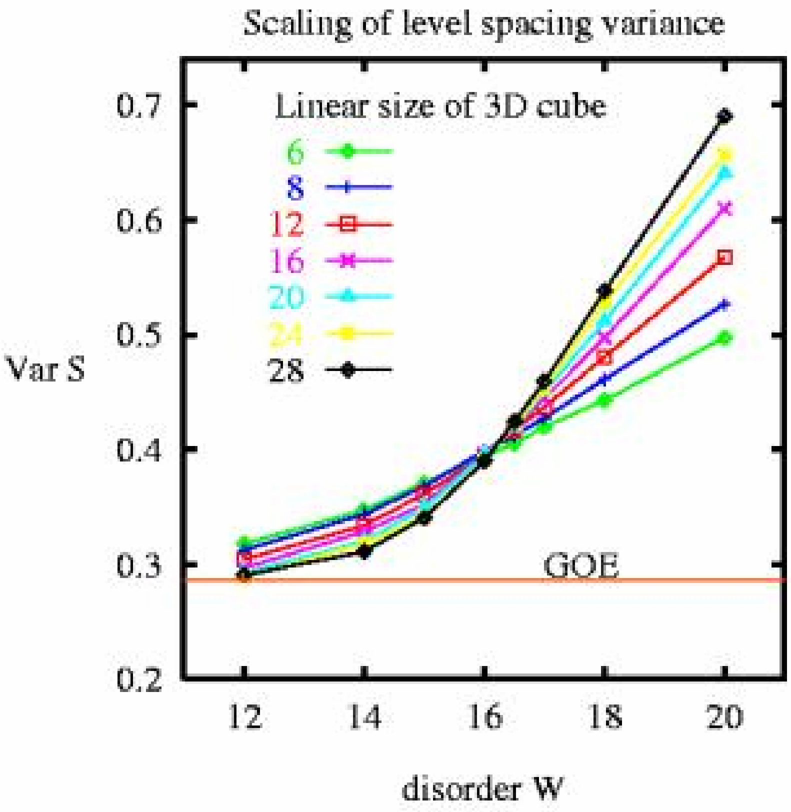
Conductance ***g***

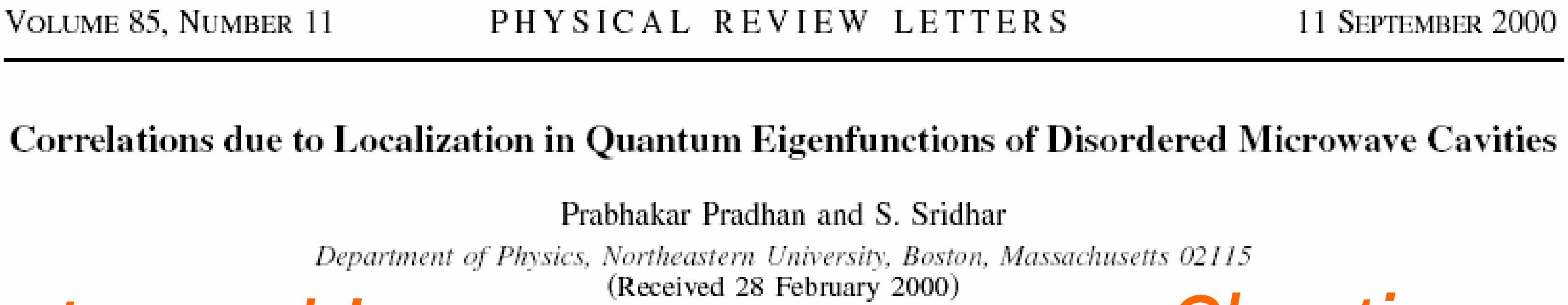
$$100\times100\times100$$

Anderson model cube

Anderson transition in terms of pure level statistics







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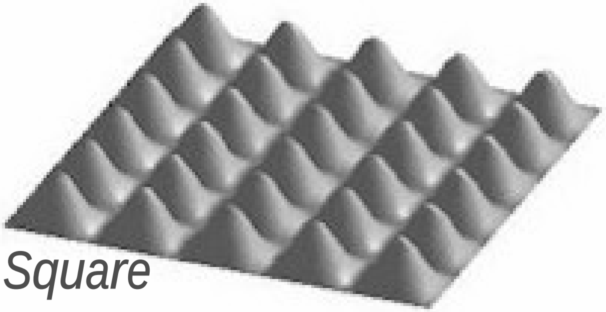
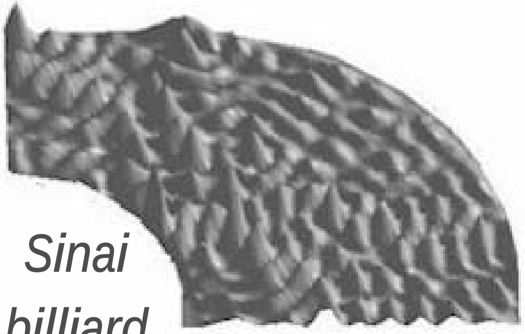
Integrable

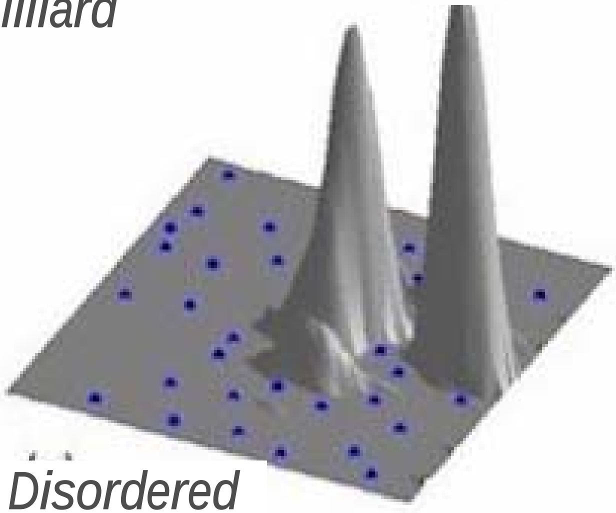
Chaotic



Square billiard

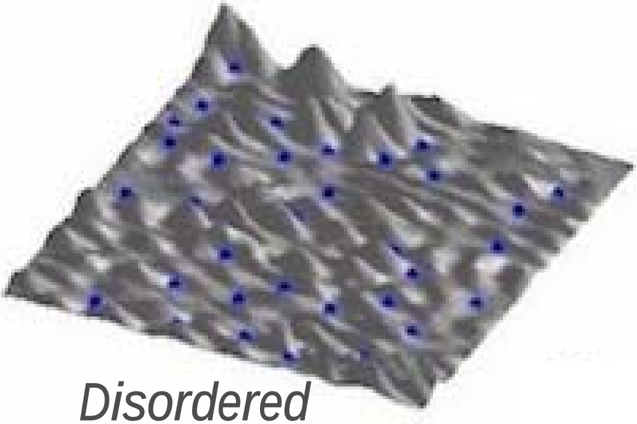


Sinai billiard



Disordered

localized



Disordered

extended

All chaotic systems resemble each other.

All integrable systems are integrable in their own way

Disordered Systems:

$E_T > \delta_1; \quad g > 1$

*Anderson **metal**; **Wigner-Dyson** spectral statistics*

$E_T < \delta_1; \quad g < 1$

*Anderson **insulator**; **Poisson** spectral statistics*

Q: *Is it a generic scenario for the Wigner-Dyson to Poisson crossover* **?**

Speculations

*Consider an **integrable** system. Each state is characterized by a **set of quantum numbers.***

*It can be viewed as a point in the **space of quantum numbers**. The whole set of the states forms a **lattice** in this space.*

*A **perturbation** that violates the integrability provides matrix elements of the **hopping** between different sites (**Anderson model** !?)*

Q: *Does Anderson localization provide a generic scenario for the Wigner-Dyson to Poisson crossover* **?**

*Consider an **integrable** system. Each state is characterized by a **set of quantum numbers.***

*It can be viewed as a point in the **space of quantum numbers**. The whole set of the states forms a **lattice** in this space.*

*A **perturbation** that violates the integrability provides matrix elements of the **hopping** between different sites (**Anderson model** !?)*

Weak enough hopping** - Localization - **Poisson
Strong hopping** - transition to **Wigner-Dyson

The very definition of the localization is **not invariant** – one should specify in which space the eigenstates are localized.

Level statistics **is invariant**:

Poissonian statistics   ∃ basis where the eigenfunctions are localized

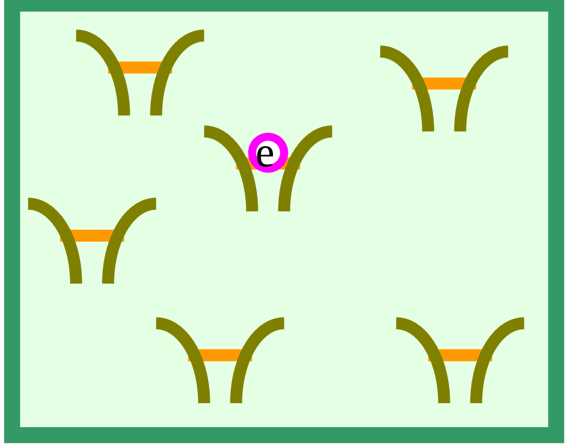
Wigner -Dyson statistics   ∀ basis the eigenfunctions are extended

Example 1

Doped semiconductor

Low concentration of donors ⇒ Electrons are localized on donors ⇨ Poisson

Higher donor concentration ⇒ Electronic states are extended ⇨ Wigner-Dyson

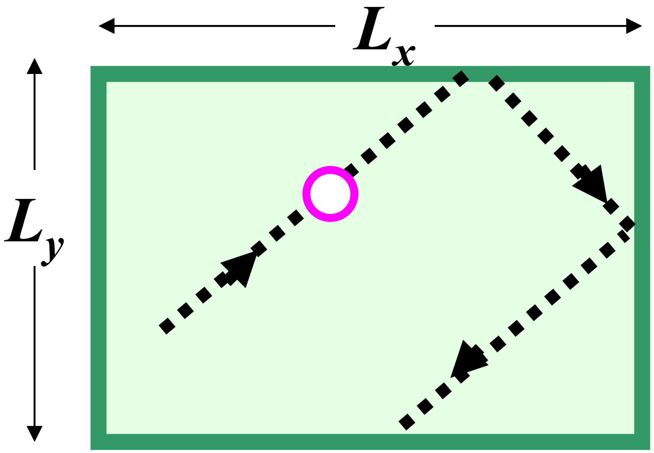


Example 2

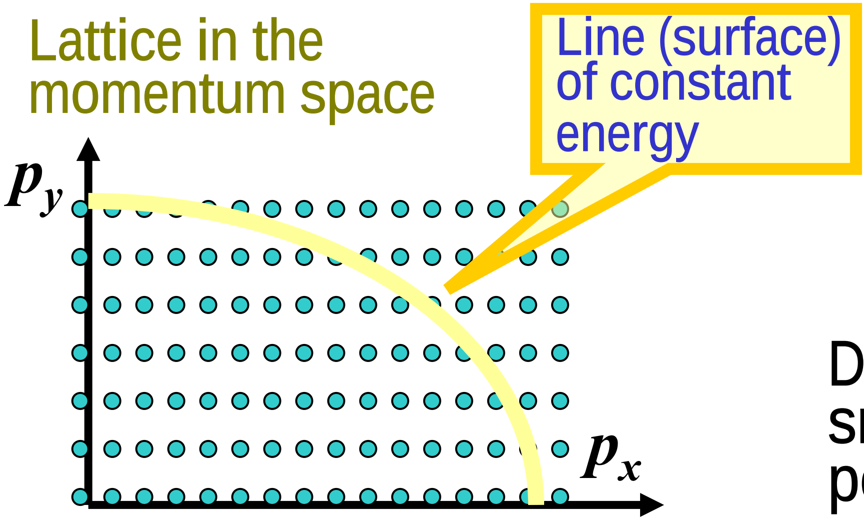
Rectangular billiard

Two integrals of motion

$$p_x = \frac{\pi n}{L_x}; \quad p_y = \frac{\pi m}{L_x}$$



Lattice in the momentum space

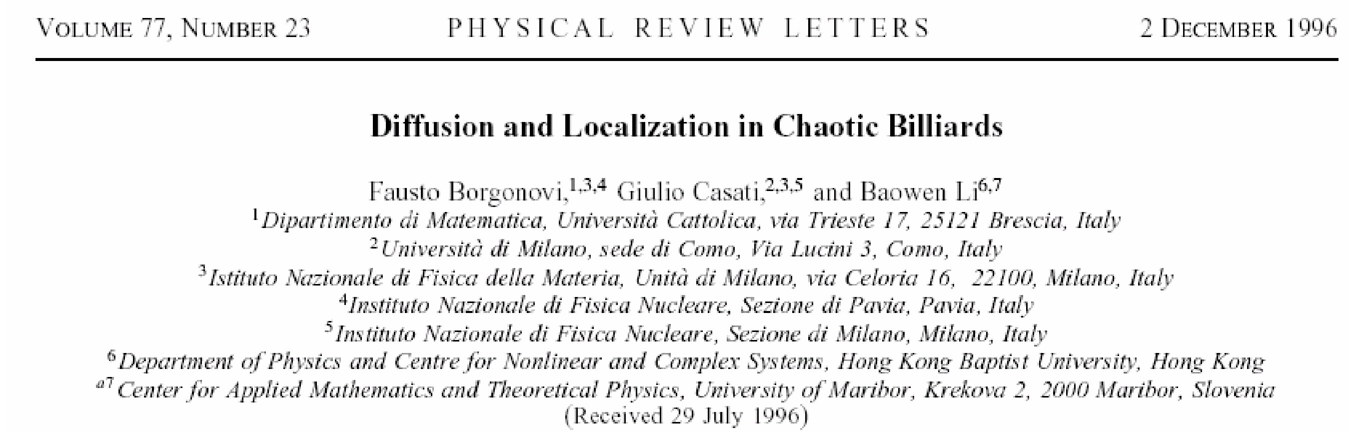


Line (surface) of constant energy

Ideal billiard – localization in the momentum space ⇨ Poisson

Deformation or smooth random potential – delocalization in the momentum space ⇨ Wigner-Dyson

