From single-particle to many-body localisation in disordered systems.

Boris Altshuler
Physics Department, Columbia University
and NEC Laboratories America

The 4th Windsor Summer School on Condensed Matter Theory

Quantum Transport and Dynamics in Nanostructures

Great Park, Windsor, UK, August 6 - 18, 2007

Lecture 3

Many Body Localization

Previous Lectures:

1. Anderson Localization as Metal-Insulator Transition
2. Anderson model. Cayley tree problem.
3. Spectral Statistics and Localization.
   Poisson versus Wigner-Dyson.
   Invariant definition of the localization
4. Chaos and Localization.
   Localization in the space of quantum numbers.

Wigner-Dyson random matrix statistics follows from the delocalization.

Q: ***Why the random matrix theory (RMT) works so well for nuclear spectra*** **?**

Many-Body excitations are delocalized !

What does it mean ?

Consider a finite system of quantum particles, e.g., fermions. Let the one-particle spectra be chaotic (Wigner-Dyson).

Q: What is the statistics of the many-body spectra**?**

a. The particles do not interact with each other.

Poisson:

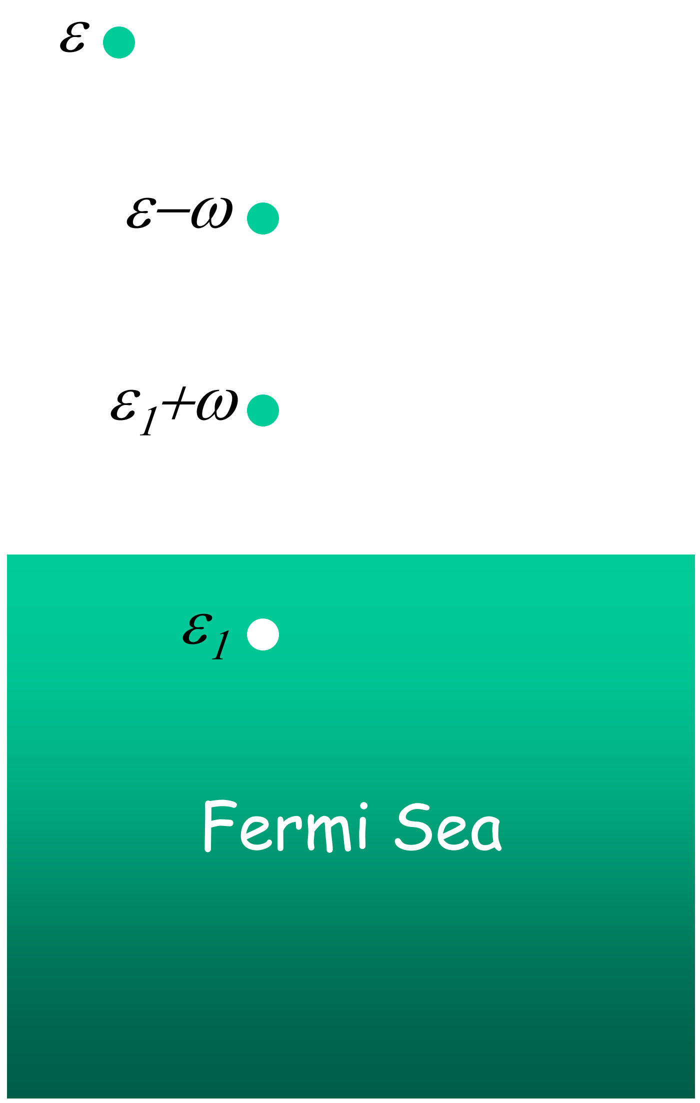
individual energies are conserving quantum numbers.

a. The particles do interact. **????**

Part 4.

Many-Body excitation in finite systems.

Decay of a quasiparticle with an energy $\mathcal{E}$ in Landau Fermi liquid

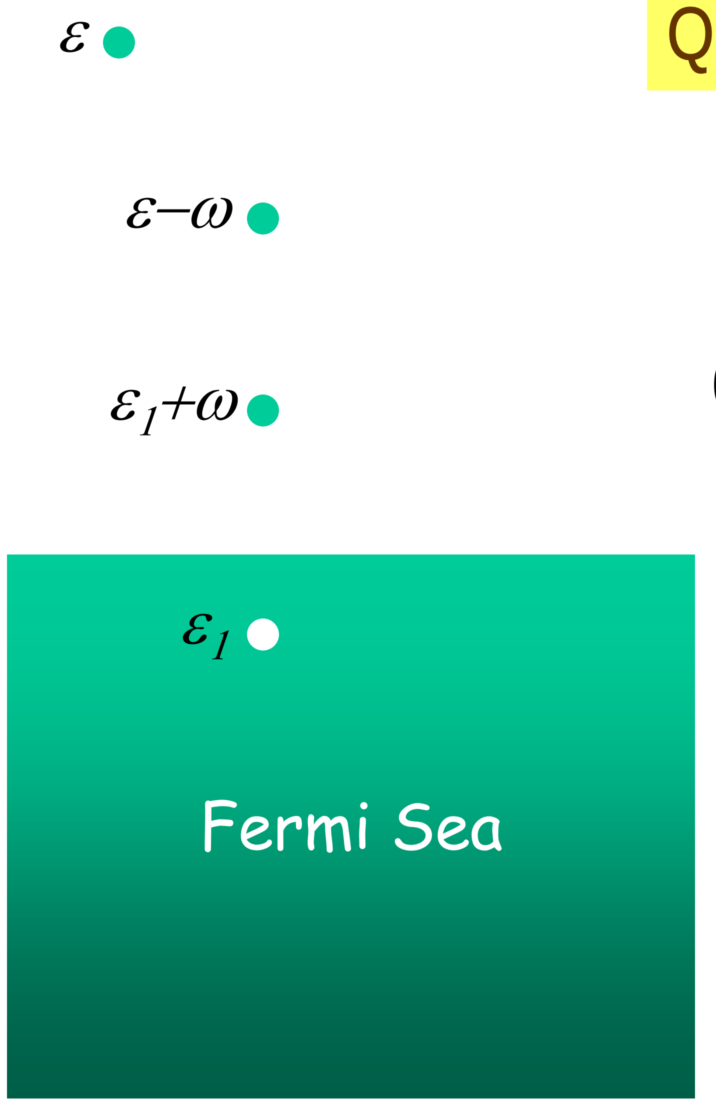


Landau Fermi liquid in a clean bulk system:

$$\gamma(\varepsilon) \propto \frac{\varepsilon^2}{\varepsilon_F}$$

Fermi energy

Decay of a quasiparticle with an energy $\mathcal{E}$ in Landau Fermi liquid



Quantum dot – zero-dimensional case ?

