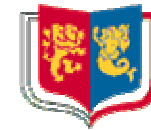


Nonlinear σ model for disordered systems: an introduction

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OUTLINE

- What is that & where to take it from?
- How to use it (1 or 2 simple examples)?
- Including interactions

3rd Windsor School on Condensed Matter Theory, 2004

What is that

$$F = \int d^d r \operatorname{Tr} \left[\frac{\pi \nu D}{8} (\nabla Q)^2 + \frac{i \pi \nu \omega}{4} \Lambda Q \right]$$

Nonlinearity condition:

$$Q^2 = 1 \quad \operatorname{Tr} Q = 0$$

How to get here – and where from?

How to use this?

Why to be bothered?

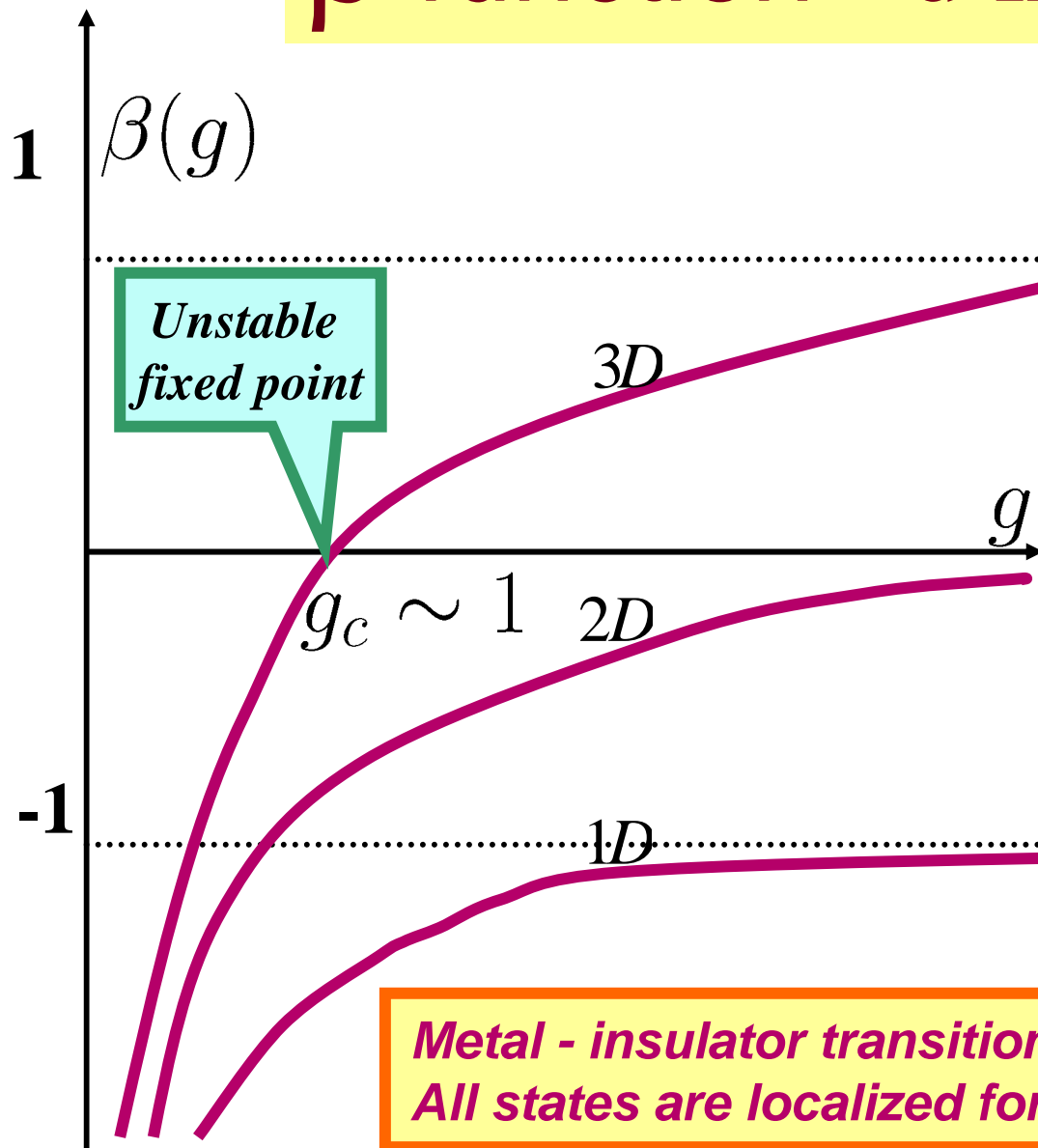
Why to be bothered?

The first application (1979-1981):

putting some foundation under the Gang-of-Four* speculations

***Abrahams, Anderson, Licciardello, and Ramakrishnan, 1979**

β -function $\equiv d \ln g / d \ln L$



Metal - insulator transition in 3D
All states are localized for 2D and 1D

Pert. Theory in 2D:

$$g/g_0 = 1 - g_0^{-1} \ln L/\ell$$

$$+ A g_0^{-2} \ln^2 L/\ell$$

RG within the NL σ M:

$$g/g_0 = 1 - g_0^{-1} \ln L/\ell$$

$$+ \dots A_n g_0^{-n} \ln^n L/\ell$$

$$+ A g^{-4} \ln L/\ell + \dots$$

And more ...

NLσM describes, apart from $g(L)$ itself

- mesoscopic distribution functions (of UCF *etc*)
- correlations of energy levels and wavefunctions
- long-time asymptotics of different observables
- statistics of rare events

and changing of the above with increasing disorder.


Also, it's a natural tool for describing some non-perturbative (ie non-analytic in g^{-1}) effects

- and more ...

Where to get it from?

Starting Point: The TOE model

THE THEORY OF EVERYTHING 

$$H = \sum_i \frac{p_i^2}{2m} + \sum_i V_i + \frac{1}{2} \sum_{ij} U_{ij}$$


interactions

lattice + disorder 

A field-theoretical approach :

- addressing only low-energy modes
- averaging over *weak* disorder

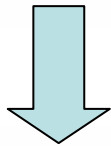
Start with a toy model (to get to the TOE one)

A generic, after diagonalization, 1-particle Hamiltonian

$$H = \sum_{\alpha} \varepsilon_{\alpha} a_{\alpha}^{\dagger} a_{\alpha} \quad \Longrightarrow \quad H |\alpha\rangle = \varepsilon_{\alpha} |\alpha\rangle$$

Green's function: $(\varepsilon - H) G = I$

$$\hat{G} = \left(\varepsilon - \hat{H} \right)^{-1} \sum_{\alpha} |\alpha\rangle \langle \alpha| = \sum_{\alpha} \frac{|\alpha\rangle \langle \alpha|}{\varepsilon - \varepsilon_{\alpha}}$$



$$G_{\varepsilon}^{\pm}(\mathbf{r}, \mathbf{r}') \equiv \langle \mathbf{r} | \hat{G}^{\pm} | \mathbf{r}' \rangle = \sum_{\alpha} \frac{\langle \mathbf{r} | \alpha \rangle \langle \alpha | \mathbf{r}' \rangle}{\varepsilon - \varepsilon_{\alpha} \pm i\delta} \equiv \sum_{\alpha} \frac{\varphi_{\alpha}^{*}(\mathbf{r}) \varphi_{\alpha}(\mathbf{r}')}{\varepsilon^{\pm} - \varepsilon_{\alpha}}$$

Gaussian Integrals

$$\int_{-\infty}^{\infty} dx e^{i(\varepsilon^+ - \varepsilon)x^2} = \sqrt{\frac{\pi}{i(\varepsilon^+ - \varepsilon)}} \implies \frac{1}{\varepsilon^+ - \varepsilon} = \int \frac{dc^* dc}{2\pi} e^{ic^*(\varepsilon^+ - \varepsilon)c}$$

Represent $\det(\varepsilon - H)^{-1} = \prod_{\alpha} (\varepsilon - \varepsilon_{\alpha})^{-1}$ as a Gaussian integral:

$$\begin{aligned} \det(\varepsilon - H)^{-1} &= \prod_{\alpha} \int \frac{dc_{\alpha}^* dc_{\alpha}}{2\pi} e^{ic_{\alpha}^*(\varepsilon^+ - \varepsilon_{\alpha})c_{\alpha}} \\ &\equiv \int \mathcal{D}c^* \mathcal{D}c e^{i \sum_{\alpha} c_{\alpha}^*(\varepsilon^+ - \varepsilon_{\alpha})c_{\alpha}} \end{aligned}$$

$\mathcal{D}c^* \mathcal{D}c$ is a symbolic notation for the product over all dc_{α}

Functional Integral

Transform the exp:

$$\begin{aligned} \sum_{\alpha} c_{\alpha}^{*}(\varepsilon^{+} - \varepsilon_{\alpha})c_{\alpha} &= \sum_{\alpha\beta} c_{\alpha}^{*}(\varepsilon^{+} - \varepsilon_{\alpha})c_{\beta} \int d^d r \varphi_{\alpha}^{*} \varphi_{\beta} \\ &= \int d^d r \underbrace{\sum_{\alpha} c_{\alpha}^{*} \varphi_{\alpha}^{*}}_{\psi^{*}(\mathbf{r})} (\varepsilon^{+} - \hat{H}) \underbrace{\sum_{\beta} c_{\beta} \varphi_{\beta}}_{\psi(\mathbf{r})} \equiv \underbrace{\int d^d r \psi^{*}(\mathbf{r}) (\varepsilon^{+} - \hat{H}) \psi(\mathbf{r})}_{\text{“Action” } S^{+}} \end{aligned}$$

