Nonlinear bosonization

Hydrodynamic description of 1D fermions

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Goal:

to express fermionic objects

$$H = c^{\dagger}(-\frac{\nabla^2}{2m} - \mu)c + \text{interaction}$$

through hydrodynamics modes:

$$\rho(x) = \sum_i \delta(x - x_i) = c^{\dagger}(x)c(x)$$

$$j(x) = \sum_i \dot{x}_i \delta(x - x_i) = c^{\dagger}(x) i\nabla c(x)$$

Same is applied to bosons

$$H = b^{\dagger}(-\frac{\nabla^2}{2m} - \mu)b + \text{interaction}$$

- Why it is possible?
- When it is effective?

Math: appearence of complex structure

Physics: If there is no, or just few channels of scattering (energy dissipation)

Two major examples:

1) 1D- electrons - forward-backward scattering;

2) Superconductor

Linear bosonization:

Linearisation of the spectrum: $$\frac{p^2}{2} - \varepsilon_F \sim \pm v_F (p \pm k_F)$$

$$c(x) \sim \psi_R(x) e^{ik_F x} + \psi_L(x) e^{-ik_F x}$$

Rules:

$$\rho(x) = c^\dagger c \to -\nabla\varphi$$

$$j(x) = ic^\dagger \nabla c \to -\dot{\varphi}$$

$$\psi_{R,L} \sim e^{2\pi i \varphi_{RL}}$$

$$K = \psi_R^\dagger \nabla \psi_R + \psi_L^\dagger \nabla \psi_L \sim (\nabla\varphi_R)^2 + (\nabla\varphi_L)^2$$

Free Fermions=Free bosons

$$c^\dagger (i\partial_t - \frac{\nabla^2}{2} - \varepsilon_F) c \sim (\dot{\varphi})^2 - (\nabla\varphi)^2$$

Accuracy? $O(1/N) = O(1/k_F)$

example

$$<< \rho(1)\rho(2)\rho(3) >> \sim \partial_1\partial_2\partial_3 << \varphi(1)\varphi(2)\varphi(3) >= 0$$

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$$<< \rho(x_1)...\rho(x_n) >>= (k_F)^{-n+2} f_n(x_1,...,x_n)$$

Linear "bosonization" fails when spectrum asymmetry is important, and when one goes beyond a linear response.

Sakita 1974

recognized in string theory as collective field theory

First quantization

$$H_F = -\sum_i \frac{\partial^2}{\partial x_i^2} - \text{fermions: antisymmetric w.f.}$$

$$H_B = -\sum_i \frac{\partial^2}{\partial x_i^2} - \text{bosons: symmetric w.f.}$$

Second Quantization

$$H = c^\dagger(-\frac{\nabla^2}{2m} - \mu)c + \text{interaction}$$

$$H = b^\dagger(-\frac{\nabla^2}{2m} - \mu)b + \text{interaction}$$

Collective variables (modes)

$$\phi_k = \sum_{i=1}^N x_i^k \qquad \pi_k = \frac{\partial}{\partial \phi_k} \qquad v_k = -ik\pi_k$$

$$[\mathrm{v_k} \phi_\mathrm{n}] = -\mathrm{i}\mathrm{k}\delta_\mathrm{kn}.$$

Density

$$u_-(z) = \sum_i \frac{1}{z - x_i} = \sum_i z^{-k-1}\phi_k$$

$$u_-(x+i0) - u_-(x-i0) = 2\pi i \sum_i \delta(x - x_i) = 2\pi i \rho(x)$$

Current, velocity

$$u_+(z) = -iv(z) \qquad v(z) = \sum_{k \geq 0} z^{k-1} v_k.$$

$$j(x) = \sum_i \dot{x}_i \delta(x - x_i) = \rho(x) v(x)$$

Continuity equation $\qquad \dot{\rho} + \nabla j = 0$

Bose field

$$u(z) = u_+(z) + u_-(z) = \sum_{k>0} v_k z^{k-1} + \sum_i \phi_k z^{-k-1}$$

bosonic notations

$$a_n = i\mathrm{v}_\mathrm{n}, \quad \mathrm{a}_{-\mathrm{n}} = \phi_\mathrm{n}, \quad , \mathrm{n} > 0$$

$$u(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad [a_n, a_k] = n\delta_{n+k}$$

$$u(z) = \partial_z \varphi(z)$$

Fermions (Bosons) in collective variables

$$H_F = \frac{\rho v^2}{2} + \frac{\pi^2}{3}\rho^3$$

$$H_B = \frac{\rho v^2}{2} - \frac{1}{8}\frac{(\nabla\rho)^2}{\rho}$$

Accuracy? $O(e^{-N})$

$$H_F = \frac{\rho v^2}{2} + \frac{\pi^2}{3}\rho^3 \qquad [v(x), \rho(y)] = -i\delta'(x-y)$$

1) Kinetic energy: Galileian inv.