Fermi golden rule

(**U.Sivan, Y.Imry & A.Aronov,1994**):

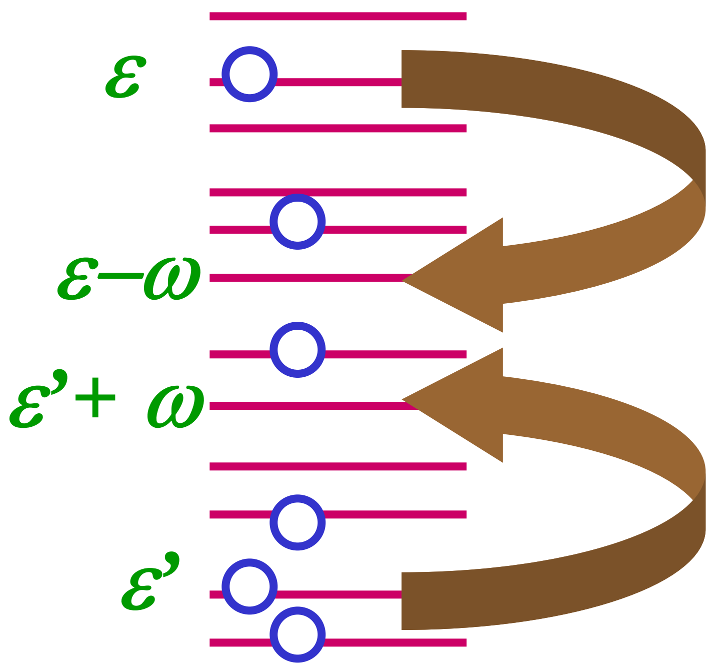
$$\gamma(\varepsilon) \propto \delta_1 \left(\frac{\varepsilon}{E_T} \right)^2$$

Mean level spacing

Thouless energy

Inelastic relaxation rate in 0D case

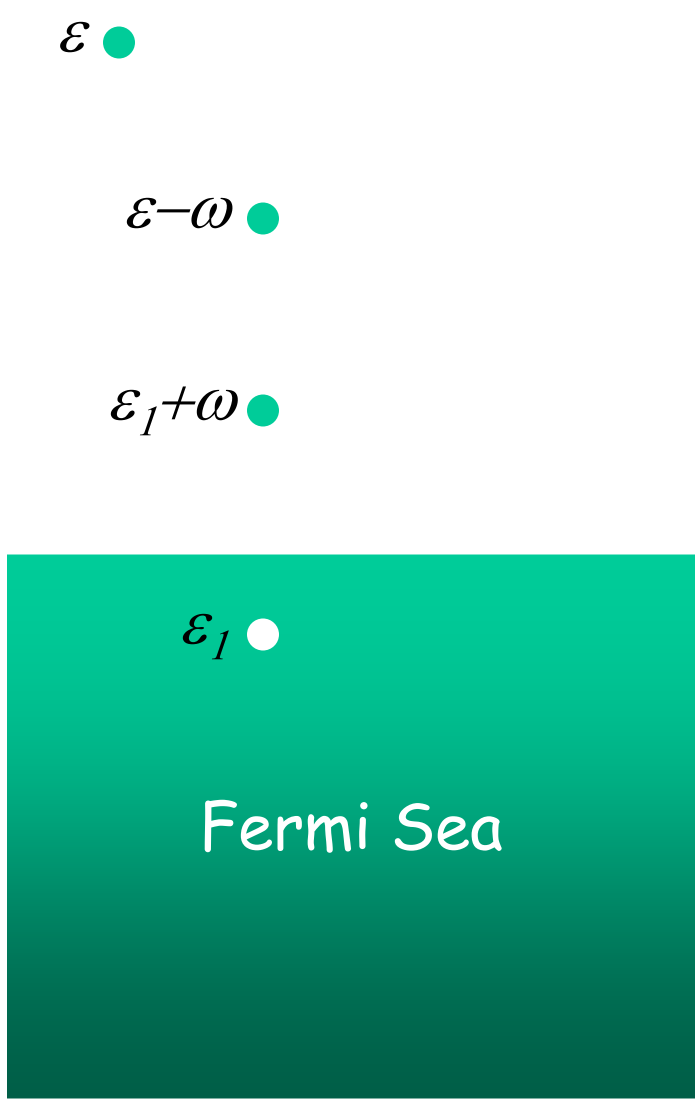
T=0



Offdiagonal matrix element

$$M\left(\omega,\varepsilon,\varepsilon'\right)\propto\frac{\delta_1}{g} << \delta_1$$

Decay of a quasiparticle with an energy $\mathcal{E}$ in Landau Fermi liquid



zero-dimensional case

↓

one-particle spectrum is discrete

↓

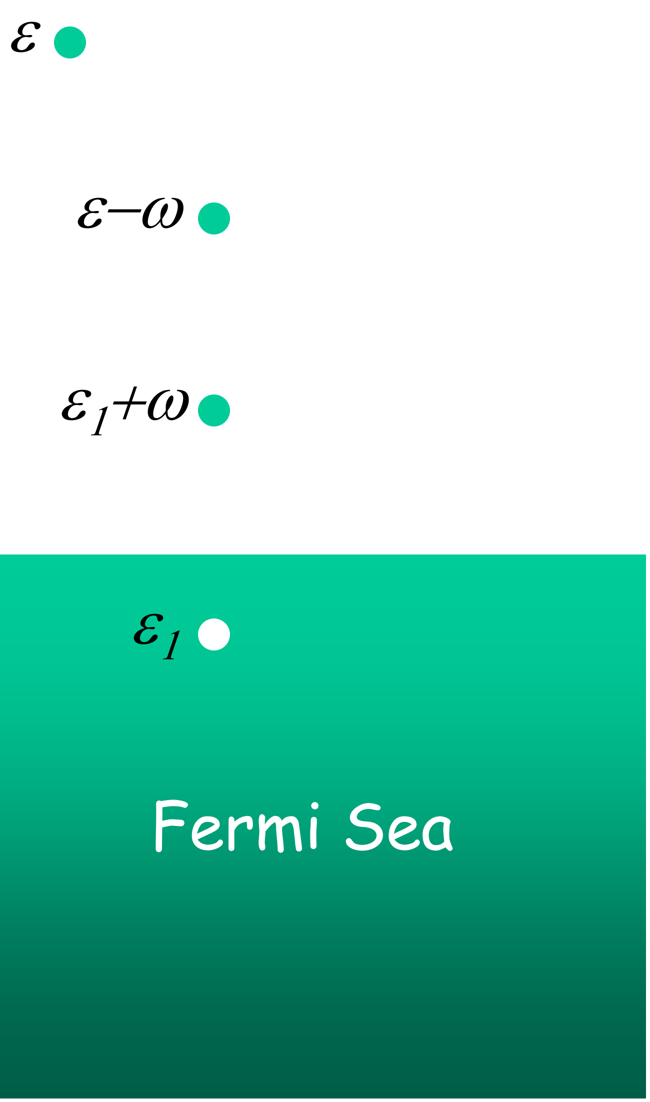
equation

$$\varepsilon_1+\varepsilon_2=\varepsilon'_1+\varepsilon'_2$$

can not be satisfied exactly

???