“Partition function”:

$$Z^{+} \equiv \det(\varepsilon^{+} - \hat{H})^{-1} = \int \mathcal{D}\psi^{*} \mathcal{D}\psi e^{iS^{+}}$$

Integration over “all fields” means integration over all c & c^{*}

Green's Functions

obtained by variable's shift in a Gaussian integral: $x \rightarrow x + h/\alpha$

$$\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} dx e^{-\alpha x^2 + 2hx} = e^{h^2/\alpha}$$

Apply to the functional integral, a product of Gaussian ones:

$$Z_h^+ \equiv \int \mathcal{D}\psi^* \mathcal{D}\psi e^{iS + i(\psi^* \cdot h + h^* \cdot \psi)} = Z^+ e^{-ih^* \cdot G^+ \cdot h}, \text{ using}$$

$$\psi \mapsto \psi - G^+ \cdot h \quad \psi^* \mapsto \psi^* - h^* \cdot G^+ \quad (\epsilon^+ - \hat{H})G^+ = I$$

Thus,

$$\hat{G} = \frac{i}{Z} \left. \frac{\partial^2 Z_h}{\partial h \partial h^*} \right|_{\substack{h=0 \\ h^*=0}} = \frac{i}{Z} \int \psi^* \psi e^{iS} \mathcal{D}\psi^* \mathcal{D}\psi$$

Fermions do it like Bosons

... but upside down

$$Z^+ \equiv \det(\varepsilon^+ - \hat{H}) = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{iS^+}$$

$$\hat{G} = \frac{i}{Z} \int \bar{\psi} \psi e^{iS} \mathcal{D}\bar{\psi} \mathcal{D}\psi \quad S = \int d^d r \bar{\psi}(\mathbf{r})(\varepsilon - H)\psi(\mathbf{r})$$

For our 1-particle toy model both fermionic and bosonic representations are equivalent.

Having the TOE model as a target, we will deal with fermions from now on.

From the Toy to Anderson model

by diagonalizing

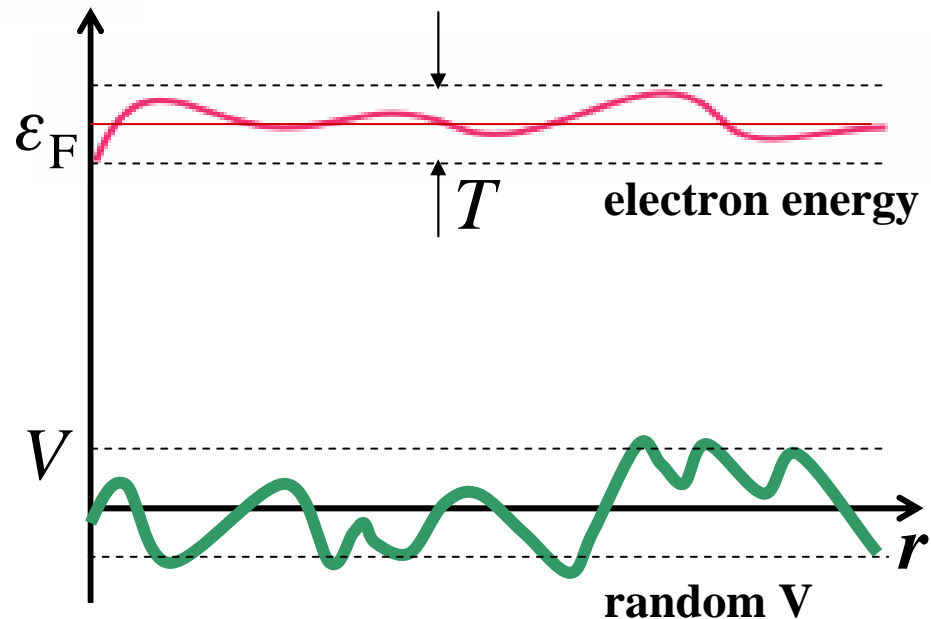
$$H = \sum_{\alpha} \varepsilon_{\alpha} a_{\alpha}^{\dagger} a_{\alpha} \quad \leftarrow H = \sum_i \varepsilon_i a_i^{\dagger} a_i + J \sum_{\langle ij \rangle} a_i^{\dagger} a_j$$

Gaussian white noise:

$$\langle V(\mathbf{r})V(\mathbf{r}') \rangle = \frac{1}{2\pi\nu\tau_{\text{el}}} \delta(\mathbf{r} - \mathbf{r}') \quad \leftarrow H = \frac{\hat{p}^2}{2m} + V(\mathbf{r})$$

A typical value of a short range ($r_c \sim \lambda_F$) potential is

$$V_{\text{typ}}^2 \sim \frac{1}{2\pi\nu\lambda_F^d\tau_{\text{el}}} = \frac{\varepsilon_F^2}{2\pi\varepsilon_F\tau_{\text{el}}}$$



Averaging Green's Function(s)

$$\langle G \rangle = \int G_V P(V) \mathcal{D}V \quad \text{with the white-noise Gaussian potential of the previous slide}$$

Explicit form of the Gaussian distribution:

$$P \{V(\mathbf{r})\} \sim \exp \left[-\pi \nu \tau_{\text{el}} \int d^d r V^2 \right]$$

Gaussian integration of G_V which is exp of V would be straightforward, if only G_V were not a fraction:

$$G_V = \frac{\int \bar{\psi} \psi e^{iS} \mathcal{D}\bar{\psi} \mathcal{D}\psi}{\int e^{iS} \mathcal{D}\bar{\psi} \mathcal{D}\psi}$$

$$S = \int d^d r \bar{\psi}(\mathbf{r}) [\varepsilon - \hat{\varepsilon}_p - V(r)] \psi(\mathbf{r})$$

Replica Method (or Trick)

$\bar{\psi} \mapsto (\bar{\psi}_1, \dots, \bar{\psi}_n) \equiv \bar{\Psi}$ n “replicas” of the field

Then
$$G = \frac{i \int \bar{\Psi} \cdot \Psi e^{iS_n} \mathcal{D}\bar{\Psi} \mathcal{D}\Psi}{Z_n}, \quad Z_n = Z_1^n$$

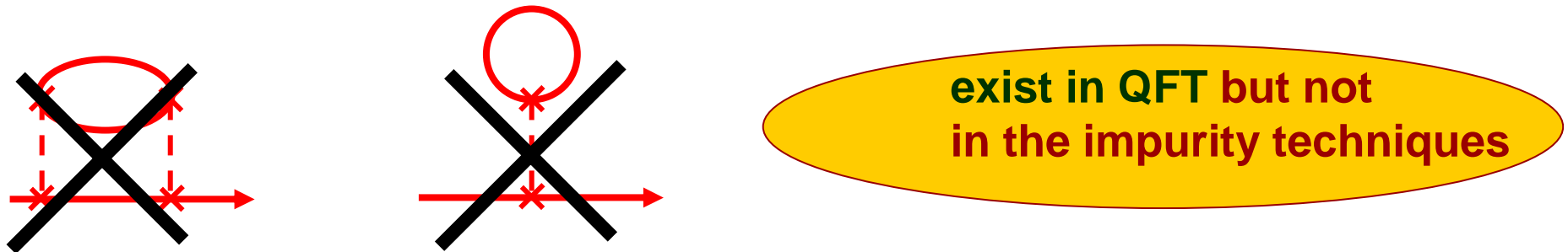
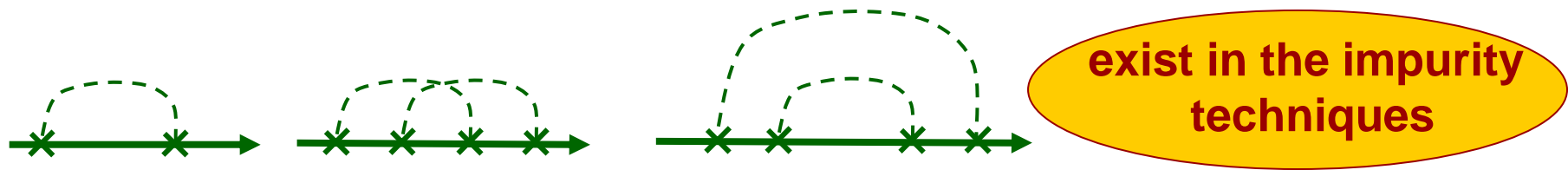
As $Z_n \rightarrow 1$ as $n \rightarrow 0$, in this “replica” limit the denominator of G can be averaged independently of the numerator:

$$\langle G \rangle = \lim_{n \rightarrow 0} \frac{1}{n} \frac{\langle \dots \rangle_V}{\langle \dots \rangle_V}$$
 The averaged 1-particle G is trivial, but $\langle GG \rangle$ is as easy:

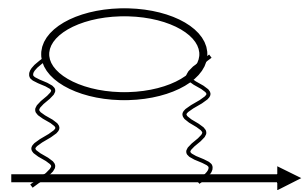
$$\left\langle G^+(\varepsilon + \frac{1}{2}\omega) G^-(\varepsilon - \frac{1}{2}\omega) \right\rangle = \lim_{n \rightarrow 0} \frac{1}{n^2} \frac{\langle \dots \rangle_V}{\langle \dots \rangle_V}$$

Why it works ?