Localization and diffusion in the angular momentum space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Diffusion and Localization in Chaotic Billiards

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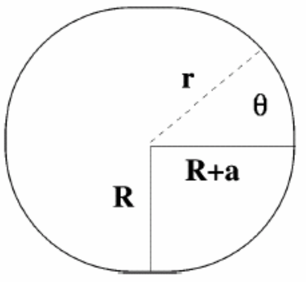
(Received 29 July 1996)

$$\varepsilon \equiv \frac{a}{R}$$



$\varepsilon > 0$ Chaotic stadium

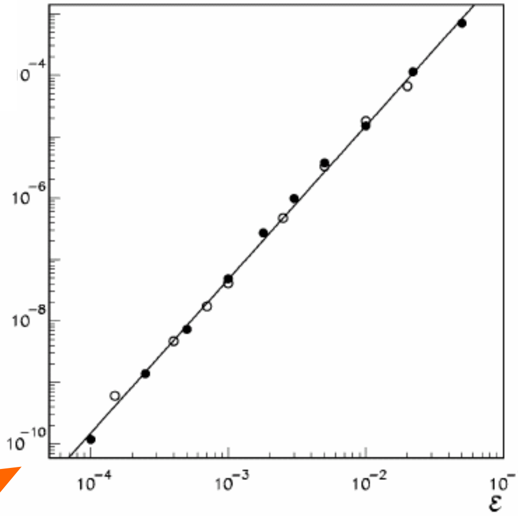
$\varepsilon \to 0$ Integrable circular billiard

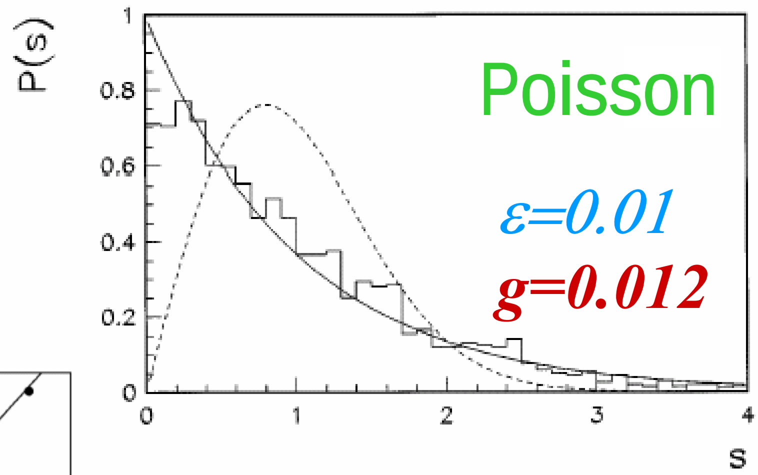
Angular momentum is the integral of motion

$\hbar = 0; \quad \varepsilon << 1$

Diffusion in the angular momentum space

$$D \propto \varepsilon^{5/2}$$

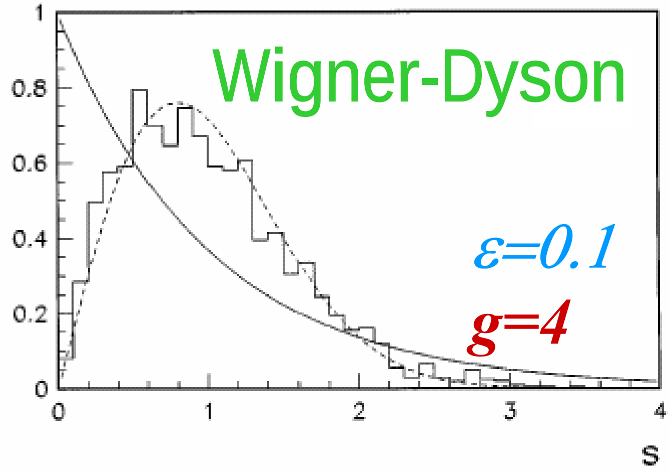




Poisson

$\varepsilon=0.01$

$g=0.012$



Wigner-Dyson

$\varepsilon=0.1$

$g=4$

D.Poilblanc, T.Ziman, J.Bellisard, F.Mila & G.Montambaux
Europhysics Letters, v.22, p.537, 1993

1D Hubbard Model on a periodic chain

$$H = t\sum_{i,\sigma}\left(c^{+}_{i,\sigma}c_{i+1,\sigma}+c^{+}_{i+1,\sigma}c_{i,\sigma}\right)+U\sum_{i,\sigma}n_{i,\sigma}n_{i,-\sigma}+V\sum_{i,\sigma,\sigma'}n_{i,\sigma}n_{i+1,\sigma'}$$

Onsite interaction

n. neighbors interaction

$V=0$ Hubbard model integrable

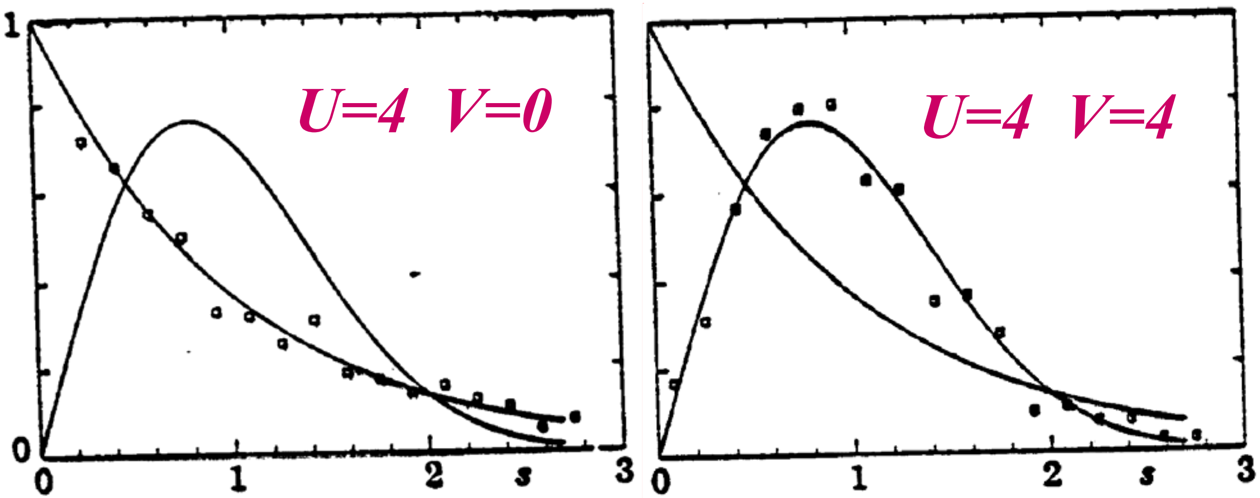
$V\neq 0$ extended Hubbard model nonintegrable

12 sites

3 particles

Zero total spin

Total momentum $\pi/6$



D.Poilblanc, T.Ziman, J.Bellisard, F.Mila & G.Montambaux
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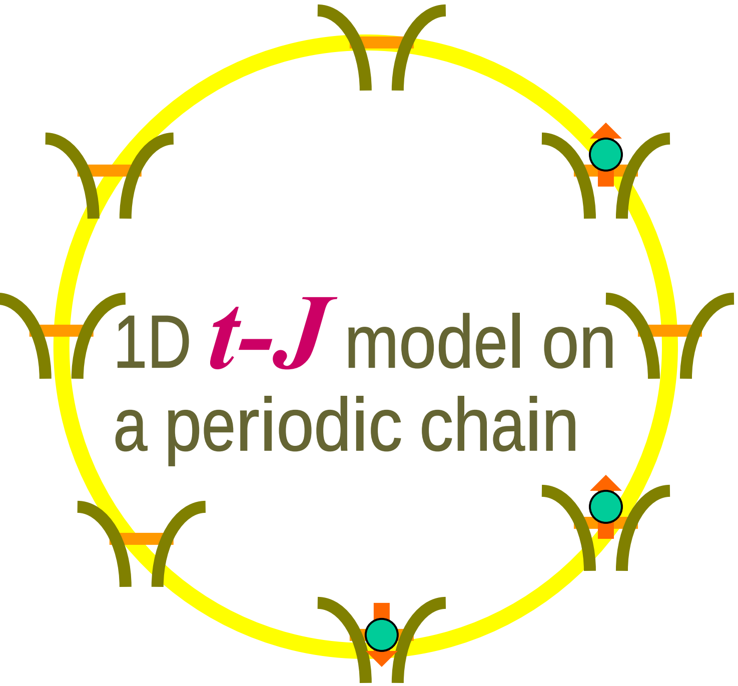
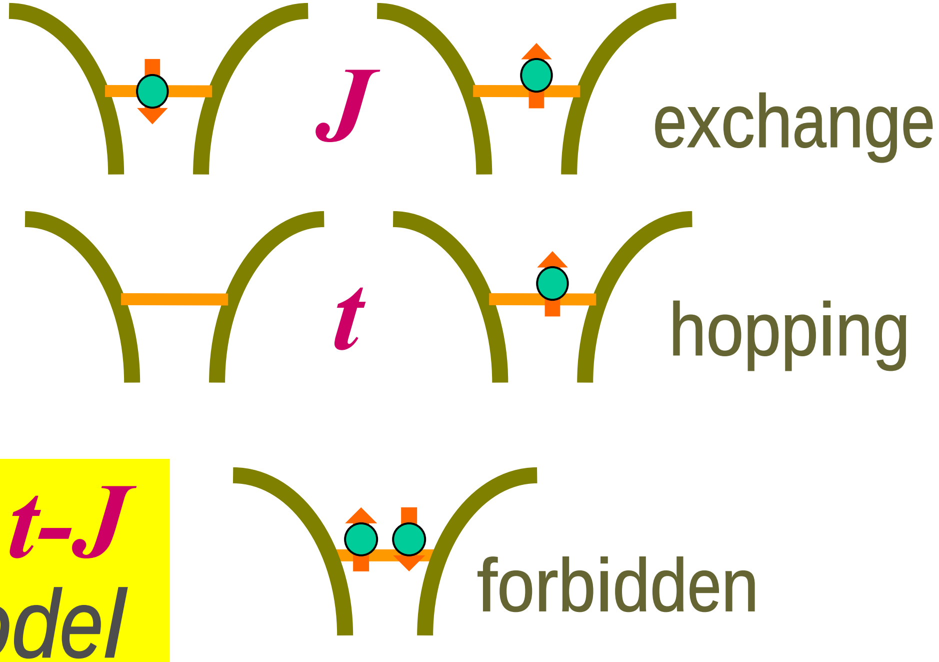
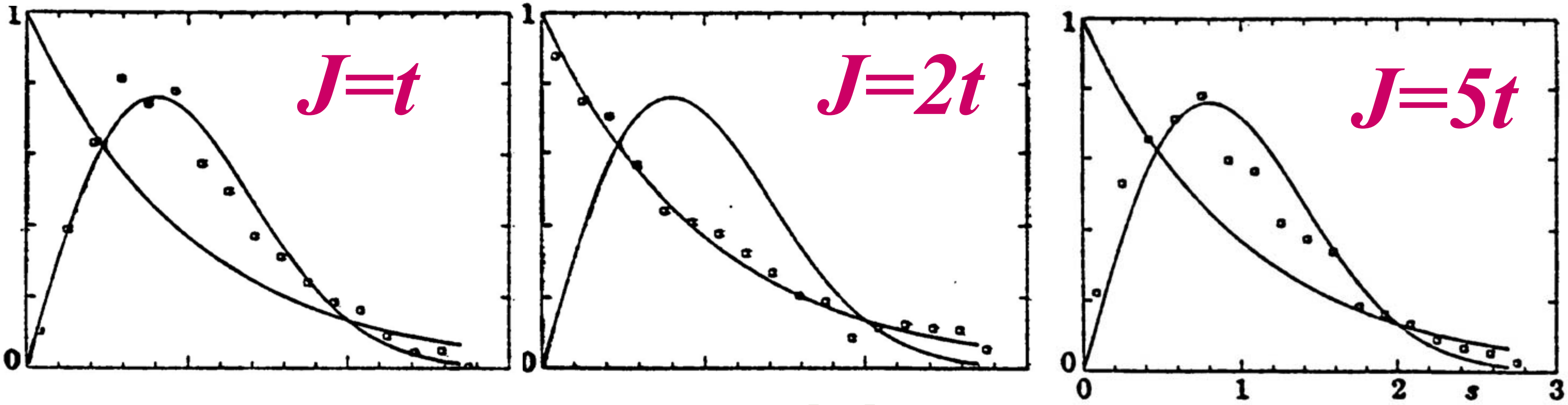
1d *t-J* *model*