Decay of a quasiparticle with an energy $\mathcal{E}$ in Landau Fermi liquid



zero-dimensional case

↓

one-particle spectrum is discrete

↓

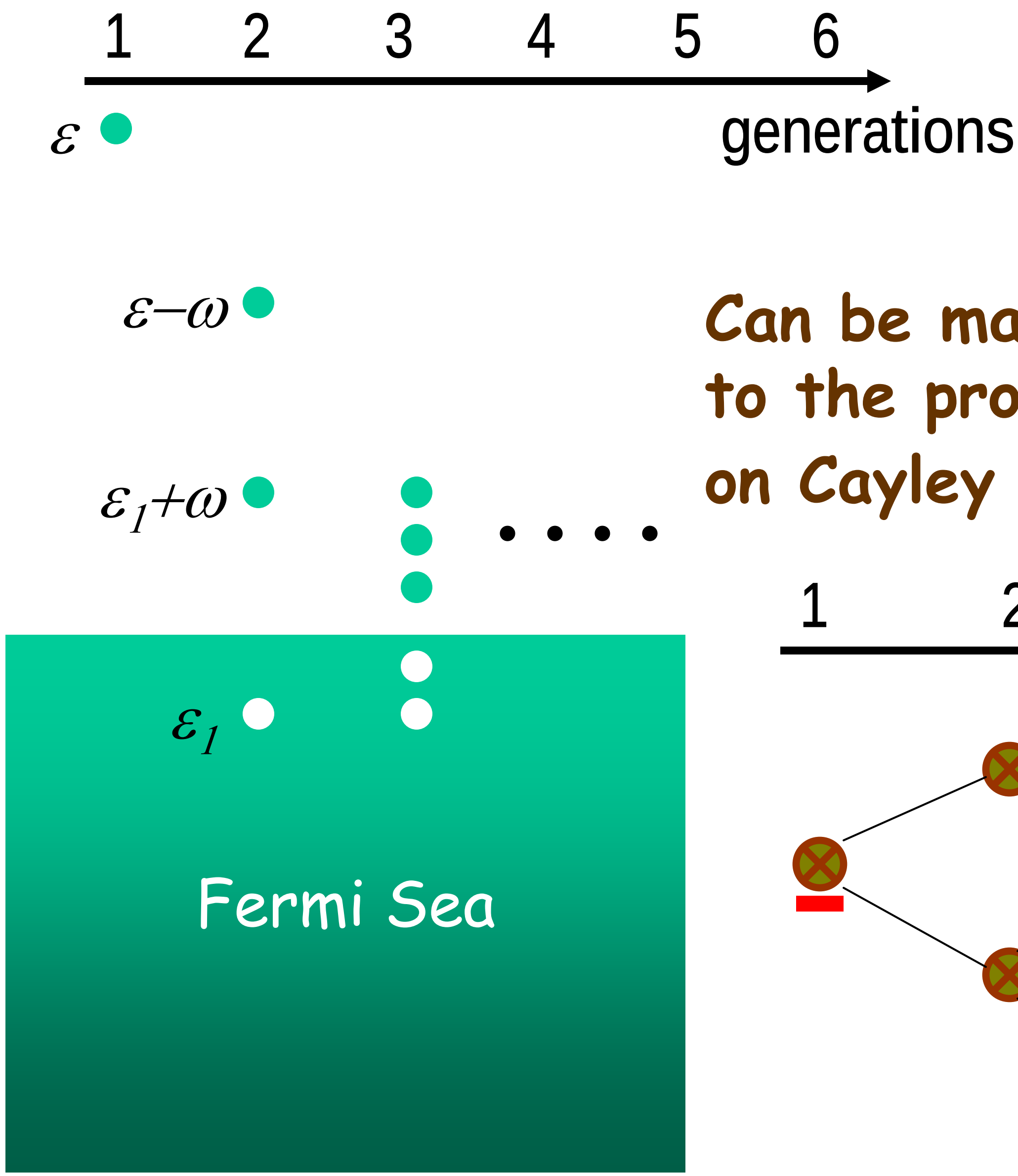
equation

$$\varepsilon_1+\varepsilon_2=\varepsilon'_1+\varepsilon'_2$$

can not be satisfied exactly

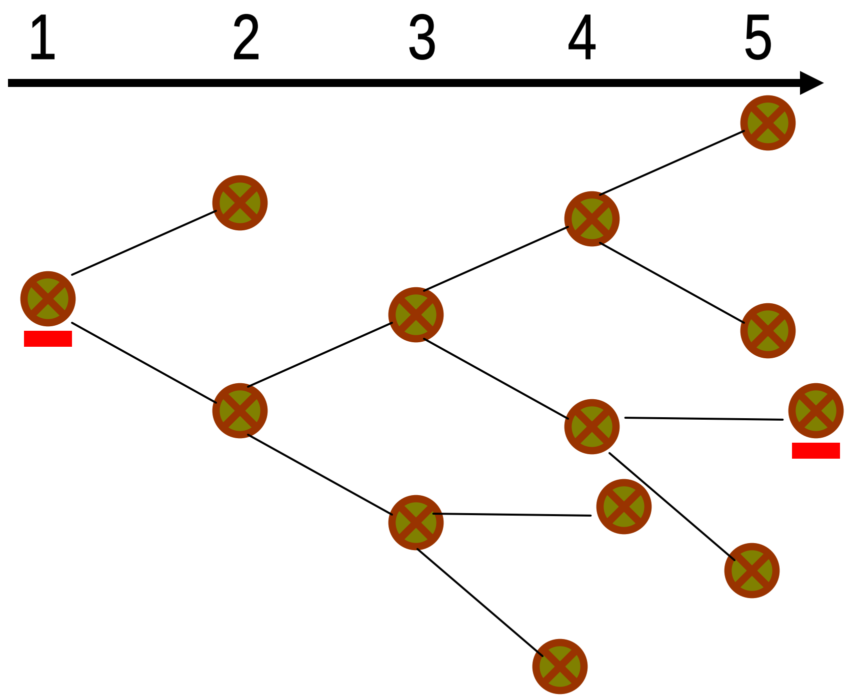
Recall: in the Anderson model the site-to-site hopping does not conserve the energy