Perturbatively, this is a method, not a trick. One expands S to get diagrams to be compared with those of direct diagrammatic technique

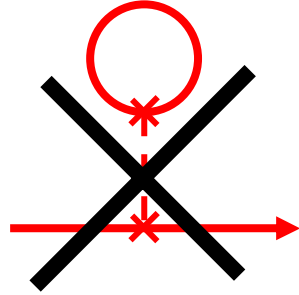
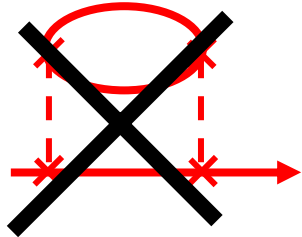


Killed by the replica trick: each closed loop has an extra n



An interaction loop does survive

Alternatives



ALSO KILLED IN SUSY TECHNIQUES AS
FERMION AND BOSON LOOPS HAVE
OPPOSITE SIGNS

NEVER APPEAR IN KELDysh TECHNIQUES

Why Replicas?

- SUSY – by far the best for non-perturbative calculations for **non-interacting** electrons **cannot** be generalised for interactions in any **meaningful** way
- Keldysh techniques would probably be better but replicas are considerably easier

How it works?

One just calculates the Gaussian integral:

$$\langle \dots \rangle_V \mapsto (\dots) \int \mathcal{D}V e^{-\int d^d r \left[i\bar{\Psi} \cdot V \cdot \Psi - \pi\nu\tau_{\text{el}} V^2 \right]} = (\dots) \exp \left[-\frac{\int d^d r (\bar{\Psi} \cdot \Psi)^2}{4\pi\nu\tau_{\text{el}}} \right]$$

Thus, one arrives at a quartic in Ψ action S . Particularly, in calculating the product $\langle G^+(\varepsilon + \frac{1}{2}\omega) G^-(\varepsilon - \frac{1}{2}\omega) \rangle$ the action is

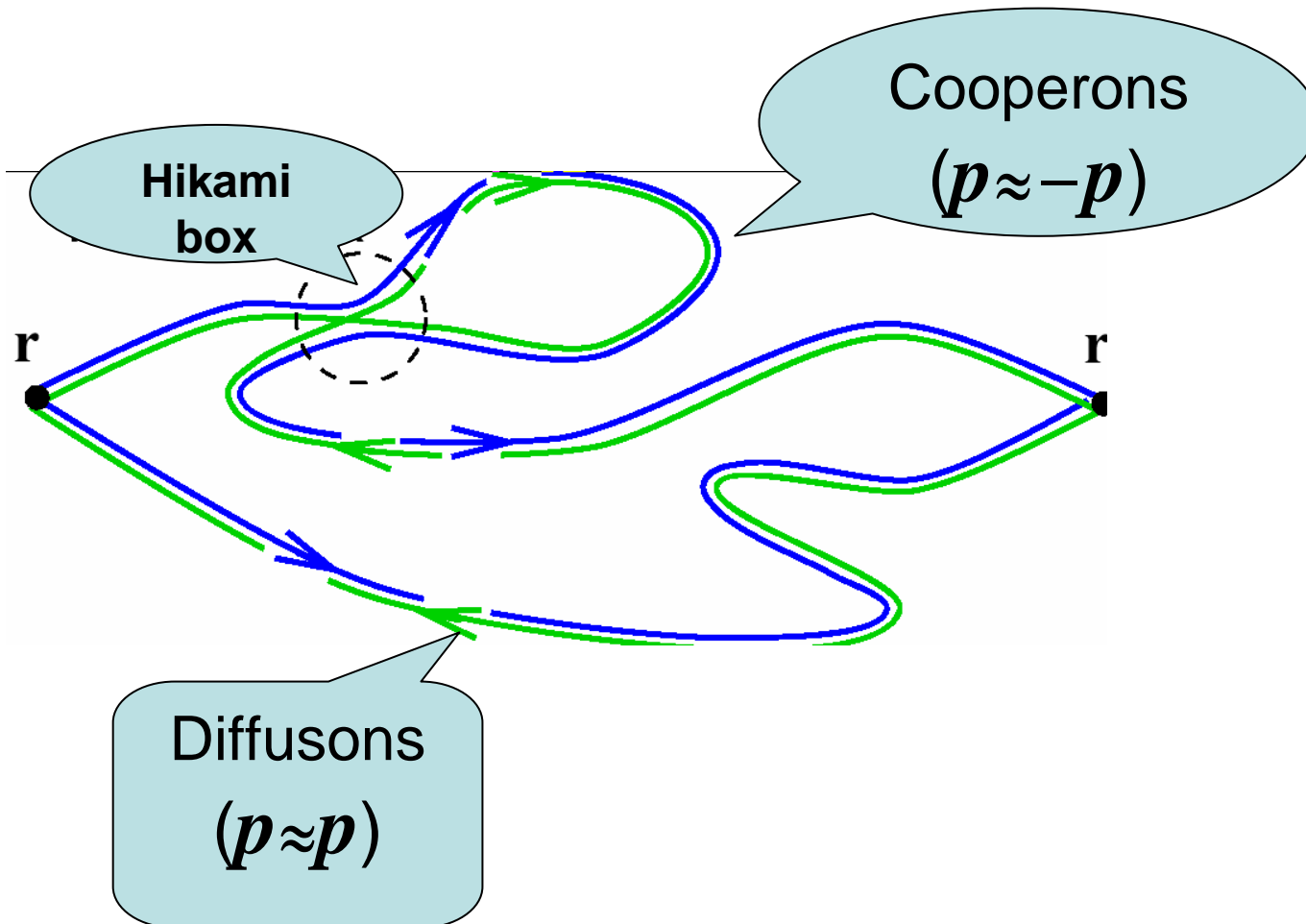
$$S = \int d^d r \left\{ \bar{\Psi} \left(-\hat{\xi} + \frac{1}{2}\omega\Lambda \right) \Psi - \frac{1}{4\pi\nu\tau_{\text{el}}} (\bar{\Psi} \cdot \Psi)^2 \right\}$$

$\xi \equiv \varepsilon_{\mathbf{p}} - \varepsilon_{\text{F}}$, $\Lambda = \text{diag}(\mathbf{1}_n, -\mathbf{1}_n)$, and Ψ is 2n-component:

$$\bar{\Psi} = (\bar{\psi}_1^+, \dots, \bar{\psi}_n^+, \bar{\psi}_1^-, \dots, \bar{\psi}_n^-)$$

Slow Modes

The full action is too complicated. A proper FT describes interacting slow modes that should be extracted from $(\overline{\Psi} \cdot \Psi)^2$



Hubbard-Stratonovich transformation

$q \equiv \frac{2}{\pi\nu} \psi \otimes \psi^\dagger$ – the “disorder” field that includes both slow channels: cooperon and diffuson

To get rid of $-\frac{1}{4\pi\nu\tau_{el}} (\bar{\Psi} \cdot \Psi)^2 \propto -\text{tr } q^2$, use

$$\int_{-\infty}^{\infty} e^{-Q^2 + 2qQ} dQ = e^{-q^2} \int_{-\infty}^{\infty} e^{-Q^2} dQ$$

Applying to the **matrix field** q above gives

$$Z = \int e^{iS} \mathcal{D}Q \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \quad Q \text{ is } 2n \times 2n \text{ matrix field and}$$

$$iS \mapsto \int d^d r \text{Tr} \left\{ -\frac{\pi\nu}{8\tau_{el}} Q^2 + i\bar{\Psi} \left[-\hat{\xi} + \frac{1}{2}\omega\Lambda + \frac{i}{2\tau_{el}} Q \right] \Psi \right\}$$

Effective Functional

Integrate out Ψ using $\det A = \int \mathcal{D}\bar{\Psi} \cdot \Psi \exp\{i\bar{\Psi}A\Psi\}$

The remarkable identity $\det A = \exp[\text{tr} \ln A]$ gives $iS \rightarrow -F$ with

$$F = \frac{\pi\nu}{8\tau} \text{Tr} Q^2 - \frac{1}{2} \text{Tr} \ln \left[-\hat{\xi} + \frac{1}{2}\omega\Lambda + \frac{i}{2\tau}Q \right], \quad Z_n = \int \mathcal{D}Q e^{-F}$$

the saddle-point approximation at $\omega=0$:

$$Q = \frac{2}{\pi\nu} \int \frac{d^d p}{(2\pi)^d} \left(-\hat{\xi} + \frac{i}{2\tau_{\text{el}}} Q \right)^{-1} \Rightarrow \begin{array}{l} Q = U^\dagger \Lambda U \\ \Lambda = \text{diag}(\mathbf{1}_n, -\mathbf{1}_n) \end{array}$$

This is equivalent to

$$Q^2 = 1 \quad \text{Tr} Q = 0$$

It makes the first term in F irrelevant (const), leaving one to deal only with $\text{Tr} \ln (\dots)$