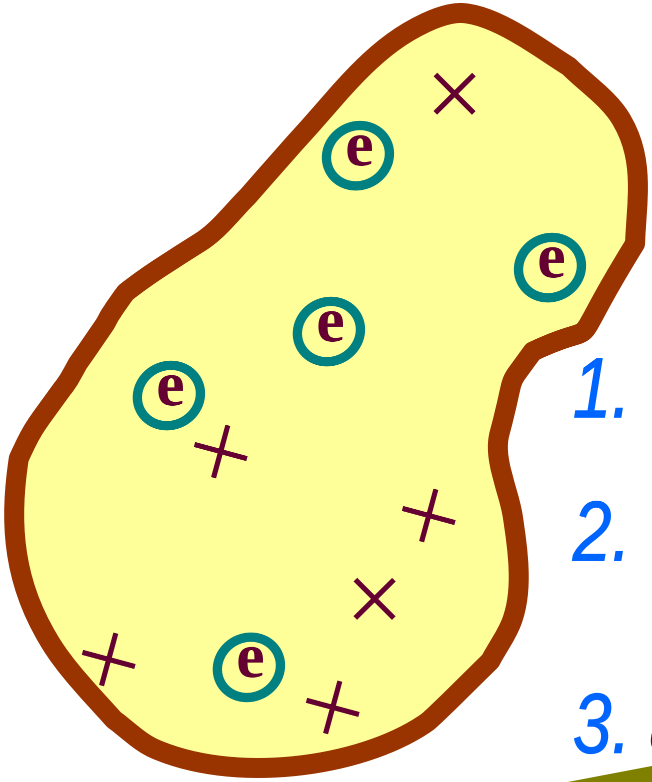
1D ***t-J*** model on a periodic chain





N=16; one hole

Quantum Dot



1. *Disorder (× – impurities)*
2. *Complex geometry*
3. ~~*e-e interactions*~~ for a while

Realizations:

- Metallic clusters
- Gate determined confinement in 2D gases (e.g. ***GaAs/AlGaAs***)
- Carbon nanotubes
-
-

Part 2 Disorder/Chaos + Interactions

Zero-Dimensional Fermi-liquid

ORIGINS

E.P. Wigner, **Conference on Neutron Physics by Time of Flight, November 1956**

P.W. Anderson, ***"Absence of Diffusion in Certain Random Lattices"*****; Phys.Rev., 1958, v.109, p.1492**

L.D. Landau, ***"Fermi-Liquid Theory"*** **Zh. Exp. Teor. Fiz.,1956, v.30, p.1058**

J. Bardeen, L.N. Cooper & J. Schriffer, ***"Theory of Superconductivity"*****; Phys.Rev., 1957, v.108, p.1175.**

~~**What does it mean - non-Fermi liquid ?**~~

What does it mean Fermi liquid ?

Fermi Liquid

- *Fermi statistics*
- *Low temperatures*
- *Not too strong interactions*
- *Translation invariance*

⇒ *Fermi Liquid*

What does it mean?

- *Fermi statistics*
- *Low temperatures*
- *Not too strong interactions*
- *Translation invariance*

→ *Fermi Liquid*

It means that

1. *Excitations are similar to the excitations in a Fermi-gas:*
   - *a) the same quantum numbers – momentum, spin ½ , charge e*
   - *b) decay rate is small as compared with the excitation energy*

2. *Substantial renormalizations. For example, in a Fermi gas*

$$\partial n/\partial\mu, \quad \gamma = c/T, \quad \chi/g\mu_B$$

are all equal to the one-particle density of states.
These quantities are different in a Fermi liquid

Signatures of the Fermi - Liquid state ?!

1. Resistivity is proportional to T^2 :

L.D. Landau & I.Ya. Pomeranchuk ***"To the properties of metals at very low temperatures"*****; Zh.Exp.Teor.Fiz., 1936, v.10, p.649**

…The increase of the resistance caused by the interaction between the electrons is proportional to T^2 and at low temperatures exceeds the usual resistance, which is proportional to T^5.

… the sum of the momenta of the interaction electrons can change by an integer number of the periods of the reciprocal lattice. Therefore the momentum increase caused by the electric field can be destroyed by the interaction between the electrons, not only by the thermal oscillations of the lattice.

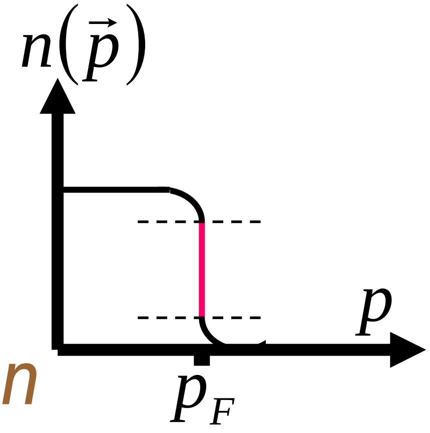
Signatures of the Fermi - Liquid state ?!

1. Resistivity is proportional to T^2 :

L.D. Landau & I.Ya. Pomeranchuk ***“To the properties of metals at very low temperatures”*****; Zh.Exp.Teor.Fiz., 1936, v.10, p.649**

Umklapp electron – electron scattering dominates the charge transport (?!)

2. Jump in the momentum distribution function at $T=0$.



2a. Pole in the one-particle Green function

$$G(\varepsilon,\vec{p})=\frac{Z}{i\varepsilon_n-\xi(\vec{p})}$$

Fermi liquid = $0<Z<1$ (?!)

Landau Fermi - Liquid theory

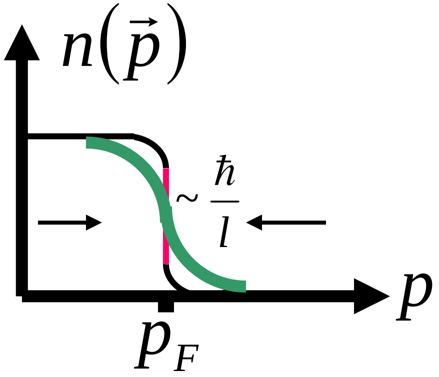
Momentum	$\vec{p}$
Momentum distribution	$n(\vec{p})$
Total energy	$E\{n(\vec{p})\}$
Quasiparticle energy	$\xi(\vec{p}) \equiv \delta E / \delta n(\vec{p})$
Landau f-function	$f(\vec{p},\vec{p}') \equiv \delta\xi(\vec{p}) / \delta n(\vec{p}')$

Q: ***Can Fermi – liquid survive without the momenta***

Does it make sense to speak about the Fermi – liquid state in the presence of a quenched disorder ?

Q: Does it make sense to speak about the Fermi – liquid state in the presence of a quenched disorder?

1. Momentum is not a good quantum number – the momentum uncertainty is inverse proportional to the elastic mean free path, l. The step in the momentum distribution function is broadened by this uncertainty



2. Neither resistivity nor its temperature dependence is determined by the umklapp processes and thus does not behave as T^2

3. Sometimes (e.g., for random quenched magnetic field) the disorder averaged one-particle Green function even without interactions does not have a pole as a function of the energy, ε. The residue , Z, makes no sense.

Nevertheless even in the presence of the disorder

I. Excitations are similar to the excitations in a disordered Fermi-gas.

II. Small decay rate

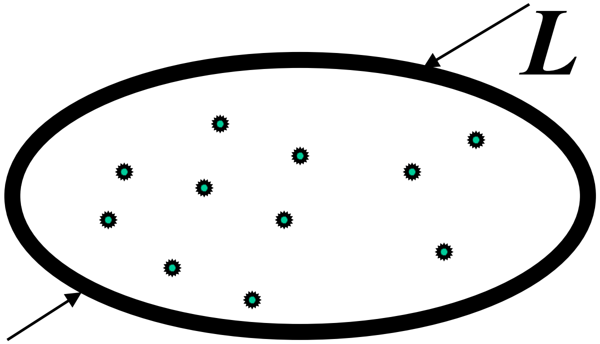
III. Substantial renormalizations

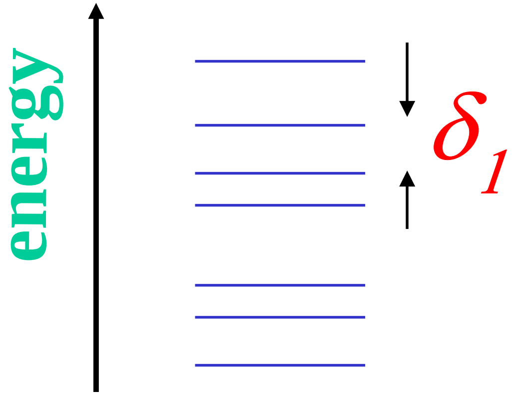
One-particle problem (*Thouless, 1972*)

Energy scales

1. Mean level spacing

$$\delta_1 = 1/\nu \times L^d$$





L is the system size;

d is the number of dimensions

2. Thouless energy

$$E_T = hD/L^2$$

D is the diffusion const

E_T has a meaning of the *inverse diffusion time* of the traveling through the system or the *escape rate* (for open systems)

$$g = E_T / \delta_1$$

dimensionless
Thouless
conductance

$$g = Gh/e^2$$

Zero DimensionalFermi Liquid

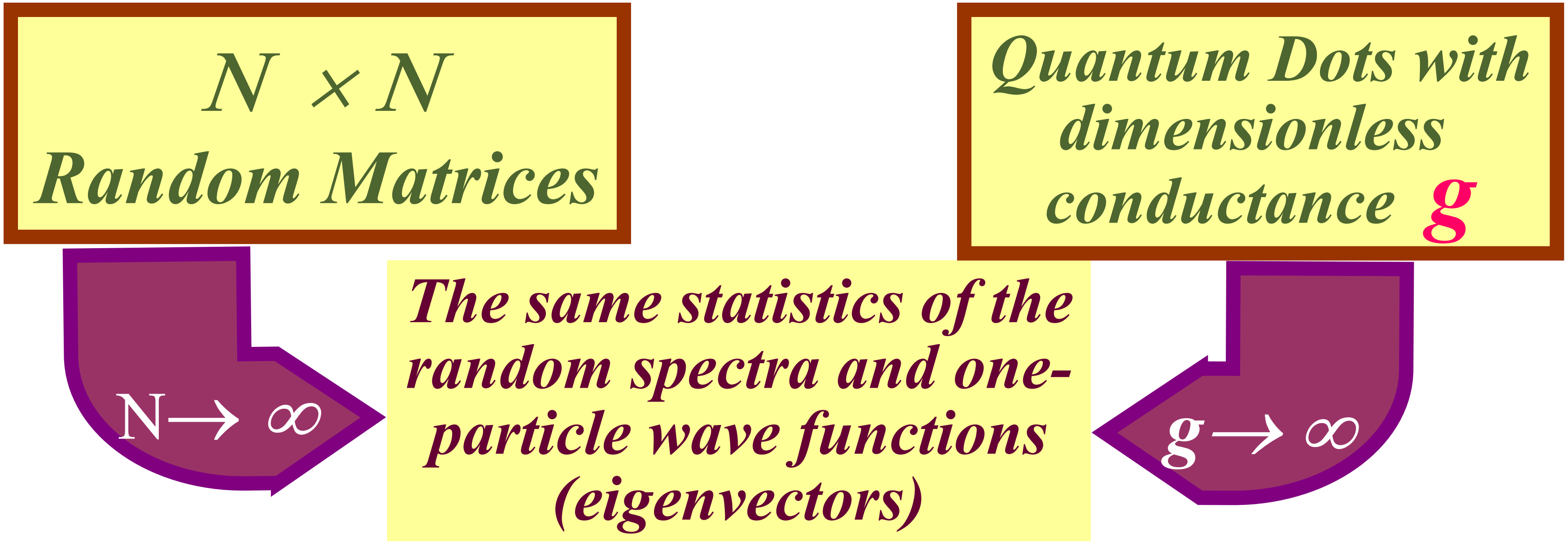
Finite System → Thouless energy E_T

$$\varepsilon << E_T \xrightarrow{def} 0D$$

At the same time, we want the typical energies, ε, to exceed the mean level spacing, δ_1:

$$\delta_1 << \varepsilon << E_T$$

$$g \equiv \frac{E_T}{\delta_1} >> 1$$



$N \times N$
Random Matrices

Quantum Dots with dimensionless conductance g

$N \to \infty$

$g \to \infty$

The same statistics of the random spectra and one-particle wave functions (eigenvectors)