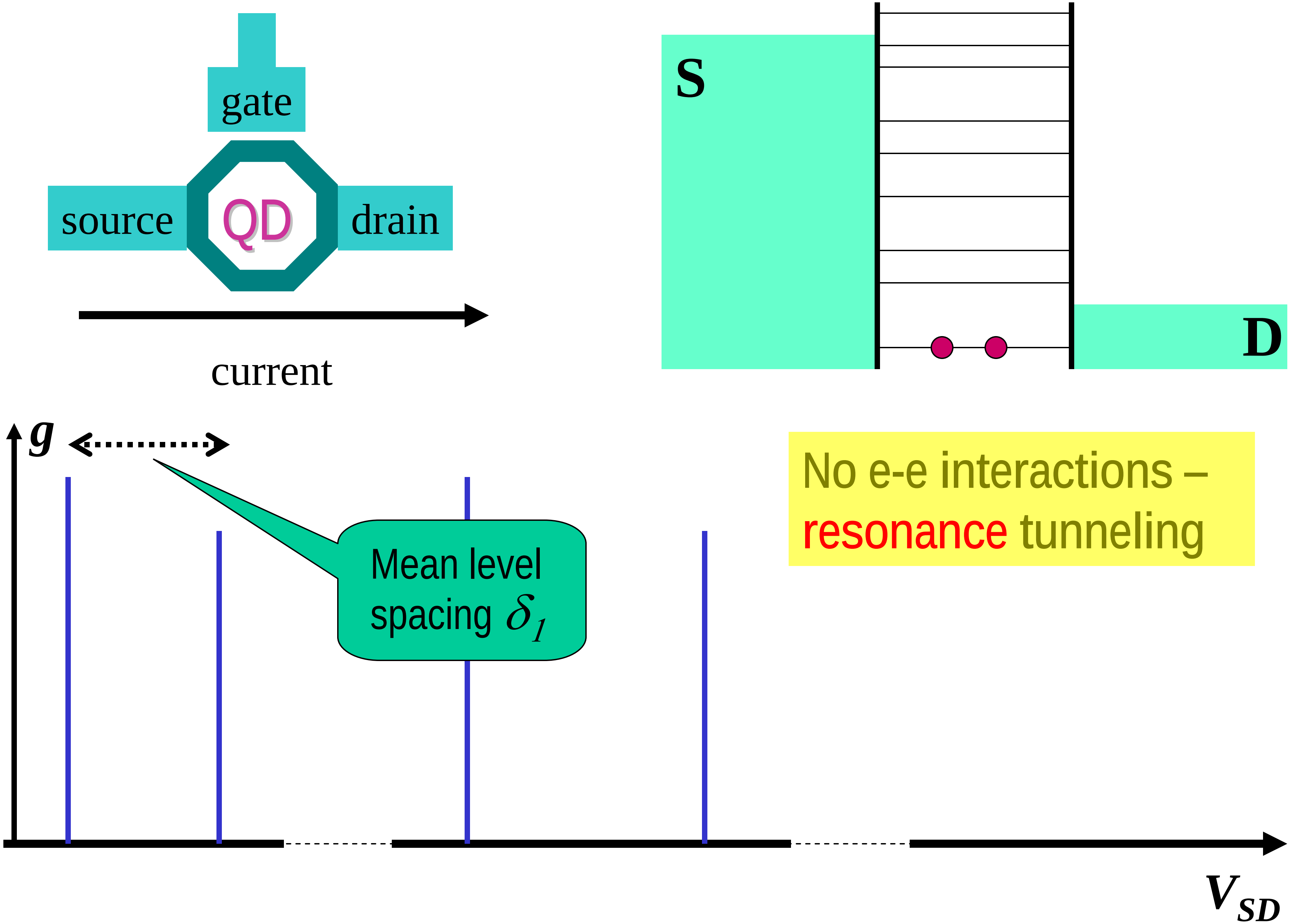
Chaos in Nuclei – Delocalization?