Gradient Expansion

Substitute $Q=U^+\Lambda U$ into $\text{Tr} \ln (U \dots U^+)$

$$F = -\frac{1}{2} \text{Tr} \ln \left[\underbrace{-\hat{\xi} + \frac{i}{2\tau} \Lambda}_{G_0^{-1}} - U \left[\hat{\xi}, U^+ \right] + \omega U \Lambda U^\dagger \right]$$

solution to the saddle-point approximation

Since $\xi \equiv \hat{p}^2/2m - \mu \approx v_F \mathbf{n} \cdot \nabla$, one expands this in powers of ∇Q and ω , with expansion parameters $q\ell$ and $\omega\tau$

$$F = \int d^d r \text{Tr} \left[\frac{\pi\nu D}{8} (\nabla Q)^2 + \frac{i\pi\nu\omega}{4} \Lambda Q \right]$$

The lowest nonvanishing orders of the expansion

Finally, the nonlinear σ model

Limits of validity

Saddle-point + gradient expansion are legitimate provided that **min** is deep enough (dimensionless $F \gg 1$):

**Dimensionless
conductance** $g_0 \equiv \nu D L^{d-2} \gg 1$

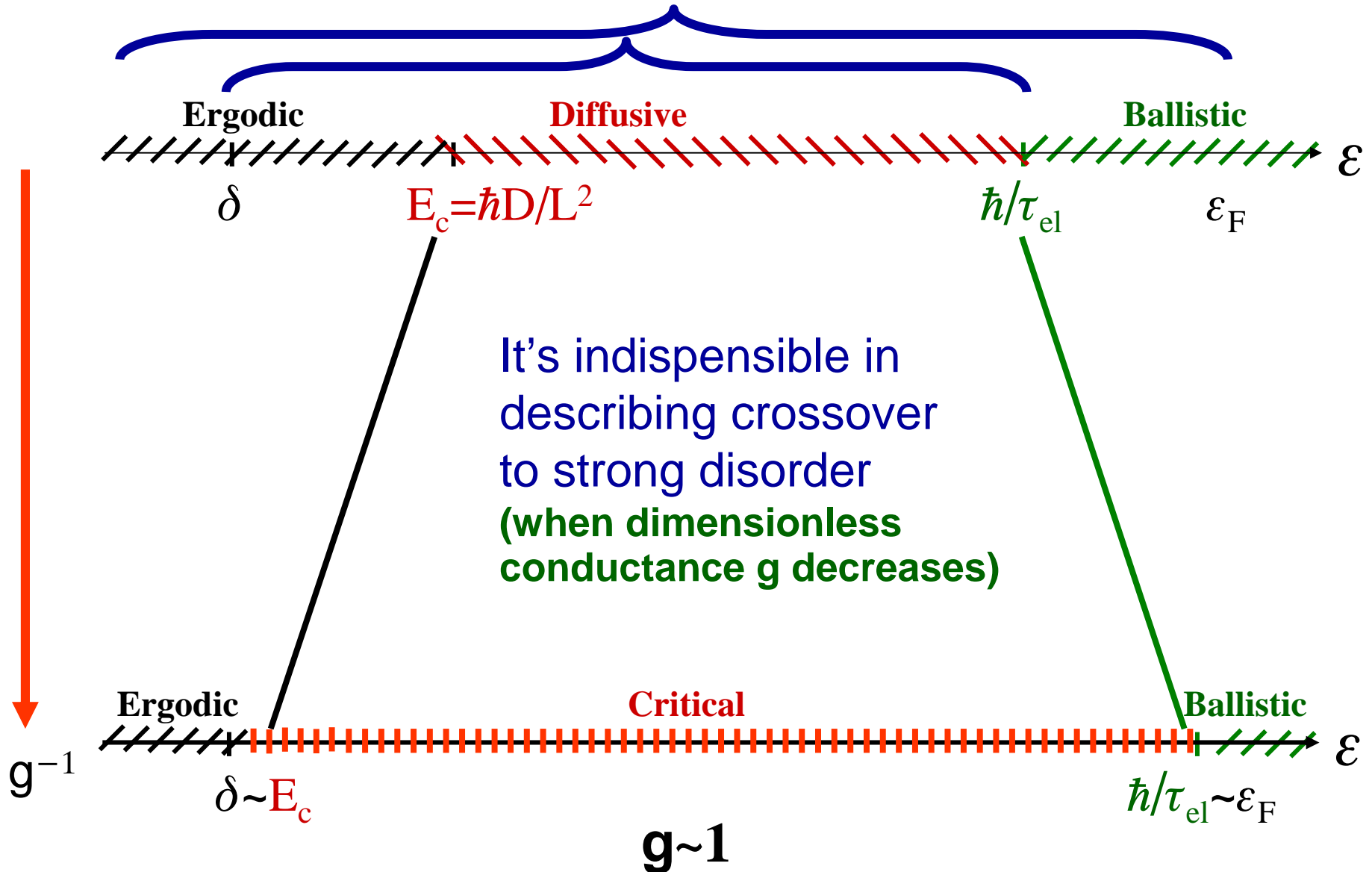
Massive modes ($Q^2 \neq 1$) can be neglected at the same condition that justifies the gradient expansion:

$$q\ell \gg 1$$

This variant of the NL σ M is not applicable to the ballistic regime, $L < \ell$

Regions of applicability

Ballistic SUSY NL σ M



Symmetry classes

<u>Matrix elements</u>	<u>Ensemble</u>	β	<u>Realization in NLσM</u>
real	orthogonal	1	symplectic space
complex	unitary	2	unitary space
<i>quarternions</i> (2×2 matrices)	symplectic	4	orthogonal space

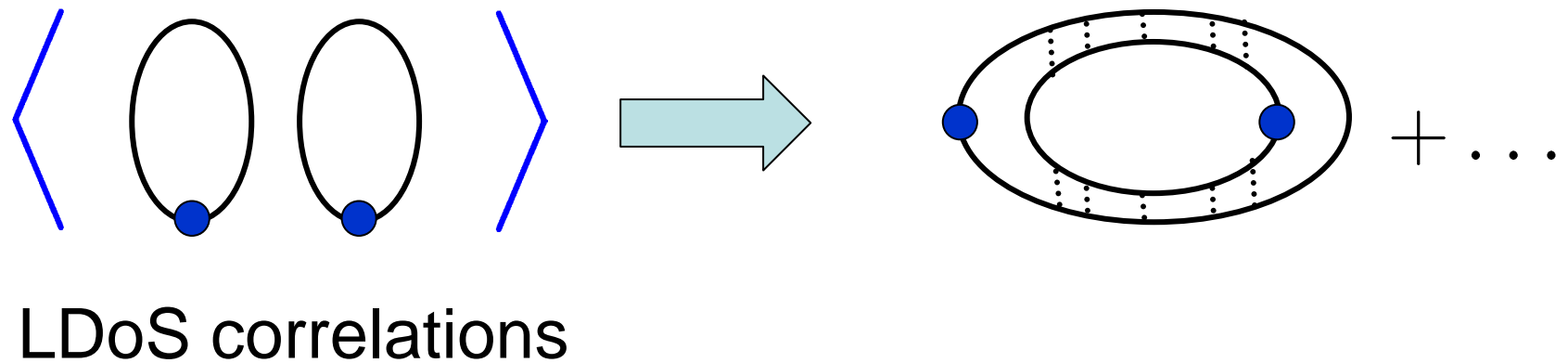
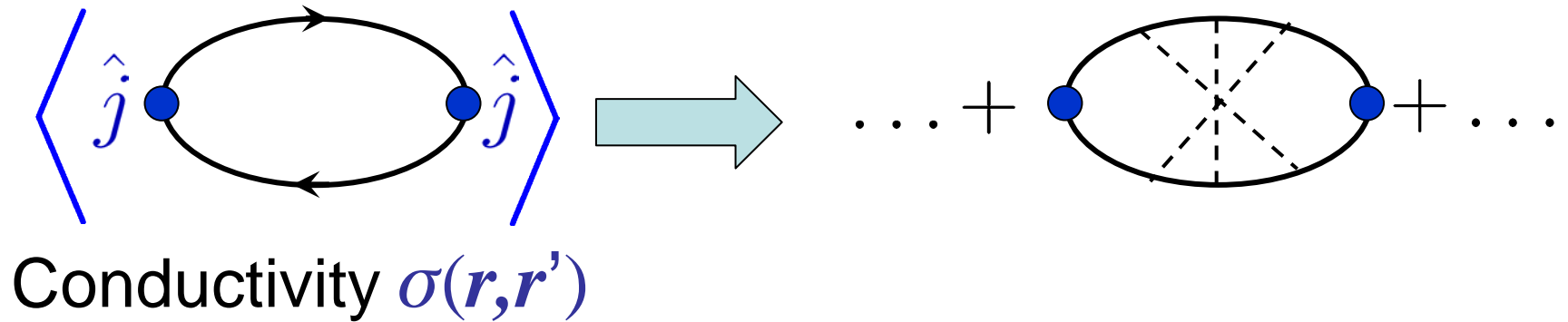
Diagrammatics in NLoM

Lecture 2: OUTLINE

- Observable quantities in NLoM
- Parameterization & diagrams
- Level Statistics
- Beyond perturbations with replicas