Two-Body Interactions

$|\alpha,\sigma>$ *Set of one particle states.* σ *and* α *label correspondingly spin and orbit.*

$$\hat{H}_0 = \sum_{\alpha} \varepsilon_\alpha a^+_{\alpha,\sigma} a_{\alpha,\sigma} \qquad \hat{H}_{\text{int}} = \sum_{\substack{\alpha,\beta,\gamma,\delta \\ \sigma,\sigma'}} M_{\alpha\beta\gamma\delta} a^+_{\alpha,\sigma} a^+_{\beta,\sigma'} a_{\gamma,\sigma} a_{\delta,\sigma'}$$

ε_α *-one-particle orbital energies* $\boldsymbol{M}_{\alpha\beta\gamma\delta}$ *-interaction matrix elements*

Nuclear Physics	ε_α	*are taken from the shell model*
	$M_{\alpha\beta\gamma\delta}$	*are assumed to be random*
Quantum Dots	ε_α	*RANDOM; Wigner-Dyson statistics*
	$M_{\alpha\beta\gamma\delta}$	*? ? ? ? ? ? ? ? ?*

Matrix Elements

$$\hat{H}_{\text{int}} = \sum_{\substack{\alpha,\beta,\gamma,\delta \\ \sigma,\sigma'}} M_{\alpha\beta\gamma\delta} a^{+}_{\alpha,\sigma} a^{+}_{\beta,\sigma'} a_{\gamma,\sigma} a_{\delta,\sigma'}$$

Matrix Elements $M_{\alpha\beta\gamma\delta}$

Diagonal - $\alpha,\beta,\gamma,\delta$ are equal *pairwise*

$\alpha=\gamma$ and $\beta=\delta$ or $\alpha=\delta$ and $\beta=\gamma$ or $\alpha=\beta$ and $\gamma=\delta$

Offdiagonal - *otherwise*

It turns out that

in the limit $g \to \infty$

- *Diagonal matrix elements are much bigger than the offdiagonal ones*

$$M_{\text{diagonal}} >> M_{\text{offdiagonal}}$$

- *Diagonal matrix elements in a particular sample do not fluctuate - selfaveraging*

Toy model:

Short range ***e-e*** *interactions*

$$U(\vec{r}) = \frac{\lambda}{\nu}\delta(\vec{r})$$

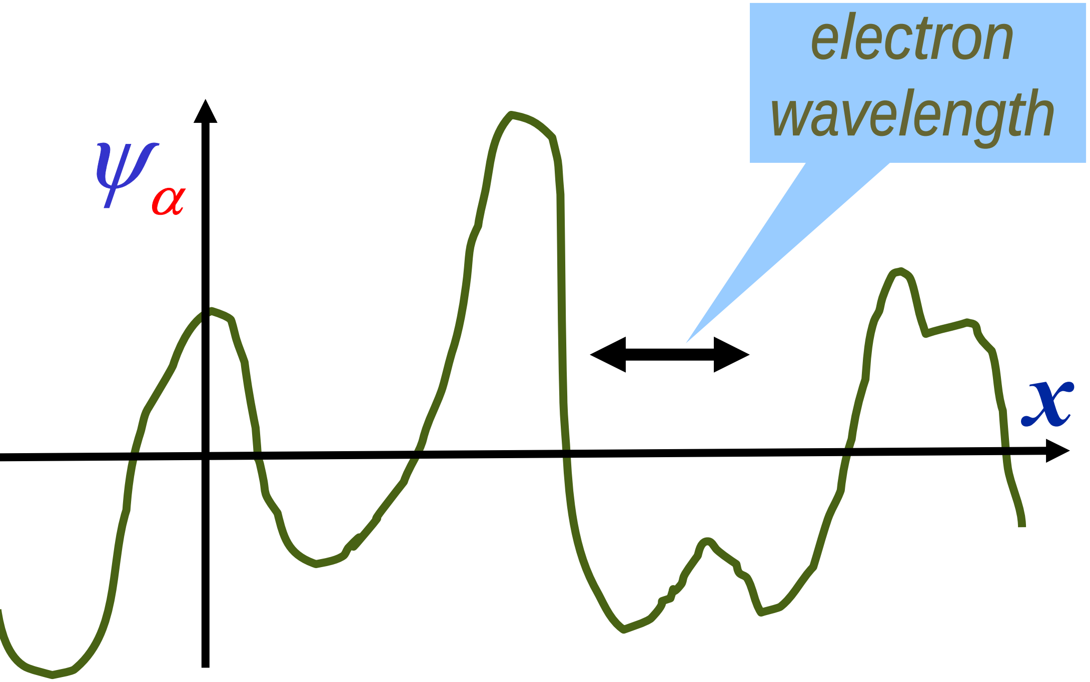
λ is dimensionless coupling constant

ν is the electron density of states

$$M_{\alpha\beta\gamma\delta} = \frac{\lambda}{\nu}\int d\vec{r}\,\psi*_{\alpha}(\vec{r})\psi*_{\beta}(\vec{r})\psi_{\gamma}(\vec{r})\psi_{\delta}(\vec{r})$$

$\psi_\alpha(\vec{r})$

one-particle eigenfunctions



$\Psi_\alpha(\boldsymbol{x})$ is a random function that rapidly oscillates

$$|\psi_\alpha(\boldsymbol{x})|^2 \geq 0$$

$$\psi_\alpha(\boldsymbol{x})^2 \geq 0$$

*as long as **T**-invariance is preserved*

In the limit

$$g \to \infty$$

- *Diagonal matrix elements are much bigger than the offdiagonal ones*

$$M_{\text{diagonal}} >> M_{\text{offdiagonal}}$$

- *Diagonal matrix elements in a particular sample do not fluctuate - selfaveraging*

$$M_{\alpha\beta\alpha\beta} = \frac{\lambda}{\nu} \int d\vec{r} \left|\psi_{\alpha}(\vec{r})\right|^{2} \left|\psi_{\beta}(\vec{r})\right|^{2}$$

$$\left|\psi_{\alpha}(\vec{r})\right|^{2} \Rightarrow \frac{1}{\text{volume}}$$

$$\Rightarrow \boxed{M_{\alpha\beta\alpha\beta} = \lambda\delta_{1}}$$

More general: *finite range interaction potential* $U(\vec{r})$

$$\boxed{M_{\alpha\beta\alpha\beta} = \frac{\lambda}{\nu} \int \left|\psi_{\alpha}(\vec{r}_1)\right|^{2} \left|\psi_{\beta}(\vec{r}_2)\right|^{2} U(\vec{r}_1 - \vec{r}_2) d\vec{r}_1 d\vec{r}_2}$$

The same conclusion

Universal (Random Matrix) limit - Random Matrix **symmetry** of the correlation functions:

All correlation functions are invariant under arbitrary orthogonal transformation:

$$\boxed{\tilde{\psi}_{\mu}(\vec{r}) = \sum_{\nu} \int d\vec{r}_1 O_{\mu}^{\nu}(\vec{r}, \vec{r}_1) \psi_{\nu}(\vec{r}_1)}$$

$$\boxed{\int d\vec{r}_1 O_{\mu}^{\nu}(\vec{r}, \vec{r}_1) O_{\nu}^{\eta}(\vec{r}_1, \vec{r}\,') = \delta_{\mu\eta} \delta(\vec{r} - \vec{r}\,')}$$

There are only three operators, which are quadratic in the fermion operators a^{+}, a , and invariant under RM transformations:

$$\hat{n} = \sum_{\alpha,\sigma} a^{+}_{\alpha,\sigma} a_{\alpha,\sigma}$$

total number of particles

$$\hat{S} = \sum_{\alpha,\sigma_1,\sigma_2} a^{+}_{\alpha,\sigma_1} \vec{\sigma}_{\sigma_1,\sigma_2} a_{\alpha,\sigma_2}$$

total spin

$$\hat{T}^{+} = \sum_{\alpha} a^{+}_{\alpha,\uparrow} a^{+}_{\alpha,\downarrow}$$

????

Charge conservation (gauge invariance) -no $\hat{T}$ or $\hat{T}^{+}$ only $\hat{T}\hat{T}^{+}$

Invariance under rotations in spin space - no $\hat{S}$ only $\hat{S}^{2}$

Therefore, in a very general case

$$\boxed{\hat{H}_{\text{int}} = eV\hat{n} + E_c\hat{n}^2 + J\hat{S}^2 + \lambda_{BCS}\hat{T}^{+}\hat{T}.}$$

Only three coupling constants describe all of the effects of e-e interactions

In a very general case only three coupling constants describe all effects of electron-electron interactions:

$$\hat{H} = \sum_{\alpha} \varepsilon_{\alpha} n_{\alpha} + \hat{H}_{\text{int}}$$

$$\hat{H}_{\text{int}} = eV\hat{n} + E_c \hat{n}^2 + J\hat{S}^2 + \lambda_{BCS} \hat{T}^{+} \hat{T}.$$

I.L. Kurland, I.L.Aleiner & B.A., 2000

See also

P.W.Brouwer, Y.Oreg & B.I.Halperin, 1999

H.Baranger & L.I.Glazman, 1999

H-Y Kee, I.L.Aleiner & B.A., 1998

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For a short range interaction with a coupling constant λ

$$E_c = \frac{\lambda \delta_1}{2} \qquad J = -2\lambda\delta_1 \qquad \lambda_{BCS} = \lambda\delta_1(2-\beta)$$

where δ_1 is the one-particle mean level spacing

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}$$

$$\hat{H}_0 = \sum_{\alpha} \varepsilon_\alpha n_\alpha$$

$$\hat{H}_{\text{int}} = eV\hat{n} + E_c\hat{n}^2 + J\hat{S}^2 + \lambda_{BCS}\hat{T}^+\hat{T}.$$

Only one-particle part of the Hamiltonian, $\hat{H}_0$, contains randomness !

$$\boxed{\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}}$$

$$\boxed{\hat{H}_0 = \sum_{\alpha} \varepsilon_\alpha n_\alpha}$$

$$\boxed{\hat{H}_{\text{int}} = eV\hat{n} + E_c\hat{n}^2 + J\hat{S}^2 + \lambda_{BCS}\hat{T}^+\hat{T}.}$$

E_c	determines the charging energy (Coulomb blockade)
J	describes the spin exchange interaction
λ_{BCS}	determines effect of superconducting-like pairing