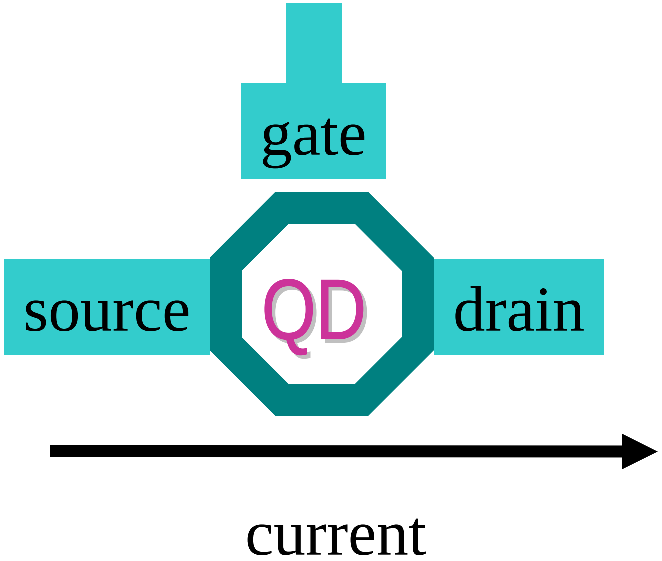


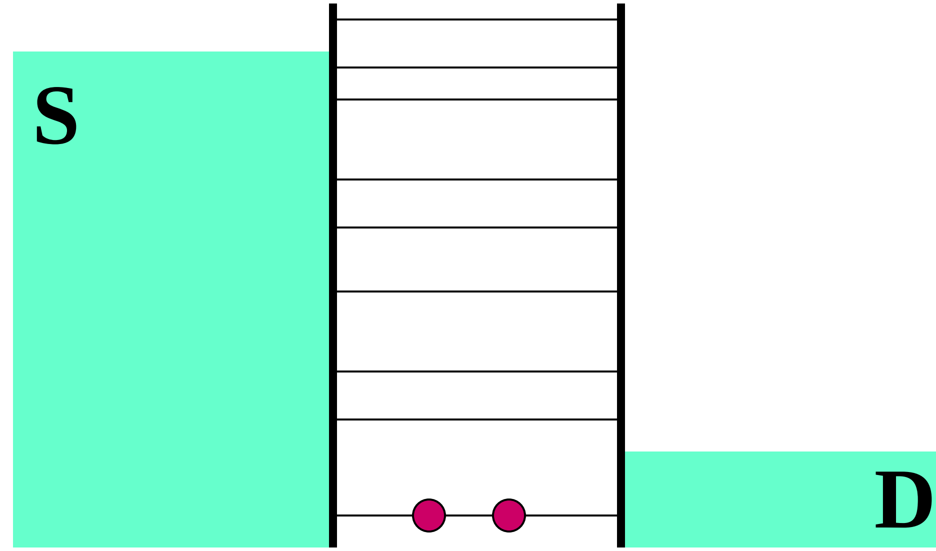
Delocalization in Fock space

Can be mapped (approximately) to the problem of localization on Cayley tree

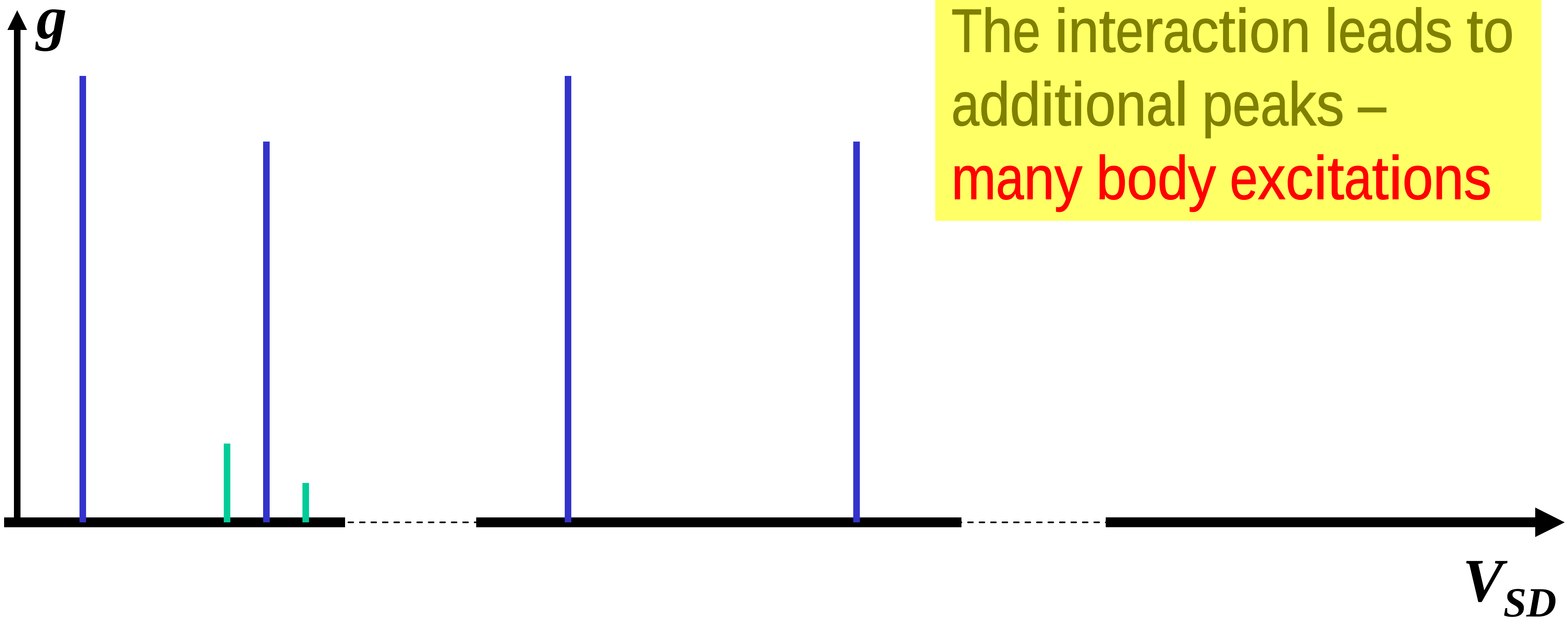


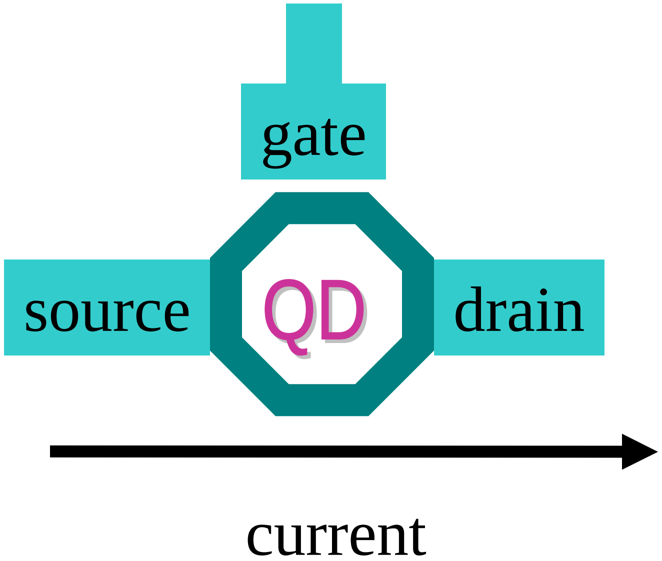

gate
source
QD
drain
current
S
D
g
Mean level spacing δ_1
No e-e interactions –
resonance tunneling
V_{SD}