Observable quantities

Response functions



Observables & source fields

A product of any number of Green functions can be written as a pre-exponential factor in the ψ - $\bar{\psi}$ field theory

A better alternative is to introduce source fields that allow one to exponentiate these factors “in groups”:

$$\begin{array}{c} \swarrow \\ \bullet \\ \searrow \\ \text{wavy} \end{array} \equiv \bar{\psi} h \psi \mapsto \text{tr} (h \cdot \psi \otimes \bar{\psi}) \mapsto \left. \frac{\partial}{\partial h} e^{\text{tr} (h \cdot \psi \otimes \bar{\psi})} \right|_{h=0}$$

Any number of such groups is obtained by repeated dif. :

$$\begin{array}{c} \curvearrowright \\ \bullet \quad \bullet \\ \curvearrowleft \end{array} \propto \left. \frac{\delta^2}{\delta h^2} e^{iS + \text{tr} (h \cdot \psi \otimes \bar{\psi})} \right|_{h=0}$$

As $\psi \otimes \bar{\psi} \rightarrow Q$, this structure is preserved in NL σ M

Conductance in NL σ M

$\sigma(\mathbf{r}, \mathbf{r}')$ is a response to an external \mathbf{E} field introduced by

$$\mathbf{p} \rightarrow \mathbf{p} - e\mathbf{A}_\omega \quad \Longrightarrow \quad \nabla \rightarrow \partial \equiv \nabla - e\mathbf{A}$$

Current density $\mathbf{j} \propto \delta/\delta\mathbf{A}$ so that $\sigma(\mathbf{r}, \mathbf{r}')$ is given by

$$\left\langle \sigma_{\alpha\beta}(\mathbf{r}, \mathbf{r}') \right\rangle = \frac{1}{16\pi n^2} \frac{\partial^2 \ln Z[\mathbf{A}]}{\partial A_\alpha(\mathbf{r}) \partial A_\beta(\mathbf{r}')} \Bigg|_{\substack{\mathbf{A}=0 \\ n=0}} \quad \text{where}$$

$$Z[\mathbf{A}] \equiv \int \mathcal{D}Q e^{-F[\mathbf{A}]}, \quad F[\mathbf{A}] = \text{Tr} \left[\frac{\pi\nu D}{8} (\partial Q)^2 + \frac{i\pi\nu}{4} \omega \Lambda Q \right]$$

$$\text{Finally, } G_{\alpha\beta} = \frac{1}{L^2} \int \sigma_{\alpha\beta}(\mathbf{r}, \mathbf{r}') d^d r d^d r'$$

DoS in NL σ M

Since DoS by itself is not affected by disorder, we consider DoS correlation function

$$R_2(\omega) = \frac{1}{\langle \nu \rangle^2} \langle \nu(\varepsilon + \omega) \nu(\varepsilon) \rangle - 1$$

Although no source field is required, it's better to have one

$$R_2(\omega) = \frac{1}{(\pi n L^d)^2} \Re \left. \frac{\partial^2 \ln Z[\Omega]}{\partial \Omega^2} \right|_{\substack{\Omega = 0 \\ n = 0}}, \quad F[\Omega] = F + \frac{i\pi\nu\Omega}{4} \text{Tr } \Lambda Q$$

By $\Omega \rightarrow \Omega(\mathbf{r})$ this can be generalised to a spatial correlation function of two LDoS

How to carry out calculations?

Parameterization

First one resolves constraints $Q^2=1$ & $\text{Tr } Q=0$. Examples:

$$Q = \Lambda e^W \text{ or } Q = \Lambda \left(W + \sqrt{1 + W^2} \right) \text{ or } Q = \left(1 - \frac{W}{2} \right) \Lambda \left(1 - \frac{W}{2} \right)^{-1}$$

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \mathbb{I}_n, \quad W = \begin{pmatrix} 0 & B_n \\ -B_n^\dagger & 0 \end{pmatrix} \quad B_n \text{ is unconstrained } n \times n \text{ matrix field}$$

Expanding Q in W , one finds $F=F_0(W)+\delta F(W)$ where

$$F_0 = -\frac{\pi\nu}{8} \text{Tr} \left[D (\nabla W)^2 - i\omega W^2 \right]$$

$$\delta F = \frac{i\pi\nu}{8} \text{Tr} \Omega W^2 + \text{higher powers in } W$$

Note that higher powers depend on the parameterization!

How it works perturbatively

- Expand e^{-F} in δF and δF in W
- Calculate Gaussian integrals $\langle (\delta F)^m \rangle_0$

Omitted matrix indices are vital for taking the $n=0$ replica limit

$$\langle WW \rangle_0 \equiv \frac{1}{Z} \int WW e^{-F_0} \propto \frac{1}{Dq^2 - i\omega} \equiv \text{wavy line}$$

In the lowest order $\delta F_\Omega = \frac{i\pi\nu}{8} \text{Tr} \Omega W^2 \Rightarrow \Omega \bullet \begin{array}{c} W \\ \triangle \\ W \end{array}$

In this order R_2 is contributed by $\frac{1}{2} \langle (\delta F_\Omega)^2 \rangle_0$:

$$R_2 = \Omega \bullet \begin{array}{c} W \\ \triangle \\ W \end{array} \begin{array}{c} W \\ \triangle \\ W \end{array} \bullet \Omega \iff \text{Diagram with two blue dots and two concentric ovals with vertical dashed lines}$$

R_2 in the diffusive regime

$$R_2(\omega) = \frac{\Delta^2}{\pi^2} \Re \sum_{\mathbf{q}} \frac{1}{(D\mathbf{q}^2 - i\omega)^2}, \quad \Delta \equiv \frac{1}{\langle \nu \rangle L^d}$$

In a finite system, $\mathbf{q} = (2\pi/L)(n_x, n_y, n_z)$, integers $n_i \geq 0$

Dif. regime: $q^2 \ll \frac{\omega}{D} \equiv L_\omega^{-2} \Leftrightarrow E_c \equiv \frac{D}{L^2} \ll \omega, \Rightarrow \sum_{\mathbf{q}} \rightarrow L^d \int \frac{d^d q}{(2\pi)^d}$

$$R_2(\omega) \sim \frac{L^{d-4}}{g^2} \Re \int \frac{d^d q}{(q^2 - iL_\omega^{-2})^2} \simeq \frac{C_d}{g^2} \left(\frac{L_\omega}{L} \right)^{4-d} \equiv \frac{C_d}{g^{d/2}} \left(\frac{\Delta}{\omega} \right)^{2-d/2}$$

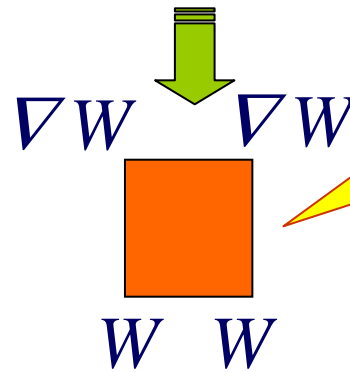
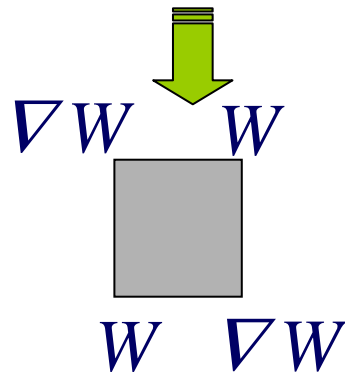
For $d=2$, this does not work:

$$\Re \int \frac{d^d q}{(q^2 - iL_\omega^{-2})^2} = 2\pi \Re \int_0^\infty \frac{q dq}{(q^2 - iL_\omega^{-2})^2} = \pi \Re \int_{-\infty}^\infty \frac{dz}{(z - iL_\omega^{-2})^2} = 0$$

Higher orders

$$\delta F_{\Omega} = \frac{i\pi\nu}{8} \left[\text{Tr} \Omega W^2 - \frac{1}{2} \text{Tr} \Omega W^4 \right] + \dots \implies \Omega \bullet \begin{array}{c} W \\ \triangle \\ W \end{array} + \Omega \bullet \begin{array}{c} W \quad W \\ \square \\ W \quad W \end{array} + \dots$$