2) Fermi energy (pressure)

$$\rho\varepsilon(\rho) = \frac{\pi^2}{6}\rho^3, \quad P = \rho^2 \frac{\partial \varepsilon}{\partial \rho} = \frac{\pi^2}{3}\rho^4$$

$$E_F = \int_{-k_F}^{k_F} \frac{k^2}{2}\frac{dk}{2\pi} = \frac{k_F^3}{6\pi} = \frac{\pi^2}{6}\rho^3 \qquad \rho = \int_{-k_F}^{k_F} \frac{dk}{2\pi} = \frac{k_F}{\pi}$$

Small waves - linear bosonization (linear hydrodynamics)

$$\rho = \rho_0 + \delta\rho$$

density

$$K = \frac{\rho v^2}{2} \sim \rho_0 \frac{v^2}{2}$$

Kinetic energy

$$\Pi = \frac{2}{6}\rho^3 \sim \Pi_0 + \pi^2 \rho_0^2 (\delta\rho)^2$$

Potential energy (Fermi pressure)

$$H = K + \Pi \sim \frac{1}{2\rho_0}(j^2 + v_0^2(\delta\rho)^2) = \frac{\kappa}{2}[(\dot{\varphi})^2 + v_0^2(\nabla\varphi)^2]$$

$$j = \dot{\varphi}, \quad \delta\rho = -\nabla\varphi$$

Equations: Euler's Hydrodynamics

$$H_F = \frac{\rho v^2}{2} + \frac{\pi^2}{3}\rho^3$$

$$\rho_t = i[H, \rho] \qquad \partial_t \rho + \partial_x(\rho v) = 0$$

$$\dot{v} = i[H, v] \qquad \partial_t v + v\partial_x v = \partial_x \left[\epsilon + \rho \frac{\partial \epsilon}{\partial \rho}\right] = \frac{1}{\rho}\partial_x P,$$

$$\epsilon(\rho) = \frac{\pi^2}{6}\rho^2, \qquad P = \frac{\pi^2}{3}\rho^3$$

Hopf Equation: $u(x+i0) = v + i\rho,$

$$\partial_t \rho + \partial_x(\rho v) = 0$$

$\Rightarrow$

$$\dot{v} + v \cdot \nabla v = \pi^2 \rho \nabla \rho$$

$$\dot{u} + u \cdot u_x = 0$$

$\Rightarrow$

$$[u(x), u(y)] = i\delta'(x-y)$$

$\Rightarrow$ $\dot{u} + u_x = 0$ – linearized version

Solution of Hopf equation

$$\dot{u} + u \cdot u_x = 0$$

$$u(x,t) = u_0(x - u(x,t) \cdot t), \quad u_0(x) = u(x,0)$$

Solution of wave equation

$\dot{u} + u_x = 0$ – linearized version

$$u(x,t) = u_0(x - t), \quad u_0(x) = u(x,0)$$

Hamiltonian in terms of Bose field: $$H_F = \frac{\rho v^2}{2} + \frac{\pi^2}{3}\rho^3$$

$$u(z) = u_+(z) + u_-(z) = \sum_{k>0} v_k z^{k-1} + \sum_i \phi_k z^{-k-1}$$

$$\phi_k = \sum_{i=1}^N x_i^k$$

$$u(z) = \partial_z \varphi(z)$$

$$H_F = \frac{1}{12}\oint_C u^3(z)\frac{dz}{2\pi i} \qquad = \int [\frac{\rho v^2}{2} + \pi^2\frac{\rho^3}{6}]dx$$

$$u(z) = \partial_z \varphi(z)$$

$$H_F = \frac{1}{12}\oint_C (\partial\varphi)^3 \frac{dz}{2\pi i}$$

$$\varphi = 2k_F x + \phi \qquad H_F \to k_F \int (\partial\phi)^2 \frac{dx}{2\pi}$$

Calogero-model

$$H = -\sum_i \partial_i^2 + \sum_{i \neq j} \frac{\beta/2(\beta/2-1)}{(x_i - x_j)^2}$$

Ground state

$$\Psi_0(x_1, \ldots, x_N) = \Delta^\beta(x), \quad \Delta(x) = \prod_{i>j}(x_i - x_j)$$

Excited states

$$\Psi(x_1, \ldots, x_N) = \Delta^\beta(x) J(x_1, \ldots x_N)$$

J(x) - Jack -polynomial - symmetric function

β=0 – bosons, β=2 – fermions

$$H = \frac{1}{2}\sum_i (-\partial_i + A(x_i))(\partial_i + A(x_i)) :$$

$$A(x) = -\frac{\beta}{2}\sum_i \frac{1}{x - x_i}$$

$$(\partial_i + A(x_i))\Psi_0(x) = 0, \quad \Psi_0(x) = \Delta^{\beta/2}(x)$$

Goal: rewrite everything in terms of collective modes

$$\tilde{H} = -\Delta^{-\beta/2} H \Delta^{\beta/2} = \sum_i (-\partial_i + \beta \sum_i \frac{1}{x_i - x_j}) \partial_i$$

$$\tilde{H} J_E(x) = E J_E(x)$$

Both, the Hamiltonian and the w.f's are symmetric functions

$$\dot{\phi}_k = [\tilde{H}, \phi_k] \qquad \dot{v}_k = [\tilde{H}, v_k]$$

$$-\frac{1}{l}\dot{v}_l = \sum_k v_k v_{l-k+2} + 2v_k\phi_{k-l-2} + \alpha(l+1)v_{l+2},$$

$$\alpha = 1 - \beta/2$$

$$\frac{1}{l}\dot{\phi}_l = \sum_k \phi_k\phi_{l-k-2} + 2v_k\phi_{l+k-2} + \alpha(l-1)\phi_{l-2},$$

Goal: to rewrite these eqs. in terms of the Bose field

$$u(z) = u_+(z) + u_-(z) = \sum_{k>0} v_k z^{k-1} + \sum_i \phi_k z^{-k-1}$$

$$\dot{u} + \frac{\beta}{2} u \partial_z u + i(\frac{\beta}{2} - 1)\partial_z^2 (u_+ - u_-) = 0$$

Benjamin-Ono equation

β=0 – bosons,   β=2 – fermions

$$\dot{\varphi} + \beta(\partial\varphi)^2 + i(\frac{\beta}{2} - 1)\partial_z^2 (\varphi_+ - \varphi_-) = 0$$

$$u = \partial_z \varphi$$