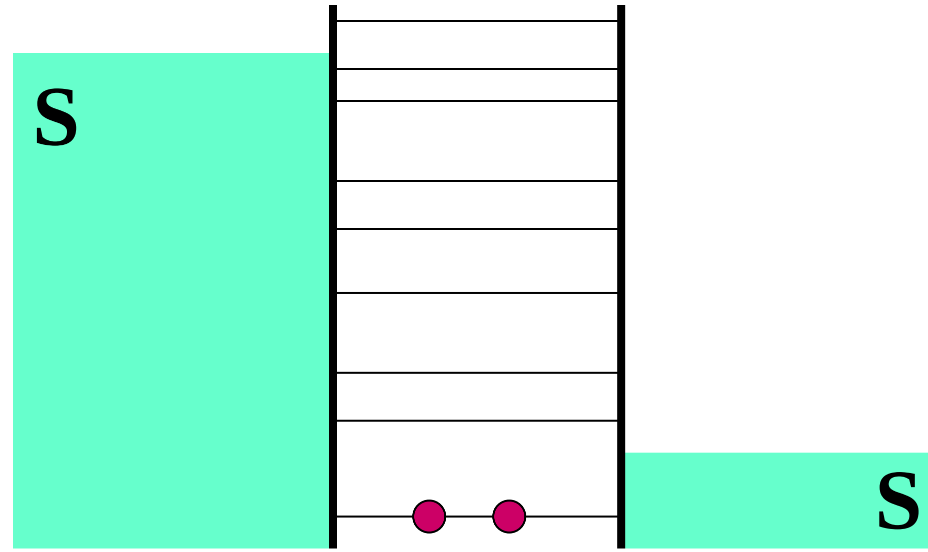




The interaction leads to additional peaks –
many body excitations







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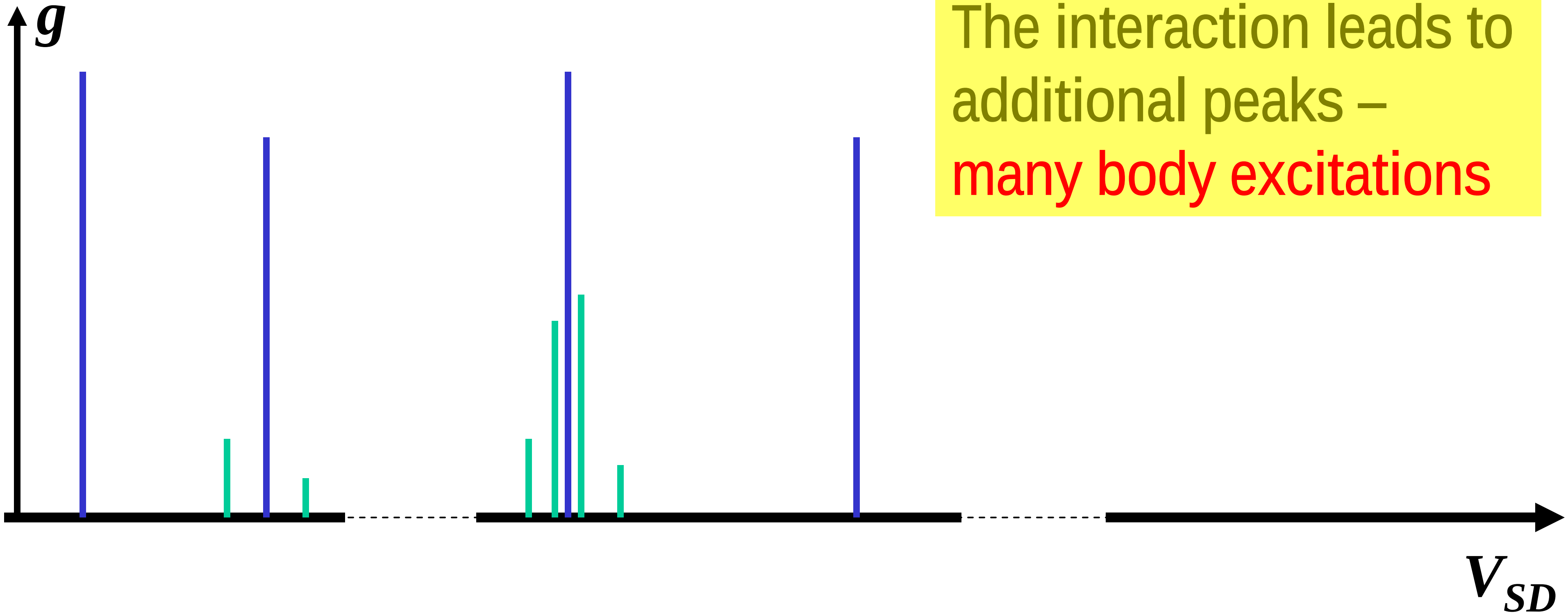
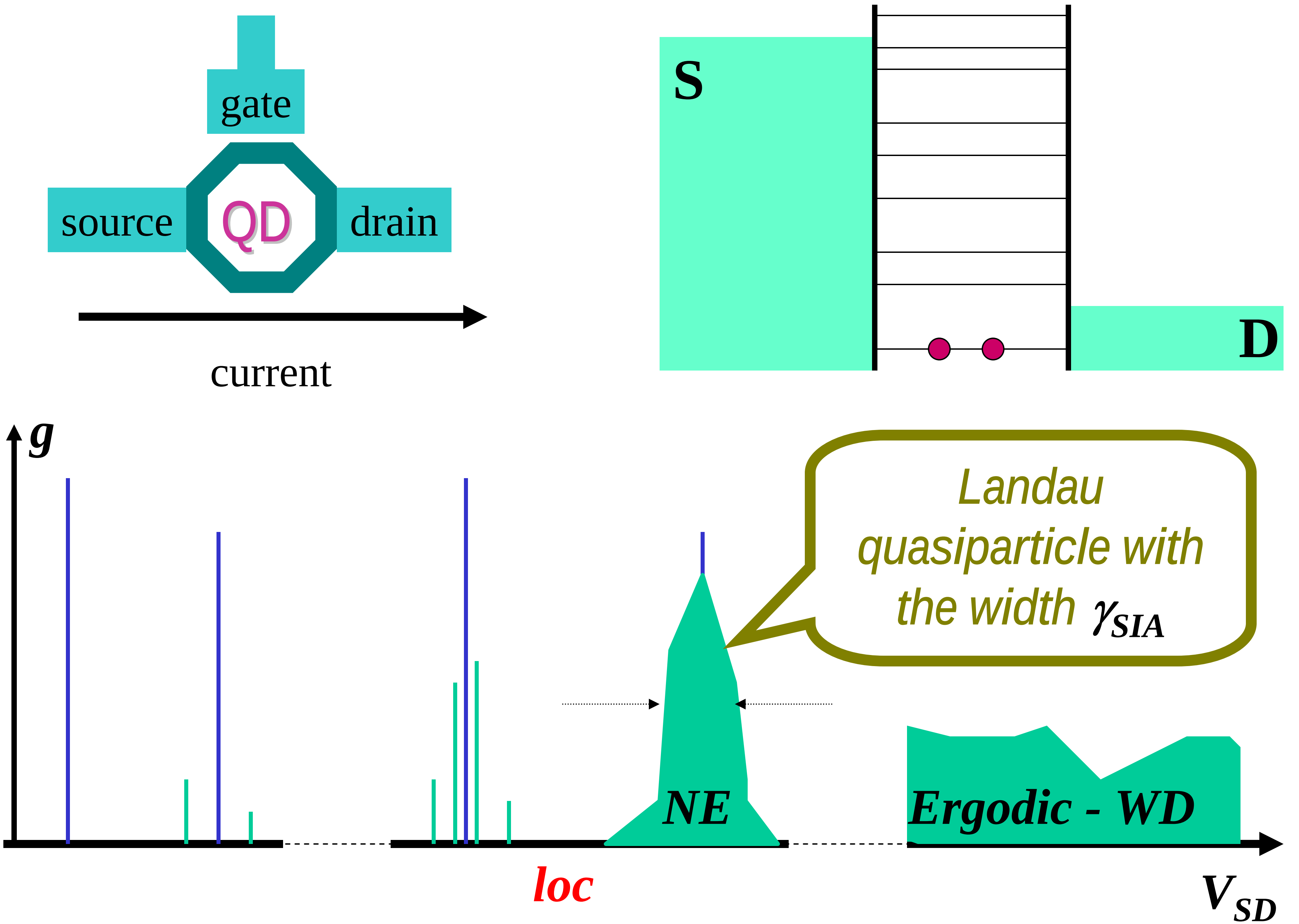



gate
source
QD
drain
current
S
D
g
Landau quasiparticle with the width γ_{SIA}
NE
Ergodic - WD
loc
V_{SD}

nonergodic states

Such a state occupies infinitely many sites of the Anderson model but still negligible fraction of the total number of sites

N total number of sites in the system

n support of a given wave function

$$N \to \infty$$

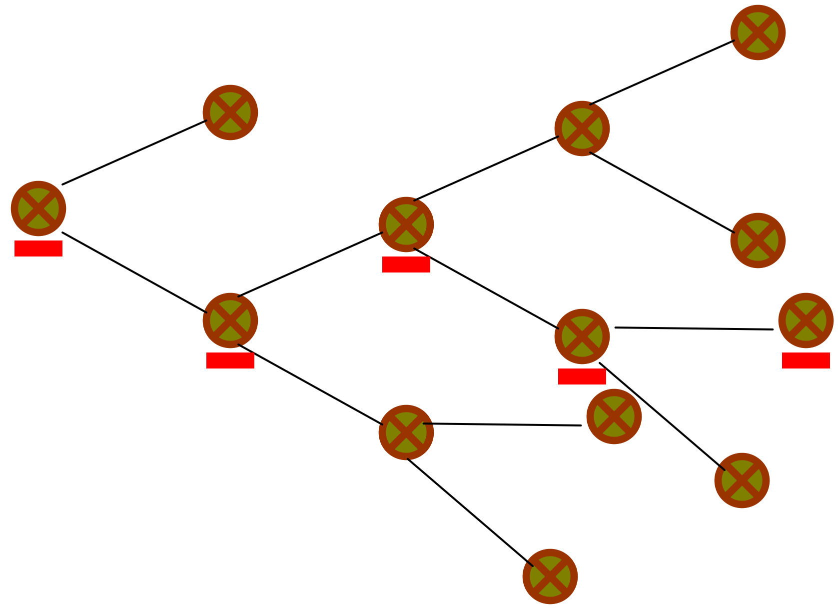
$n \to const$ localized

$n \to \infty$ extended

$W > I > W/K$

There is of order one resonance at every step

$n \sim \ln N$ **nonergodic**



Part 5.