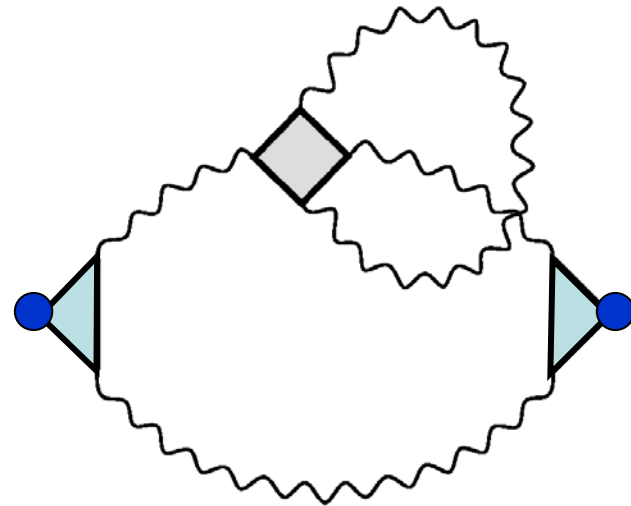
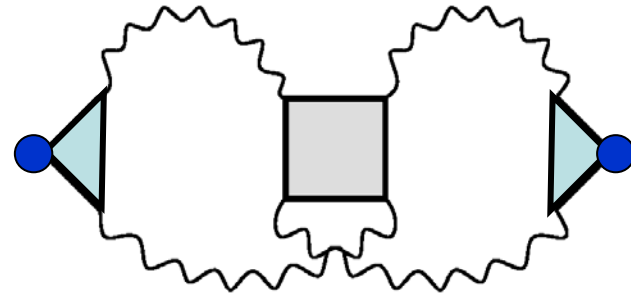
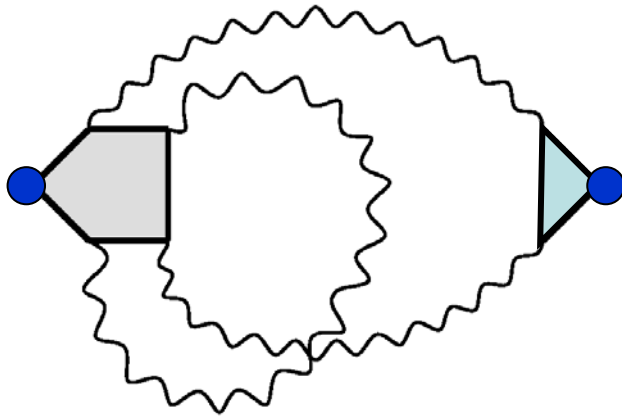
$$\delta F_{(\nabla Q)^2} = \frac{\pi\nu D}{16} \left[\text{Tr} (\nabla W W^2) - \text{Tr} (\nabla W)^2 W^2 \right] + \dots$$



does not contribute in the 2nd order

Contributions come from $\langle (\delta F_{\Omega})^2 \rangle_0$ and $\langle (\delta F_{\Omega})^2 \delta F_{(\nabla Q)^2} \rangle_0$. Both replica limit and angular integration severely cut the number of contributing diagrams

R_2 in the diffusive regime in 2D



Results:

$$R_2(\omega) \sim \frac{1}{g^2} \left(\frac{\Delta}{\omega} \right)$$

R_2 in the ergodic regime, $\omega \ll E_c$

$$R_2(\omega) = \frac{\Delta^2}{\beta\pi^2} \Re \sum_q \frac{1}{(Dq^2 - i\omega)^2} \mapsto -\frac{\Delta^2}{\beta\pi^2\omega^2}$$

Higher order perturbative corrections $\sim (\Delta/\omega)^m$:

Perturbation theory breaks down for $\omega \ll \Delta$

It is not that good at $E_c \gg \omega \gg \Delta$ either. Exact result

$$R_2(\omega) = - \left(\frac{\Delta}{\omega}\right)^2 \sin^2 \left(\frac{\omega}{\Delta}\right)^2 \quad \begin{array}{l} \text{unitary case} \\ \beta=2 \end{array}$$

Non-analytic in $\Delta/\omega \Rightarrow$ nonperturbative.

Exploring Replica Symmetry

- For $\omega < E_c$ the NL σ M contains the ω term only:

$$Z_n(\omega) = \int \mathcal{D}Q \exp \left[-i \frac{\omega}{2} \text{Tr} \Lambda Q \right]$$

- The saddle-point: Q obeys $[Q, \Lambda] = 0$, *i.e.* Q is a diagonal matrix with equal (as $\text{Tr} Q = 0$) number of ± 1 elements.
- \therefore the saddle-point is highly degenerate.

$$Z_n(\omega) = \sum_{p=0}^n (C_n^p)^2 \int \mathcal{D}Q \exp \left[-i \frac{\omega}{2} \text{Tr} \Lambda_p Q \right]$$

$$\Lambda_p = \text{diag}(1_{n-p}, -1_p, 1_p, -1_{n-p})$$

- Normal choice $p=0$; any p gives the same perturbatively

Breaking Replica Symmetry

- Calculating the integrals yields

$$Z_n(\omega) = \sum_{p=0}^n [F_n^p]^2 \cdot \frac{e^{i\omega(2p-n)}}{(2\omega)^{(n-p)^2+p^2}} \quad F_n^p \equiv C_n^p \prod_{j=1}^p \frac{\Gamma(1+j)}{\Gamma(n+2-j)}$$

- Symmetry is broken by extending summation to $p=\infty$ (as $F_n^p=0$ for $n>p$), and **taking the replica limit $n \rightarrow 0$** .

- It works: $F_n^p \propto n^p$ as $n \rightarrow 0$ and only $p=0,1$ terms contribute:

$$Z_n(\omega) = \frac{e^{-i\omega n}}{\omega n^2} + n^2 \frac{e^{i\omega(2-n)}}{4\omega(n-1)^2+1}$$

gives the exact result for S_2 and thus R_2 .

Breaking replica symmetry leads to correct non-perturbative results

Summary

- It's up and running
- It's much easier to use than to derive, but this is much easier to declare than to demonstrate

Next time

- Including interactions (Coulomb and BCS)
- Mapping to various known models
- Describing superconductor-insulator transition

Announced at the last lecture:

- Including interactions (Coulomb and BCS)
- Mapping to various known models
- Describing superconductor-insulator transition

Lecture 3: changing gears to

Functional bosonization for Luttinger liquid

Hubbard-Stratonovich transformation: change from fermionic to bosonic representation

$$\mathcal{S} = \int d\mathbf{x} \bar{\psi}_\eta(\mathbf{x}) (i\partial_\tau + \hat{\xi} - V(\mathbf{r})) \psi_\eta(\mathbf{x}) \quad (\mathbf{x} \equiv \mathbf{r}, \tau)$$
$$+ \frac{1}{2} \int d\mathbf{x} d\mathbf{x}' \bar{\psi}_\eta(\mathbf{x}) \bar{\psi}_{\eta'}(\mathbf{x}') V_0(\mathbf{x} - \mathbf{x}') \psi_{\eta'}(\mathbf{x}') \psi_\eta(\mathbf{x})$$

Note that we use the linearized spectrum

$$\xi = \frac{p^2}{2m} - \varepsilon_F \approx v_F (|p| - p_F)$$

but in $d > 1$ it does not split e's into 2 species

$$\frac{1}{2} \text{Tr} \left(\Phi U^{-1} \Phi + \frac{\pi\nu}{4\tau_{\text{el}}} Q^2 \right) - \frac{1}{2} \text{Tr} \ln \left(-i\partial_\tau - \hat{\xi} + \frac{iQ}{2\tau_{\text{el}}} + i\Phi \right)$$

NL σ M would follow from the saddle point approximation

Pure Interaction in 1D

$$H = H_0 + H_{\text{int}}, \quad H_0 = \int dx \hat{\psi}_\eta^\dagger(x) \hat{\xi} \hat{\psi}_\eta(x)$$

$$H_{\text{int}} = \frac{1}{2} \int dx dx' \hat{\psi}_\eta^\dagger(x) \hat{\psi}_{\eta'}^\dagger(x') V_0(x - x') \hat{\psi}_{\eta'}(x') \hat{\psi}_\eta(x)$$

In the simplest spinless case, keeping only forward-scattering local interaction, one gets from a linearised Tomonaga-Luttinger Hamiltonian a naïve bosonic one:

$$H = \int_{-L/2}^{L/2} dx \left[\frac{v}{4} \left(\frac{d}{dx} \tilde{\rho} \right)^2 + g \tilde{\rho}^2 \right]$$

repulsion: $g < 1$; free Fermions $g = 1$ (see Giamarchi's lectures)

Hubbard-Stratonovich in $D > 1$ is an entirely different “bosonisation” procedure than the operator one in 1D

The aims of this talk

- To introduce a “functional bosonisation” for the Luttinger liquid in the spirit of the approach used in the derivation of NL σ M
- As an application, to derive the LDoS $\nu(\varepsilon, x)$ at an arbitrary distance x from a single impurity (“end”) in the Luttinger liquid

Effective Functional

$$S[\psi^*, \psi] = \int d\xi \left[\psi_\eta^*(\xi) \partial_\tau \psi_\eta(\xi) - H(\psi^*, \psi) \right]$$

No spin!; $\eta = (L, R) \equiv \pm$, $\xi \equiv (x, \tau)$; $\int d\xi \equiv \int_{-L/2}^{L/2} dx \int_0^\beta d\tau$

H is obtained from the Hamiltonian by substituting fermionic operators by continuous fields in the Matsubara representation

$$Z = \int e^{-S[\psi^*, \psi]} \mathcal{D}\psi^* \mathcal{D}\psi$$

Hubbard-Stratonovich transformation

Introduce $\phi \propto \rho$ – the bosonic field to decouple the ρ^2 term

$$e^{-\frac{1}{2}\rho^*V_0\rho} = \frac{\int \mathcal{D}\phi e^{i\rho^*\phi} e^{-\frac{1}{2}\phi^*V_0^{-1}\phi}}{\int \mathcal{D}\phi e^{-\frac{1}{2}\phi^*V_0^{-1}\phi}}$$