ORIGINS

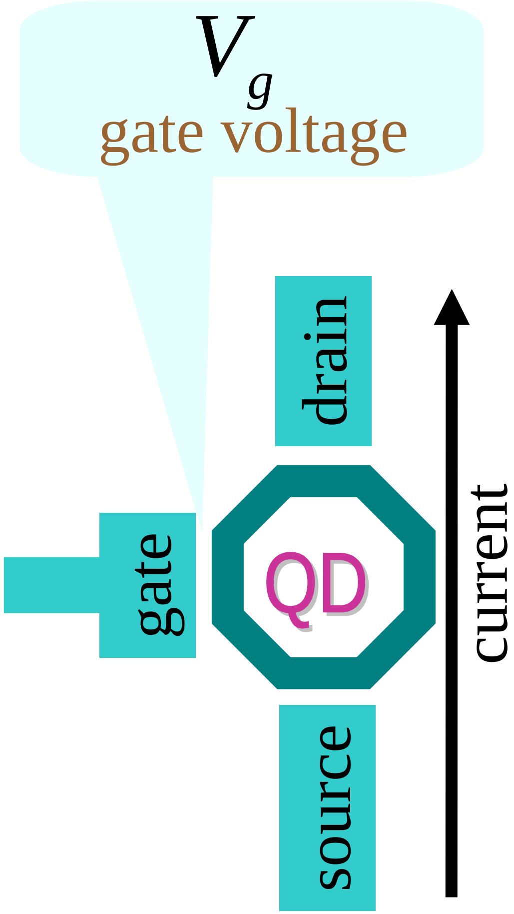
E.P. Wigner, Conference on Neutron Physics by Time of Flight, November 1956

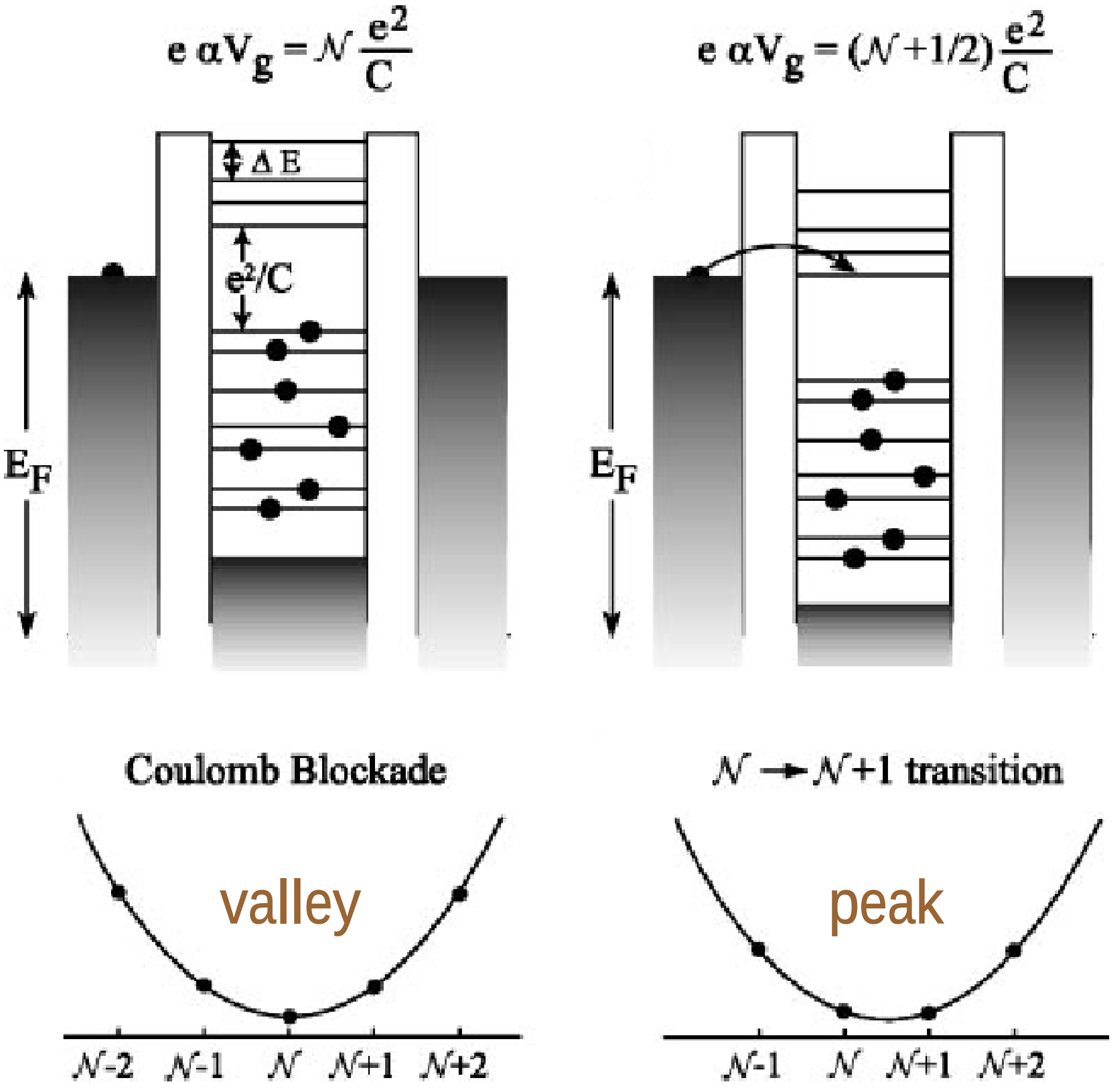
P.W. Anderson, “*Absence of Diffusion in Certain Random Lattices*”; Phys.Rev., 1958, v.109, p.1492

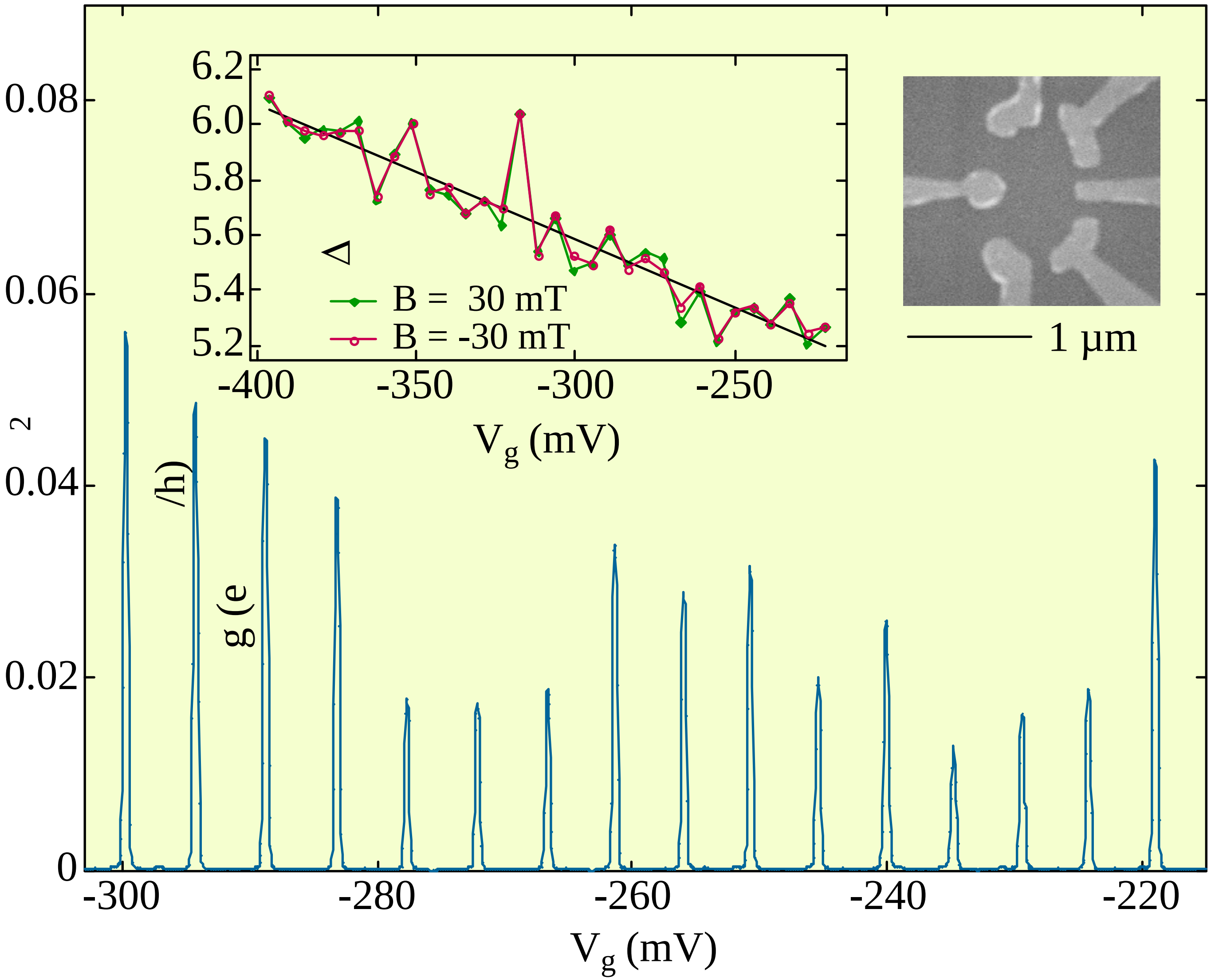
L.D. Landau, ”*Fermi-Liquid Theory*” Zh. Exp. Teor. Fiz.,1956, v.30, p.1058

J. Bardeen, L.N. Cooper & J. Schriffer, “*Theory of Superconductivity*”; Phys.Rev., 1957, v.108, p.1175.

Example 1: Coulomb Blockade





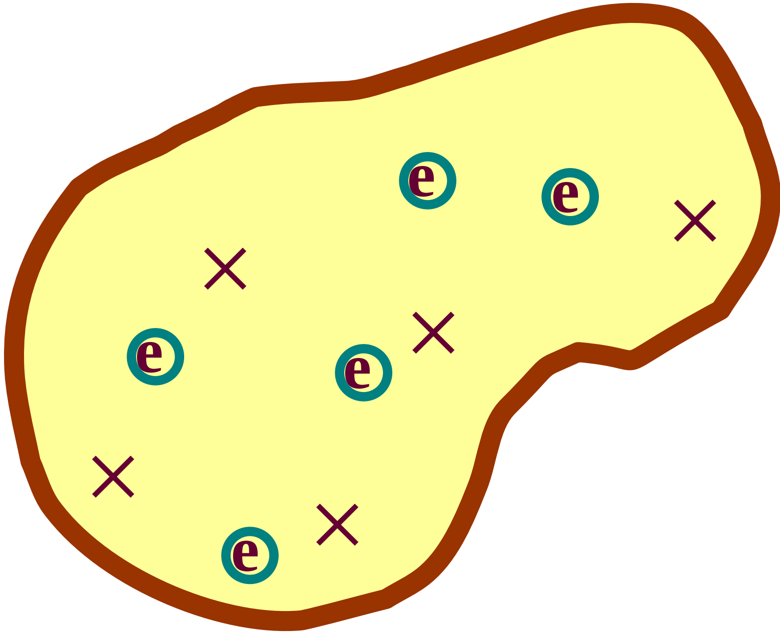


Coulomb Blockade Peak Spacing
Patel, et al. PRL 80 4522 (1998)
(Marcus Lab)