Many-Body Localization

Metal-insulator transition in a weakly interacting many-electron system with localized single-particle states

Annals of Physics, v. 321, p. 1126-1205 (2006)

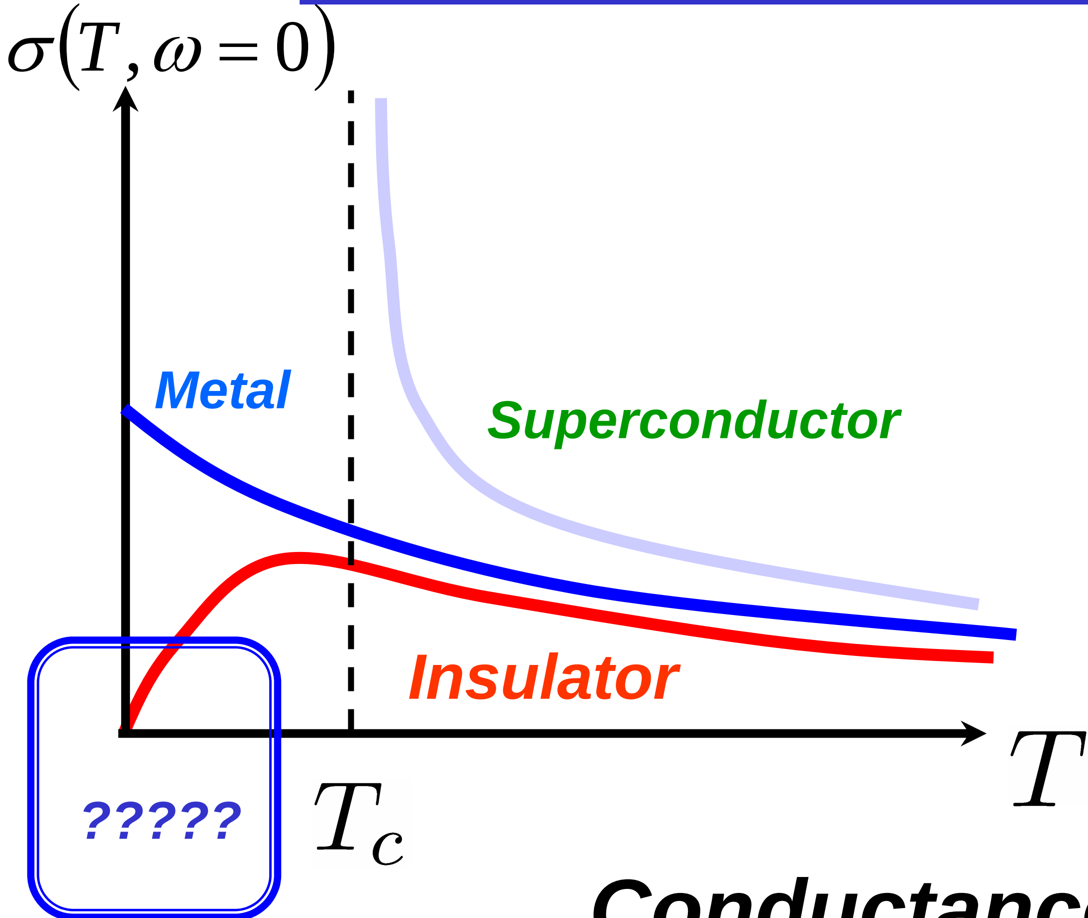
D. M. Basko & I.L. Aleiner

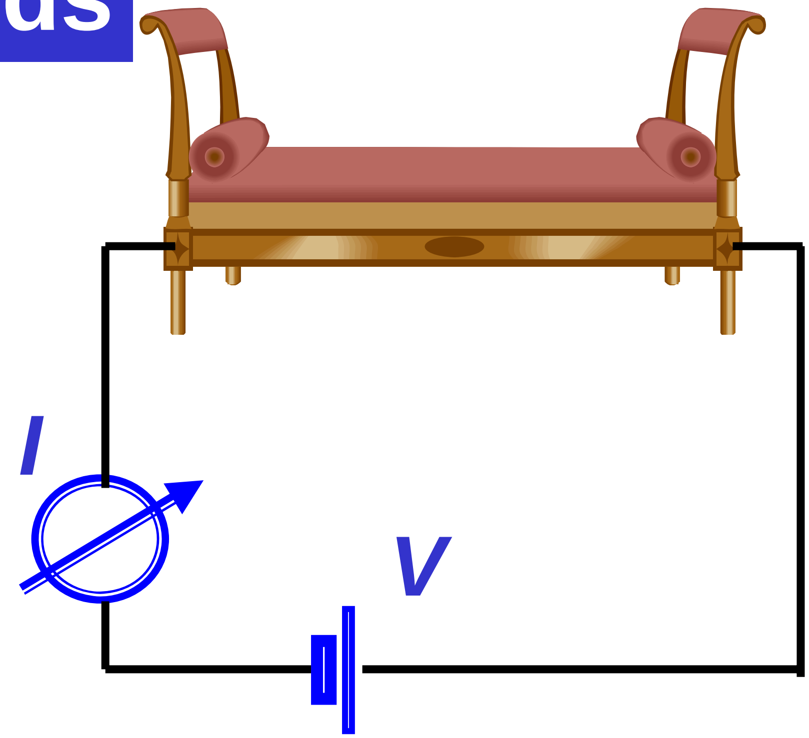
Columbia University

B. L. Altshuler

Columbia University, NEC-Laboratories America

Transport in solids

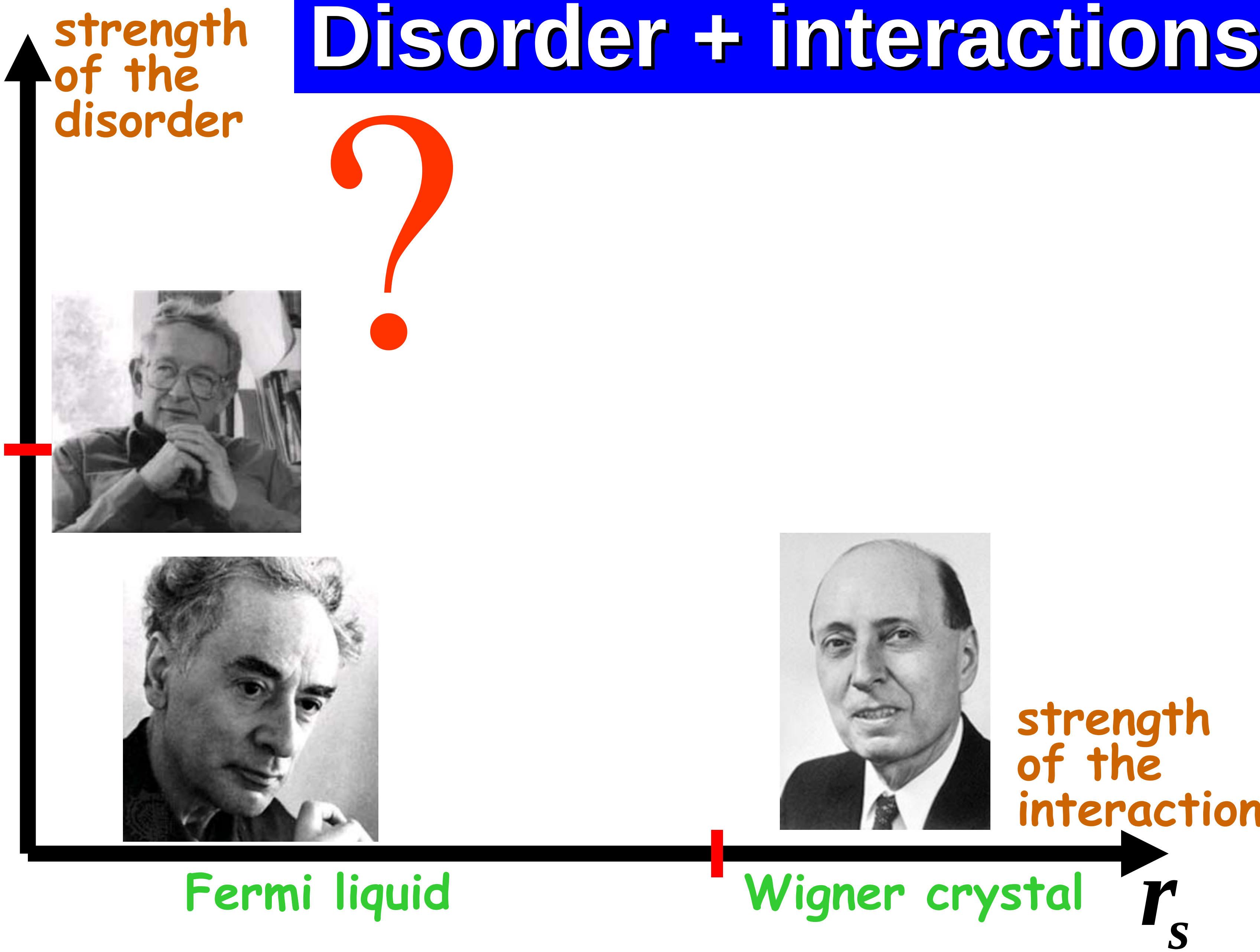




Conductance: $$G(\omega, T) = \left.\frac{I}{V}\right|_{V \to 0}$$

Conductivity: $$G(\omega, T) = \sigma(\omega, T)\frac{L_x L_y}{L_z}$$

Disorder + interactions



Can hopping conductivity exist without phonons ?

Question: can *e-e* interaction **alone** sustain hopping conduction in a localized system?

Given:

1. All one-electron states are localized