This results in the effective action (with $\partial_{\pm} = \partial_{\tau} \mp i v_F \partial_x$)

$$S[\phi, \psi] = \int d\xi \psi_{\eta}^*(\xi) (\partial_{\eta} - i\phi) \psi_{\eta}(\xi) + \frac{1}{2} \int d\xi d\xi' \phi(\xi) V_0^{-1}(\xi - \xi') \phi(\xi')$$

Fermion-Boson Decoupling

Gauge transform eliminates the mixed term in the derivative

$$\psi_\eta \mapsto \psi_\eta e^{i\theta_\eta}, \quad \psi_\eta^* \mapsto \psi_\eta^* e^{-i\theta_\eta}, \quad \partial_\eta \theta_\eta = \phi$$

Thus, fermions and bosons decouple

$$S_b = \frac{1}{2} \int d\xi d\xi' \phi(\xi) V^{-1}(\xi - \xi') \phi(\xi')$$

$$S_f = \int d\xi \psi_\eta^*(\xi) \partial_\eta \psi_\eta(\xi)$$

Where is the trick? $V \neq V_0$ because of the Jacobian of the GT

Jacobian

$$\ln J[\phi] = \sum_{\eta=\pm} \text{Tr} \ln \left| \frac{\partial_{\eta} - i\phi}{\partial_{\eta}} \right| = - \sum_{\eta=\pm} \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} (i\phi g_{\eta})^n$$

$$\ln J = -\frac{1}{2} \int d\xi d\xi' \phi(\xi) \Pi(\xi - \xi') \phi(\xi') \quad \text{(see Appendix)}$$

$$\Pi(q, \Omega) = \frac{1}{\pi v_{\text{F}}} \frac{v_{\text{F}}^2 q^2}{\Omega^2 + v_{\text{F}}^2 q^2}$$

RPA polarization operator is exact for the LL
(Dzyaloshinskii, Larkin, '73)

The **Jacobian** of the gauge transformation gives the screening!

$$V_0^{-1} \mapsto V^{-1} = V_0^{-1} + \Pi$$

Exact Green's function

Green's function is (since $\psi_\eta \mapsto \psi_\eta e^{i\theta_\eta}$):

$$\mathcal{G}_{\eta\eta'}(\xi; \xi') = \left\langle\left\langle e^{i\theta_\eta(\xi) - i\theta_{\eta'}(\xi')} \psi_\eta(\xi) \psi_{\eta'}^*(\xi') \right\rangle\right\rangle$$

$$\langle\langle \dots \rangle\rangle = \frac{1}{Z} \int \dots e^{-S} \mathcal{D}\psi^* \mathcal{D}\psi \mathcal{D}\phi$$

$$S = \frac{1}{2} \int d\xi d\xi' \phi(\xi) V^{-1}(\xi - \xi') \phi(\xi') + \int d\xi \psi_\eta^*(\xi) \partial_\eta \psi_\eta(\xi)$$

$\partial_\eta \theta_\eta = \phi \Rightarrow$ we need $\langle\langle \theta_\eta \theta_{\eta'} \rangle\rangle$ from $\langle\langle \phi_\xi \phi_{\xi'} \rangle\rangle = V(\xi - \xi')$

$$\theta_- = \theta_+^* \equiv \theta = \theta_1 + i\theta_2$$

Bosonic Averaging

From $(\partial_\tau - iv_F \partial_x)\theta(\xi) = \phi(\xi)$, it follows that, e.g.

$$(\partial_{\tau\tau}^2 + v_F^2 \partial_{xx}^2) \langle\langle \theta_1(\xi)\theta_1(\xi') \rangle\rangle = -\partial_{\tau\tau}^2 V(\xi),$$

This equation is solved by the Fourier transform:

$$\langle \theta_1(\xi)\theta_1(\xi') \rangle_\phi = \frac{1}{2} \ln \frac{|\sin(z_F - z'_F)|}{|\sin(z - z')|^{1/g}}$$

$$z_F = \pi T(\tau + ix/v_F)$$

$$z = \pi T(\tau + ix/v)$$

$$v = v_F/g$$

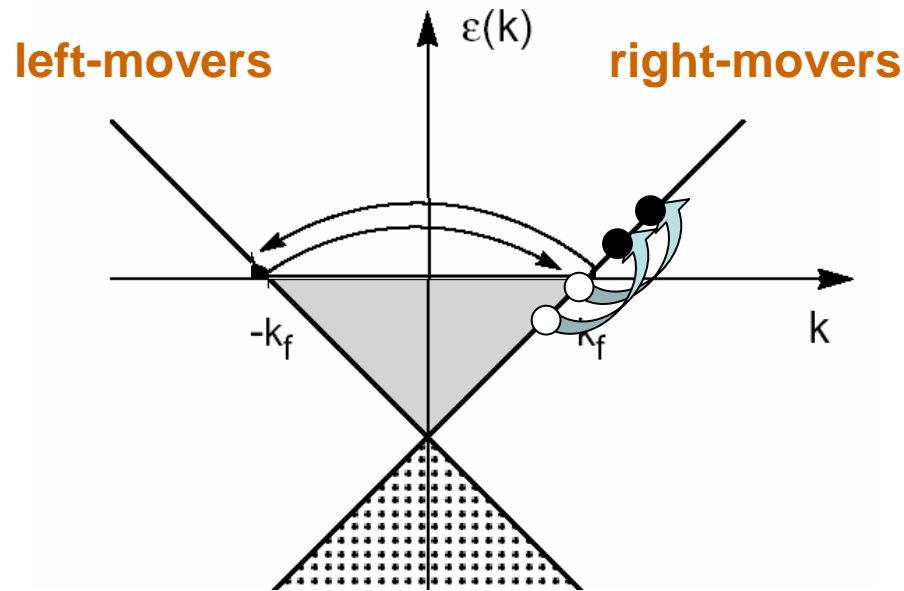
$$g \equiv \left(1 + V_0/\pi v_F\right)^{-1/2},$$

where $V_0(q) \longrightarrow V_0 = V_0(q \sim T/v_F)$

Pure Luttinger Liquid: Summary of the Approach

- Any correlation function can be calculated in terms of $\langle\langle \theta_a(\xi) \theta_b(\xi') \rangle\rangle$ and $\langle\langle \psi^*(\xi) \psi(\xi') \rangle\rangle$
- Calculating GF in this way reproduces the results for LDoS in the pure LL

Adding an Impurity



Back-scattering couples left- and right-movers: the bosonic excitations are no longer free:

Impurity adds a new “coupling” term:

$$H_{\text{imp}} = v_F \int dx \lambda(x) \left[\hat{\psi}_+^\dagger(x) \hat{\psi}_-(x) + \hat{\psi}_-^\dagger(x) \hat{\psi}_+(x) \right]$$

Impurity Coupling

$$S_{\text{imp}} = -v_F \int d\xi \lambda(x) \left[\psi_+^*(\xi) \psi_-(\xi) + \psi_-^*(\xi) \psi_+(\xi) \right]$$

couples fermionic and bosonic fields after the gauge transform:

$$S_{\text{imp}} = -v_F \int d\xi \lambda \left[e^{2\theta_2} \psi_+^*(\xi) \psi_-(\xi) + e^{-2\theta_2} \psi_-^*(\xi) \psi_+(\xi) \right]$$

– the problem is no longer exactly solvable.

Green's function (that defines the tunnelling LDoS) is now

$$\mathcal{G}_{\eta\eta'}(\xi; \xi') = \frac{1}{Z_\lambda} \left\langle \left\langle e^{i\theta_\eta(\xi) - i\theta_{\eta'}(\xi')} \psi_\eta(\xi) \psi_{\eta'}^*(\xi') e^{-S_{\text{imp}}} \right\rangle \right\rangle$$

$$Z_\lambda \equiv \left\langle \left\langle e^{-S_{\text{imp}}} \right\rangle \right\rangle \quad \text{is to be calculated}$$

Calculating Z_λ

Symbolically:

$$Z_\lambda = \sum_{n=0}^{\infty} \frac{v_F^{2n}}{(n!)^2} \left\{ \left\langle \left(\int_{\xi} \psi_+^* \psi_- \right)^{2n} \right\rangle_f \left\langle \int_{\xi\xi'} e^{n(\theta-\theta')} \right\rangle_f \right\}$$

The fermionic average cancels an unpleasant denominator of the bosonic one, both resulting in a formal expression in terms of

$$z = \pi T (\tau + ixg/v_F)$$

Formal Results

$$Z_\lambda = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left(\frac{T}{2\alpha^{1-g}} \right)^{2n} \quad (\alpha \sim T/\varepsilon_F \ll 1)$$
$$\times \prod_{k=1}^n \int d\xi_k d\xi'_k \lambda(x_k) \lambda^*(x'_k) |P_n(z)|^{2g}$$

$$P_n(z) = \frac{\prod_{i < j}^n \sin(z_i - z_j) \sin(z'_i - z'_j)}{\prod_{i=1}^n \sin(z_i - z'_i)}$$

Re-Bosonization

Z_λ in terms of a new impurity functional above is given by

$$Z_\lambda = \langle e^{-S_\lambda[\Theta]} \rangle_0, \quad S_\lambda[\Theta] = -\frac{T}{\alpha} \int d\xi \lambda(x) \cos \Theta(\xi)$$