Example 2: Spontaneous Magnetization

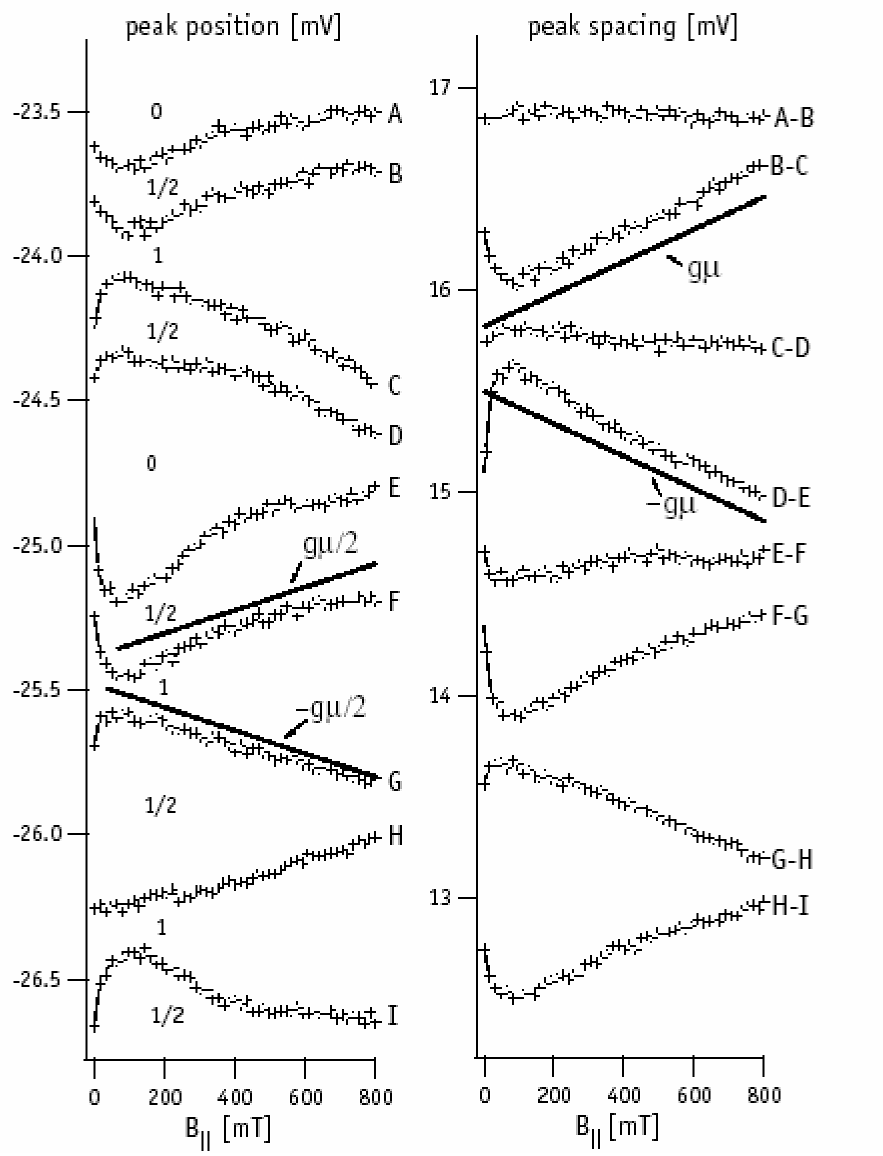
1. Disorder (×impurities)
2. Complex geometry
3. **e-e** interactions

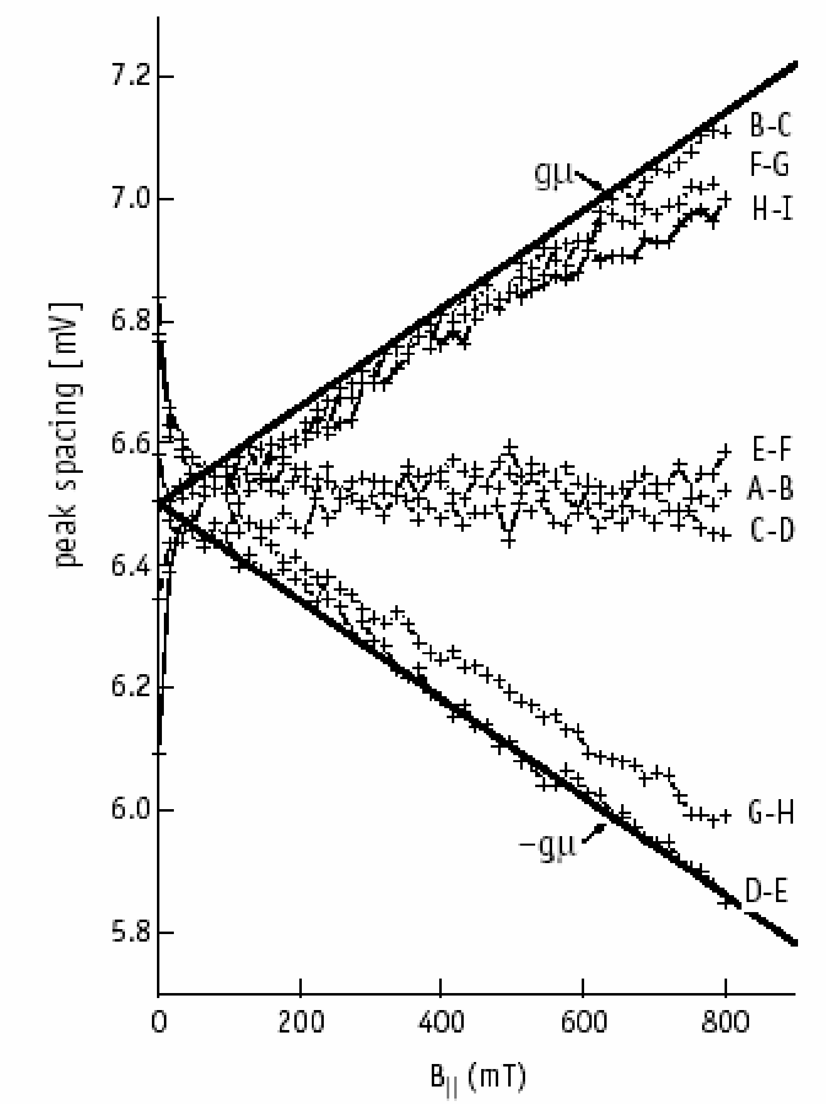
} *chaotic one-particle motion*



Q: What is the spin of the Quantum Dot in the ground state?

How to measure the Magnetization – motion of the Coulomb blockade peaks in the parallel magnetic field





In the presence of magnetic field

$$\hat{H}_{\mathbf{int}} = \sum_{\alpha} \varepsilon_{\alpha} n_{\alpha} + J\vec{\hat{S}}^{2} + \vec{B} \bullet \vec{\hat{S}}$$

Scaling:

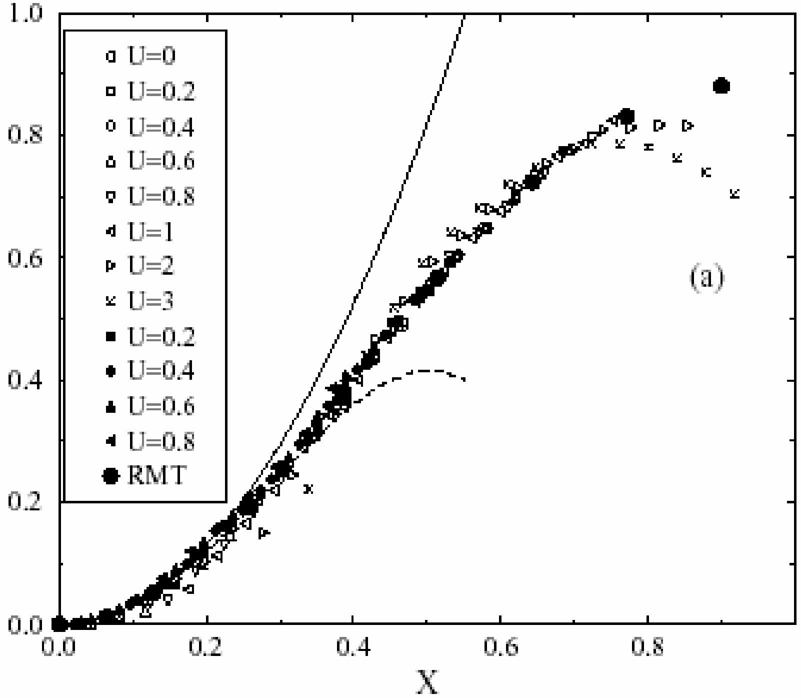
the probability to find a ground state at a given magnetic field, ***B***, with a given spin, ***S***, depends on the combination rather than on ***B*** and ***J*** separately

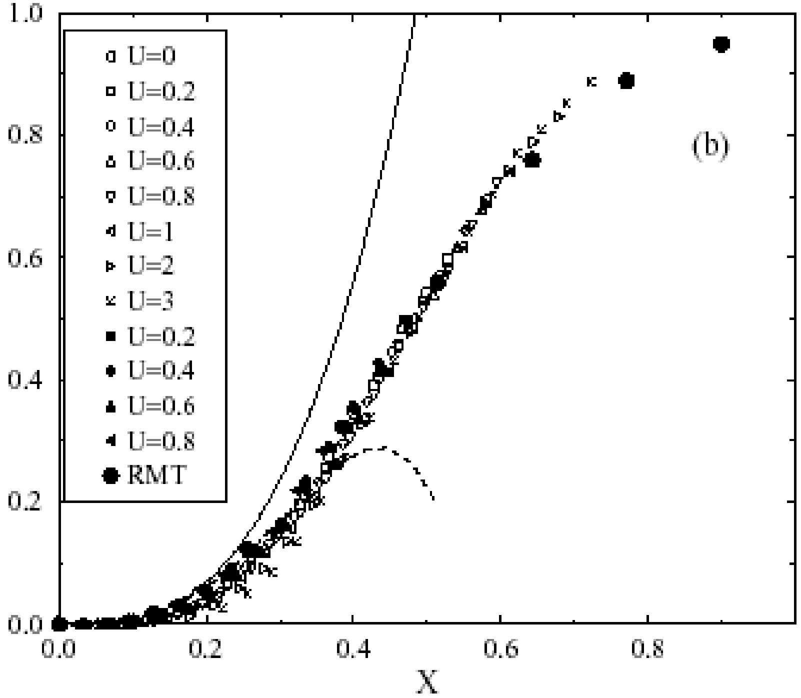
$$X = J + g\mu_{B} \frac{B}{2S}$$

Probability to observe a triplet state as a function of the parameter X

● *- results of the calculation based on the universal Hamiltonian with the RM one-particle states*

The rest – exact diagonalization for Hubbard clusters with disorder. No adjustable parameters





$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}$$

$$\hat{H}_0 = \sum_{\alpha} \varepsilon_\alpha n_\alpha$$

$$\hat{H}_{\text{int}} = eV\hat{n} + E_c \hat{n}^2 + J\hat{S}^2 + \lambda_{BCS} \hat{T}^+ \hat{T}.$$

I. *Excitations are similar to the excitations in a disordered Fermi-gas.*

II. *Small decay rate*

III. *Substantial renormalizations*

Isn't it a Fermi liquid ?

Fermi liquid behavior follows from the fact that different wave functions are almost uncorrelated

CONCLUSIONS

Anderson localization provides a generic scenario for the transition between chaotic and integrable behavior.

One-particle chaos + moderate interaction of the electrons $\mapsto$ to a rather simple Hamiltonian of the system, which can be called Zero-dimensional Fermi liquid.

The main parameter that justifies this description is the Thouless conductance, which is supposed to be large

Excitations are characterized by their one-particle energy, charge and spin, but not by their momentum.

These excitations have the lifetime, which is proportional to the Thouless conductance, i.e., is long.

This approach allows to describe Coulomb blockade (renormalization of the compressibility), as well as the substantial renormalization of the magnetic susceptibility and effects of superconducting pairing

BCS Hamiltonian

Finite systems

$$\hat{H} = \sum_{\alpha} \varepsilon_{\alpha} n_{\alpha} + E_c \hat{n}^2 + J\hat{S}^2 + \lambda_{BCS} \hat{T}^{+} \hat{T}.$$

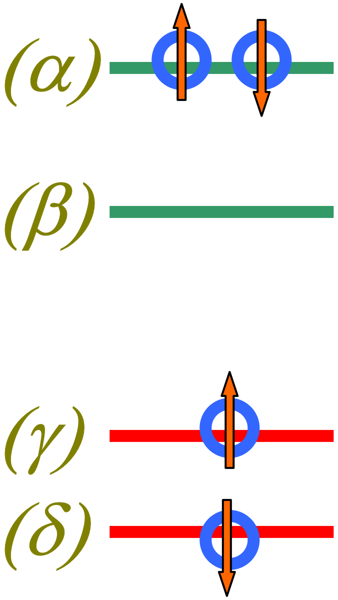
$$\hat{T}^{+} = \sum_{\alpha} a^{+}_{\alpha,\uparrow} a^{+}_{\alpha,\downarrow}$$

commute with each other

does not commute with K.E.

$$\hat{H} = \sum_{\alpha} \varepsilon_{\alpha} n_{\alpha} + \lambda_{BCS} \hat{T}^{+} \hat{T}.$$

$$\hat{H} = \sum_{\alpha} \varepsilon_{\alpha} n_{\alpha} + \lambda_{BCS} \hat{T}^{+} \hat{T}.$$



double occupied

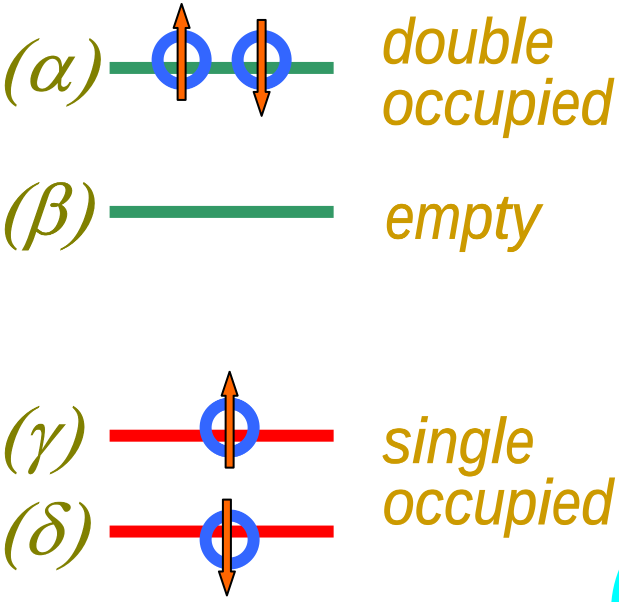
empty

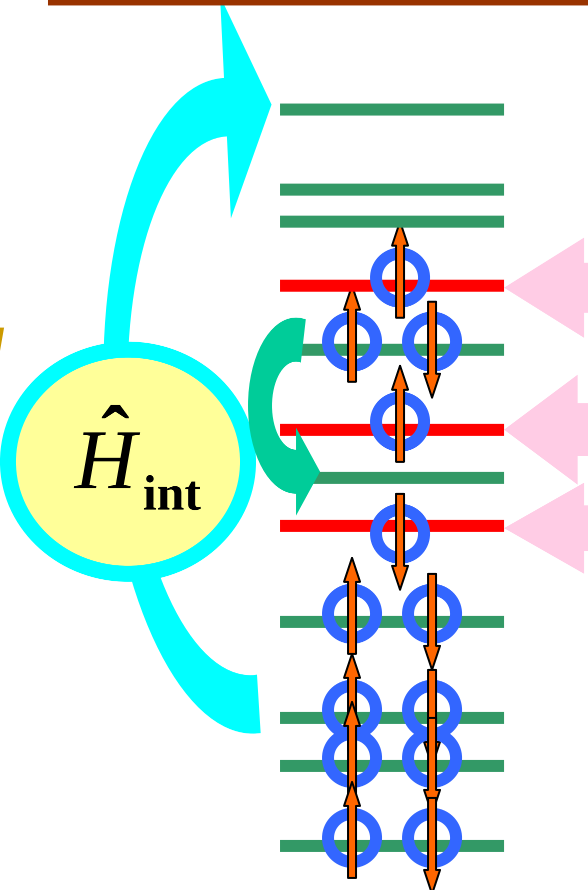
single occupied

$$\hat{H} = \sum_{\alpha} \varepsilon_{\alpha} n_{\alpha} + \lambda_{BCS} \hat{T}^{+} \hat{T}.$$

$a^{+}_{\alpha\uparrow} a^{+}_{\alpha\downarrow} a_{\beta\uparrow} a_{\beta\downarrow}$ mixes (α) and (β)

at the same time $a^{+}_{\alpha\uparrow} a^{+}_{\alpha\downarrow} a_{\beta\uparrow} a_{\beta\downarrow} (\gamma,\delta) = 0$





This single-occupied states are not effected by the interaction.

They are blocked

The Hilbert space is separated into two independent Hilbert subspaces

Blocking effect .

(V.G.Soloviev, 1961)

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Theory of Superconductivity*

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$$H_{\mathrm{red}}=2\sum_{k>k_F}\epsilon_{\mathbf{k}}b_{\mathbf{k}}^{*}b_{\mathbf{k}}+2\sum_{k<k_F}|\epsilon_{\mathbf{k}}|b_{\mathbf{k}}b_{\mathbf{k}}^{*}-\sum_{\mathbf{k}\mathbf{k}'}V_{\mathbf{k}\mathbf{k}'}b_{\mathbf{k}'}^{*}b_{\mathbf{k}}. \quad (2.14)$$

$$b_{\mathbf{k}}=c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow},$$
$$b_{\mathbf{k}}^{*}=c_{\mathbf{k}\uparrow}^{*}c_{-\mathbf{k}\downarrow}^{*}$$

One-particle energy

BCS interaction

The energy is measured relative to the Fermi level

BCS Hamiltonian; no single-occupied states

n one-particle levels; one-particle energies ε_α

$$\hat{H}_{BCS} = \sum_{\substack{0\le\alpha\le n-1 \\ \sigma=\uparrow,\downarrow}} \varepsilon_\alpha a^{+}{}_{\alpha,\sigma} a_{\alpha,\sigma} - \lambda_{BCS} \sum_{\substack{0\le\alpha,\beta\le n-1 \\ \alpha\ne\beta}} a^{+}{}_{\alpha,\uparrow} a^{+}{}_{\alpha,\downarrow} a_{\beta,\uparrow} a_{\beta,\downarrow}$$

Anderson spin chain

$$\hat{T}_\alpha^z = \frac{1}{2}\left(-1 + \sum_{\sigma=\uparrow,\downarrow} a^{+}{}_{\alpha,\sigma} a_{\alpha,\sigma}\right) \qquad \hat{T}_\alpha^+ = a^{+}{}_{\alpha,\uparrow} a^{+}{}_{\alpha,\downarrow} \qquad \hat{T}_\alpha^- = a_{\alpha,\uparrow} a_{\alpha,\downarrow}$$

SU_2 algebra

$$\hat{H}_{BCS} = \sum_{0\le\alpha\le n-1} \varepsilon_\alpha \hat{T}_\alpha^z - \lambda_{BCS} \sum_{0\le\alpha\ne\beta\le n-1} \hat{T}_\alpha^+ \hat{T}_\alpha^- = \sum_{0\le\alpha\le n-1} \varepsilon_p \hat{T}_p^z - \lambda_{BCS} \hat{L}_+ \hat{L}_-$$

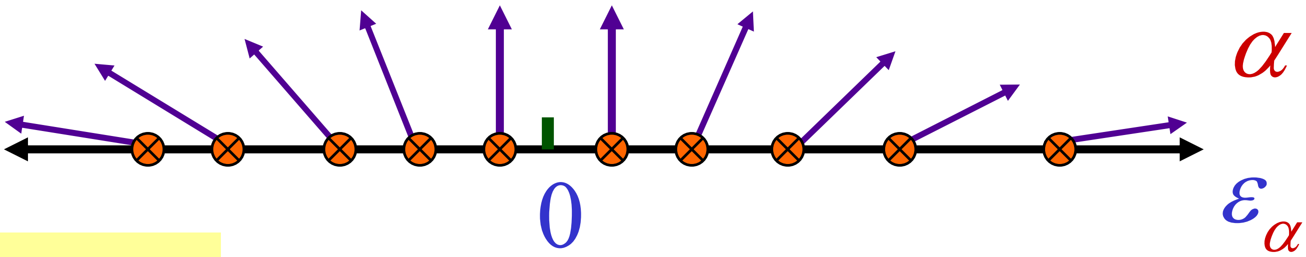
$$\hat{L}_\pm \equiv \sum_\alpha \hat{T}_\alpha^\pm$$

Anderson spin chain

$$\hat{T}_\alpha^z = \frac{1}{2}\left(-1 + \sum_{\sigma=\uparrow,\downarrow} a^+{}_{\alpha,\sigma} a_{\alpha,\sigma}\right) \qquad \hat{T}_\alpha^+ = a^+{}_{\alpha,\uparrow} a^+{}_{\alpha,\downarrow} \qquad \hat{T}_\alpha^- = a_{\alpha,\uparrow} a_{\alpha,\downarrow}$$

$$\vec{\hat{L}} \equiv \sum_\alpha \vec{\hat{T}}_\alpha$$

$$\boxed{\hat{H}_{BCS} = \sum_{0\le\alpha\le n-1} \varepsilon_\alpha \hat{T}_\alpha^z - \lambda_{BCS} \sum_{0\le\alpha\ne\beta\le n-1} \hat{T}_\alpha^+ \hat{T}_\alpha^- = \sum_{0\le\alpha\le n-1} \varepsilon_p \hat{T}_p^z - \lambda_{BCS} \hat{L}_+ \hat{L}_-}$$



$$\sum \hat{T}_\alpha^+ \equiv \frac{1}{\lambda_{BCS}} \hat{\Delta}$$

Superconducting order parameter

$$\sum \hat{T}_\alpha^z \equiv \hat{N}$$

Total number of the particles

$$\left[\hat{N}, \hat{\Delta}\right] \ne 0$$

Anderson spin chain

$$\hat{T}^z_\alpha = \frac{1}{2}\left(-1 + \sum_{\sigma=\uparrow,\downarrow} a^{+}{}_{\beta,\sigma} a_{\beta,\sigma}\right) \qquad \hat{T}^+_\alpha = a^{+}{}_{\alpha,\uparrow} a^{+}{}_{\alpha,\downarrow} \qquad \hat{T}^-_\alpha = a_{\alpha,\uparrow} a_{\alpha,\downarrow}$$

$$\hat{H}_{BCS} = \sum_{0\le\alpha\le n-1} \varepsilon_\alpha \hat{T}^z_\alpha - \lambda_{BCS} \sum_{0\le\alpha\ne\beta\le n-1} \hat{T}^+_\alpha \hat{T}^-_\alpha = \sum_{0\le\alpha\le n-1} \varepsilon_p \hat{T}^z_p - \lambda_{BCS} \hat{L}_+ \hat{L}_-$$

$$\sum \hat{T}^z_\alpha \equiv \hat{N}$$

Total number of the particles

For a fixed number of the particles (closed system we can add the term $-gN^2=const$ to the hamiltonian:

$$\hat{H}_{BCS} \sum_{0\le\alpha\le n-1} \varepsilon_\alpha \hat{T}^z_\alpha - \lambda_{BCS} \left(\hat{\vec{L}}\right)^2$$

$$\hat{\vec{L}} \equiv \sum_\alpha \hat{\vec{T}}_\alpha$$

$$\hat{H}_{BCS} \sum_{0\le\alpha\le n-1} \varepsilon_\alpha \hat{T}^z_\alpha - \lambda_{BCS}\left(\hat{\vec{L}}\right)^2$$

$$\hat{\vec{L}} \equiv \sum_{\alpha} \hat{\vec{T}}_\alpha$$

Integrable model

Richardson solution - Bethe Ansatz

$$\frac{-1}{\lambda_{BCS}} + \sum_{0\le\beta\le n-1} \frac{2}{E_\alpha - E_\beta} = \sum_{0\le\beta\le n-1} \frac{1}{E_\alpha - 2\varepsilon_\beta}$$

$$E = \sum_{0\le\alpha\le n-1} E_\alpha$$

n equations for the parameters E_p

How to describe dynamics in the time domain ?