The functional average $\langle \dots \rangle_0$ is performed with the weight

$$S_0[\Theta] = \frac{1}{8\pi g v} \int d\xi \left[(\partial_\tau \Theta)^2 + v^2 (\partial_x \Theta)^2 \right]$$

The Green's function (with $\chi(z_1) = \arg \frac{\sin(z_1 - z)}{\sin(z_1 - z')}$)

$$\tilde{\mathcal{G}}_{\eta\eta'}(\xi, \xi') = Z_\lambda^{-1} \left\langle e^{\frac{i\eta}{2}\Theta(\xi) - \frac{i\eta'}{2}\Theta(\xi')} e^{-S_\lambda[\Theta-\chi]} \right\rangle_0$$

Self-consistent harmonic approximation

Assumption: $\alpha \ll \lambda \ll 1$ ($\alpha \sim T/\varepsilon_F$)

the impurity potential $\lambda(\mathbf{x}) = \lambda\delta(\mathbf{x})$ is weak but non-perturbative

SCHA: the deviation of Θ from χ is prohibitive so that cos potential is substituted by the quadratic approximation:

$$S_\Lambda[\Theta - \chi] = \frac{\Lambda T}{2\alpha} \int d\tau_1 [\Theta(0, \tau_1) - \chi(0, \tau_1)]^2$$

What is left is a (rather gory) calculation of Gaussian integrals

Green's Function

$$\frac{2\pi}{p_F} \mathcal{G}(x, x; \tau) = \begin{cases} \frac{\alpha^{\frac{1}{g}}}{(\sin \alpha \tilde{\tau})^{\frac{1}{g}}} [\max(\Lambda^{-1}, \tilde{x})]^{\frac{1}{2}(\frac{1}{g}-g)} [1 - \cos(2p_F x + \Phi)], & \max(\tilde{x}, \Lambda^{-1}) \ll \tilde{\tau} \\ \frac{\alpha^{\frac{1}{2}(\frac{1}{g}+g)}}{(\sin \alpha \tilde{\tau})^{\frac{1}{2}(\frac{1}{g}+g)}} \left[1 - \left(\frac{\sin \alpha \tilde{\tau}}{\sinh \alpha \tilde{x}} \right)^g \cos(2p_F x + \Phi) \right], & \min(\tilde{x}, \Lambda^{-1}) \gg \tilde{\tau} \end{cases}$$

x is the distance from the impurity, $\Lambda = \lambda^{\frac{1}{1-g}}$, $\tilde{x} = gp_F|x|$, $\tilde{\tau} = \varepsilon_F|\tau|$

Friedel oscillations can be easily extracted

Well known LDoS

Suppression of the LDoS at the impurity point

$$\nu(\varepsilon) \propto \begin{cases} \varepsilon^{\frac{1}{g}-1}, & \text{at the impurity} \\ \varepsilon^{\frac{1}{2}\left(\frac{1}{g}+g\right)-1}, & \text{in the bulk} \end{cases}$$

Corresponds to “end” and bulk LDoS in Leonid Glazman’s lecture this morning

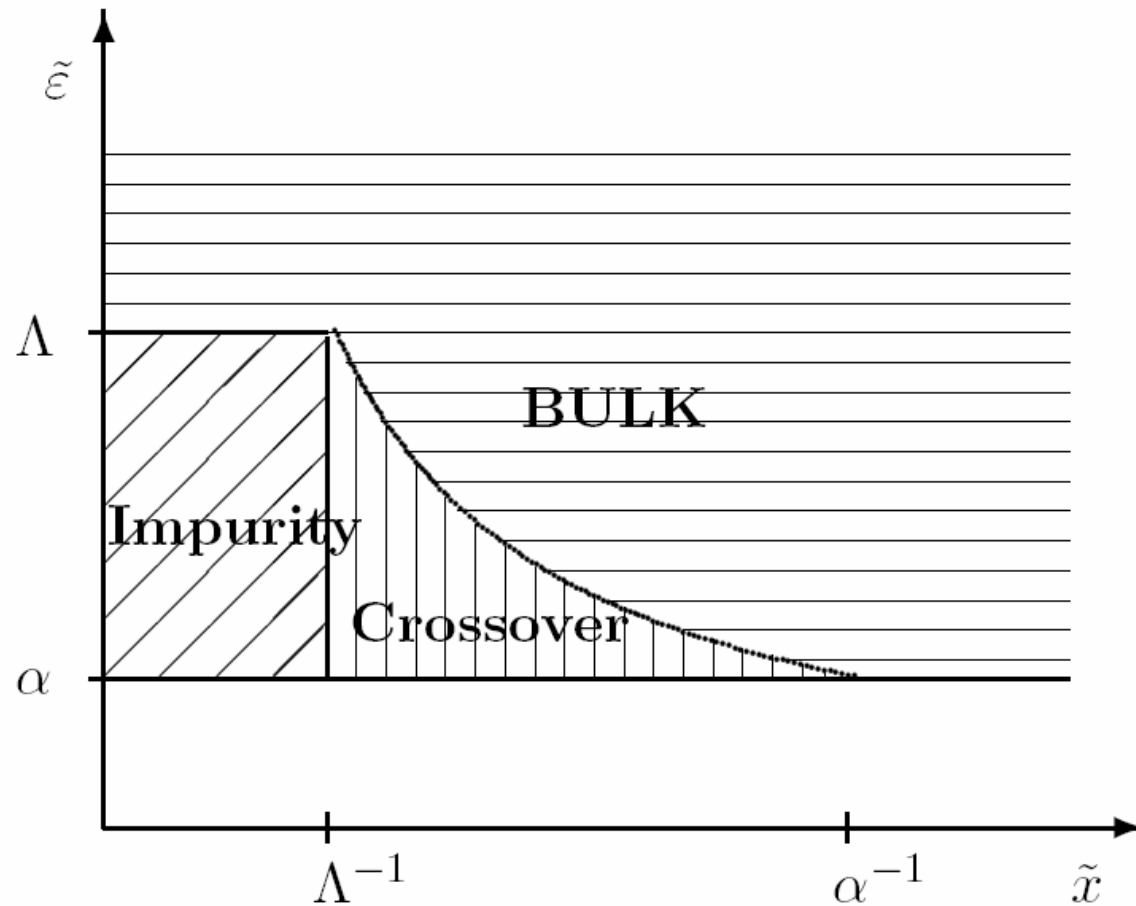
Local Tunnelling DoS

is obtained at an arbitrary distance from the impurity

$$\nu(x, \varepsilon) \sim \begin{cases} \varepsilon^{\frac{1}{g}-1} \Lambda^{-\frac{1}{2}} \left(\frac{1}{g}-g\right), & \tilde{x} \ll \Lambda^{-1} \ll \varepsilon^{-1} \\ \varepsilon^{\frac{1}{g}-1} \tilde{x}^{\frac{1}{2}} \left(\frac{1}{g}-g\right), & \Lambda^{-1} \ll \tilde{x} \ll \varepsilon^{-1} \\ \varepsilon^{\frac{1}{2}} \left(\frac{1}{g}+g\right)^{-1}, & \tilde{x} \gg \Lambda^{-1}, \varepsilon^{-1} \end{cases}$$

Local Tunnelling DoS

different regions at a distance x from the impurity



$$\alpha \sim T/\epsilon_F$$

$$\Lambda = \lambda^{\frac{1}{1-g}}$$

$$\tilde{x} = gp_F |x|,$$

$$\tilde{\tau} = \epsilon_F |\tau|$$

Summary

- Functional Bosonization works: it reproduces known results for Friedel Oscillations & LDoS and allows one to find LDoS at an arbitrary distance from the impurity
- In many way, it is analogue of field-theoretical treatment of higher-dimensional models
- The operator and functional approaches are equivalent for exact results; however, ease of use for approximations may be different

Appendix : Jacobian

$$\ln J[\phi] = \sum_{a=\pm} \text{Tr} \ln \left| \frac{\partial_a - i\phi}{\partial_a} \right| = - \sum_{a=\pm} \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} (i\phi g_a)^n$$

where $g_-(\xi, \xi') = g_+(\xi) = \frac{T}{2v_F} \frac{1}{\sin(z_F - z'_F)}$

The n^{th} term is

$$\text{Tr} (g_a \phi)^n = \int \prod_{k=1}^n dx_k d\tau_k \Gamma_n^{(a)}(z_{F1}; \dots; z_{Fn}) \prod_{i=1}^n \phi(x_i, \tau_i),$$

with Γ

$$\Gamma_n^{(a)}(z_{F1}; \dots; z_{Fn}) = \prod_{i=1}^n g_a(z_{Fi} - z_{Fi+1}) \propto \prod_{i=1}^n \frac{s_i}{s_i - s_{i+1}}, \quad s_i = e^{2iz_{Fi}}$$

2: Jacobian

Only symmetric part of the vertex contributes to the integral:

$$\text{Sym}[\Gamma_n^+(z_{F_1}; \dots; z_{F_n})] \propto \frac{\mathcal{A}_n(s_1, \dots, s_n)}{\prod_{i < j}^n (s_i - s_j)} \prod_{k=1}^n s_k$$

where \mathcal{A}_n is an absolutely anti-symmetric polynomial

By power counting, its order is $n(n-3)/2 > n(n+1)/2$ – only possible for $n=1$ and $n=2$ loops, whose calculation is straightforward (after dealing with inevitable divergences)