Gaudin Magnets

$$\hat{H}_\alpha = 2 \sum_{\substack{0 \le \beta \le n-1 \\ \beta \ne \alpha}} \frac{\hat{\vec{T}}_\alpha \hat{\vec{T}}_\beta}{\varepsilon_\alpha - \varepsilon_\beta} - A\hat{T}_\alpha^z \qquad \alpha = 0,1,2,\ldots,n-1$$

$$\left[\hat{H}_\alpha, \hat{H}_\beta\right] = 0$$

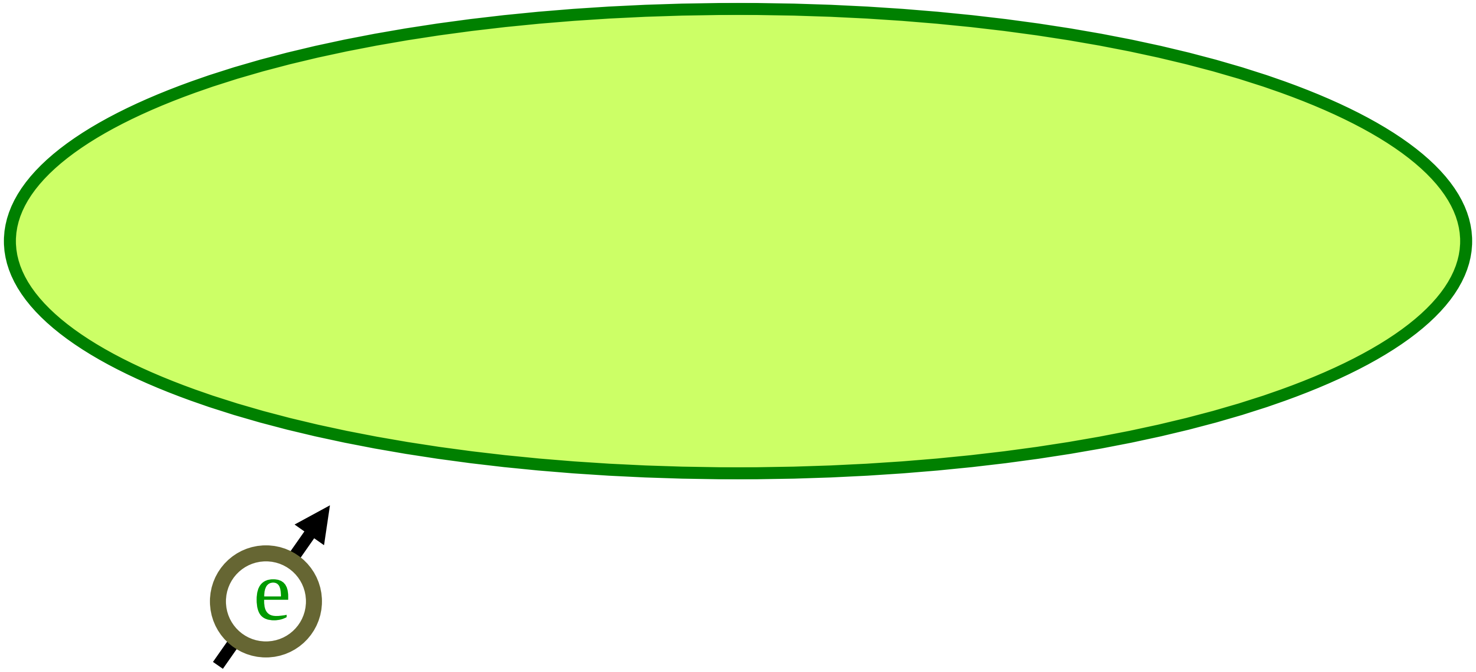
$\hat{H}_0$ – Hamiltonian

$\hat{H}_{\beta>0}$ – intgrals of motion

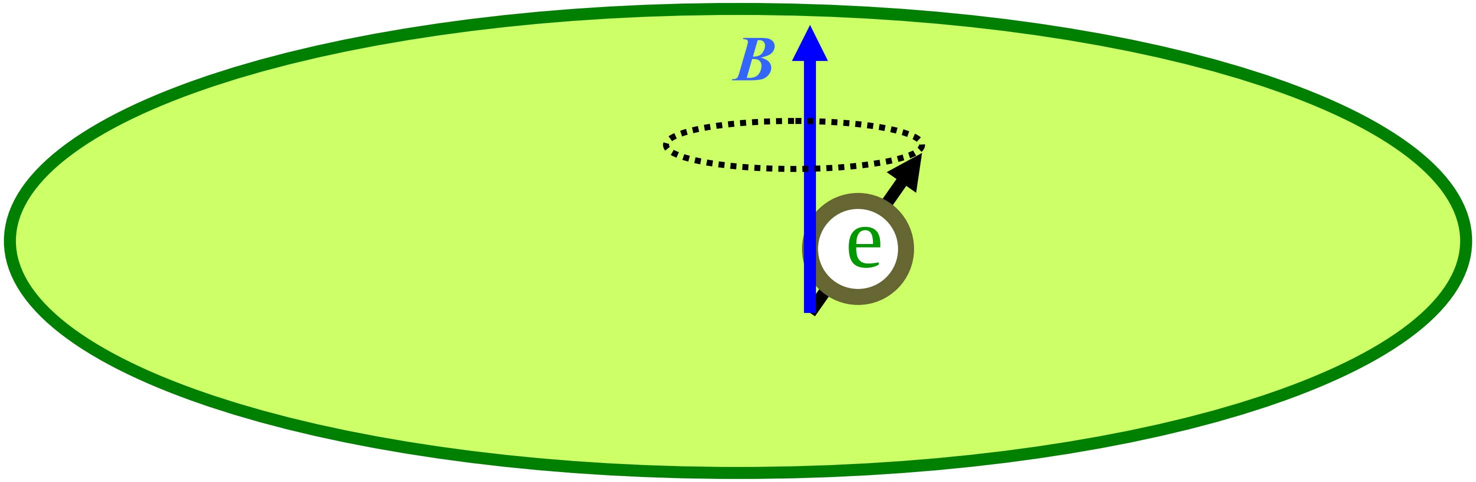
BCS Hamiltonian

$$\sum_{0 \le \alpha \le n-1} \varepsilon_\alpha \hat{H}_\alpha = \hat{H}_{BCS} + const \qquad \lambda_{BCS} = \frac{2}{A}$$

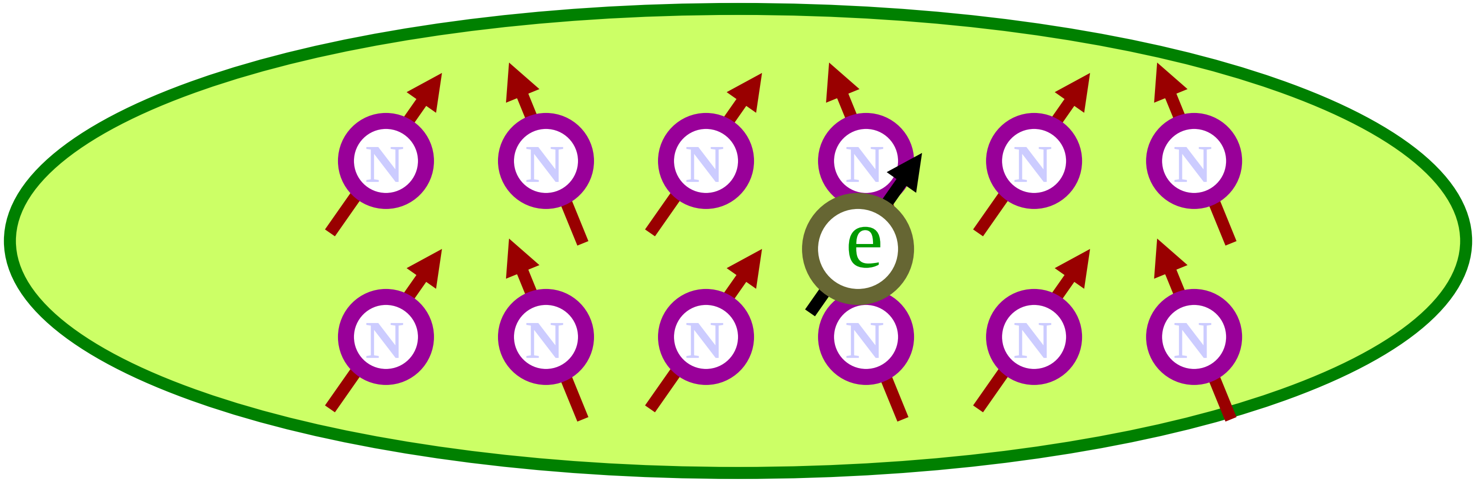
Overhauser interaction of electronic spins in quantum dots



Overhauser interaction of electronic spins in quantum dots



Overhauser interaction of electronic spins in quantum dots



- More than 10^6 of nuclear spins per electron
- Excange interaction of the electronic spin with the nuclear ones.
- Collective effect of the nuclei on the electron is pretty strong.
- No interaction between the nuclear spins

$$\boxed{\hat{H}_{eN} = \sum_{1\le\alpha\le n-1} \gamma_\alpha \hat{\vec{S}}_0 \hat{\vec{S}}_\alpha - B\hat{S}_0^z}$$

$$\gamma_\alpha \propto \left|\psi\left(\vec{r}_\alpha\right)\right|^2$$

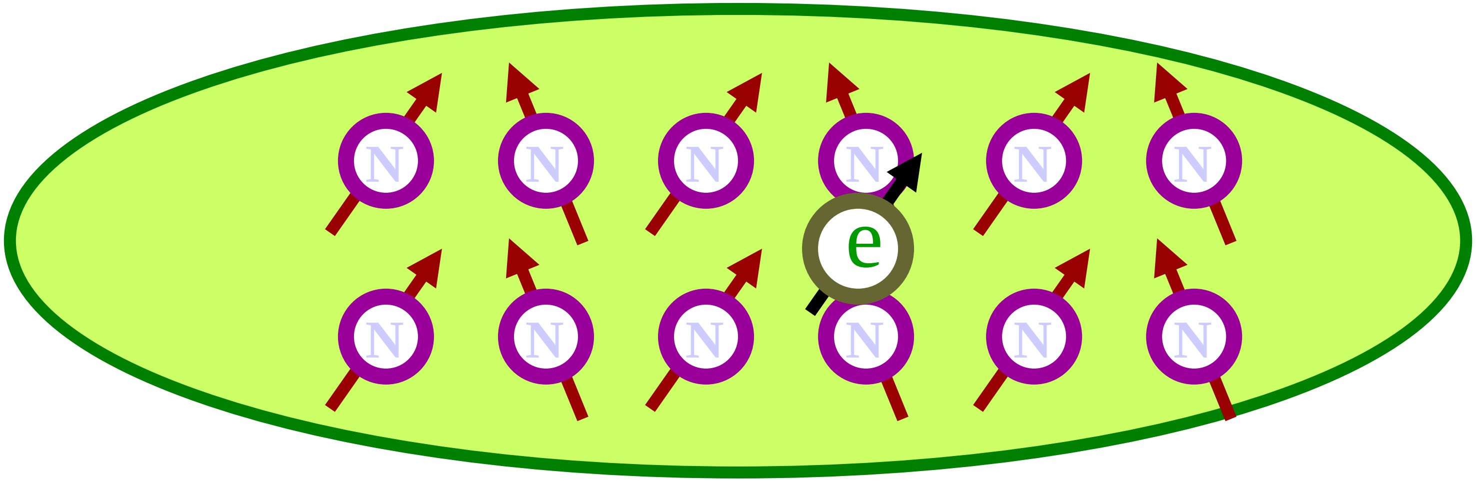
$\hat{\vec{S}}_0$ – *spin of the electron*

$\hat{\vec{S}}_{\alpha>0}$ – *nuclear spins*

$\psi_\alpha\left(\vec{r}\right)$ – *the electron w.f.*

B – *magnetic field* $\parallel z$

Overhauser interaction of electronic spins in quantum dots



$$\hat{H}_{eN} = \sum_{1\le\alpha\le n-1} \gamma_\alpha \hat{\vec{S}}_0 \hat{\vec{S}}_\alpha - B\hat{S}_0^z$$

Central spin problem

$$\hat{H}_\alpha = \sum_{0\le\beta\le n-1,\ \beta\ne\alpha} \frac{2\hat{\vec{T}}_\alpha \hat{\vec{T}}_\beta}{\varepsilon_\alpha - \varepsilon_\beta} - A\hat{T}_\alpha^z$$

Gaudin problem

$$\hat{H}_{eN} = \hat{H}_0$$

$$\hat{\vec{S}}_\alpha \Rightarrow \hat{\vec{T}}_\alpha$$

$$\gamma_\alpha \Rightarrow \frac{2}{\varepsilon_0 - \varepsilon_\alpha}$$

$$B \Rightarrow A$$

$$\hat{H}_{eN} = \sum_{1\le\alpha\le n-1} \gamma_\alpha \hat{\vec{S}}_0 \hat{\vec{S}}_\alpha - B\hat{S}_0^z$$

Central spin problem

$$\hat{H}_\alpha = \sum_{0\le\beta\le n-1,\ \beta\ne\alpha} \frac{2\hat{\vec{T}}_\alpha \hat{\vec{T}}_\beta}{\varepsilon_\alpha - \varepsilon_\beta} - A\hat{T}_\alpha^z$$

Gaudin problem

$$\hat{\vec{S}}_\alpha \Rightarrow \hat{\vec{T}}_\alpha$$

$$\hat{H}_{eN} = \hat{H}_0 \qquad \gamma_\alpha \Rightarrow \frac{2}{\varepsilon_0 - \varepsilon_\alpha}$$

$$B \Rightarrow A$$

Idea:

- Can we consider the classical dynamics of these Hamiltonians?
- Can it be described explicitly?
- What are connections between the classical and quantum dynamics?

Substitute quantum spin operators by classical vectors

$$\hat{\vec{T}}_\alpha \Leftarrow \vec{s}_\alpha$$

$$\hat{\vec{L}} \Leftarrow \vec{J}$$

Barankov, Levitov & Spivak cond-mat/0312053

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