2. Electrons interact with each other
3. The system is closed (no phonons)
4. Temperature is low but finite

Find: DC conductivity $\sigma(T,\omega=0)$

zero or finite?

"All states are localized "

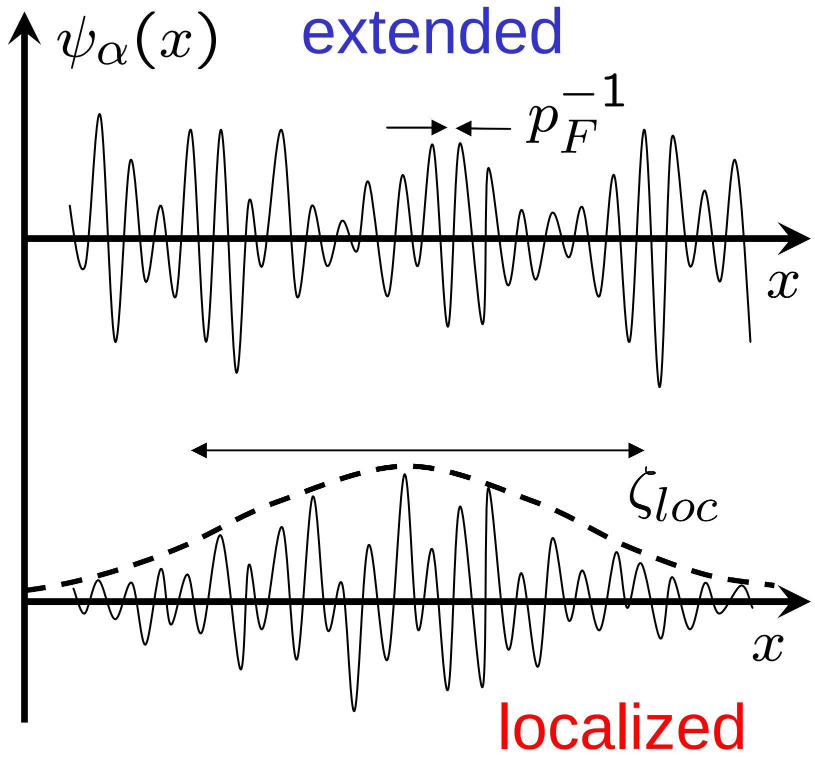
means that

probability to find a state extended over the system size L is

$$\mathcal{P}_{ext} \propto \exp\left(-\#\frac{L}{\zeta_{loc}}\right)$$

1. Localization of single-electron wave-functions:

$$\left[-\frac{\boldsymbol{\nabla}^2}{2m} + U(\boldsymbol{r}) - \epsilon_F\right]\psi_\alpha(\boldsymbol{r}) = \xi_\alpha \psi_\alpha(\boldsymbol{r})$$



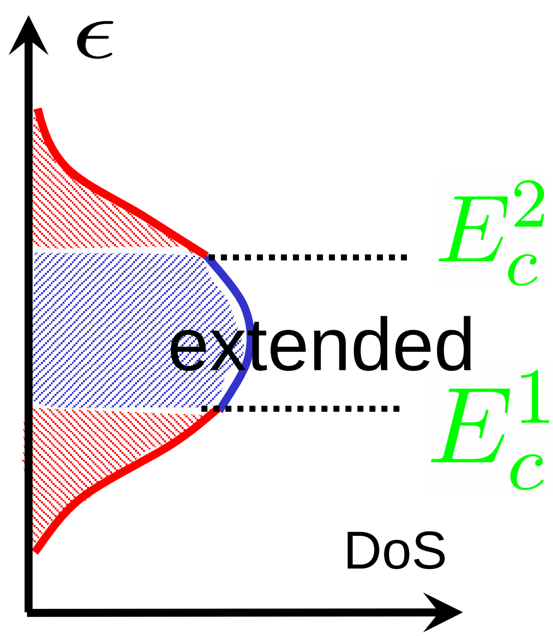
d=1; All states are *localized*

d=2; All states are *localized*

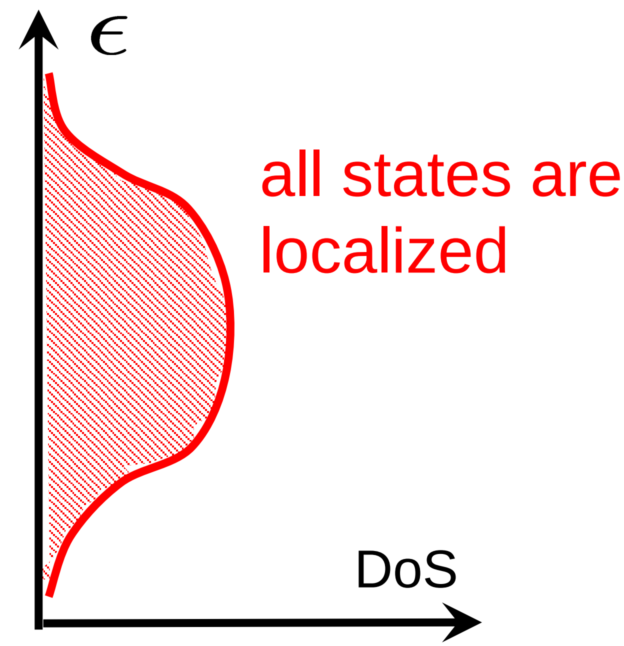
d>2; ***Anderson transition***

Anderson Transition

$I > I_c$



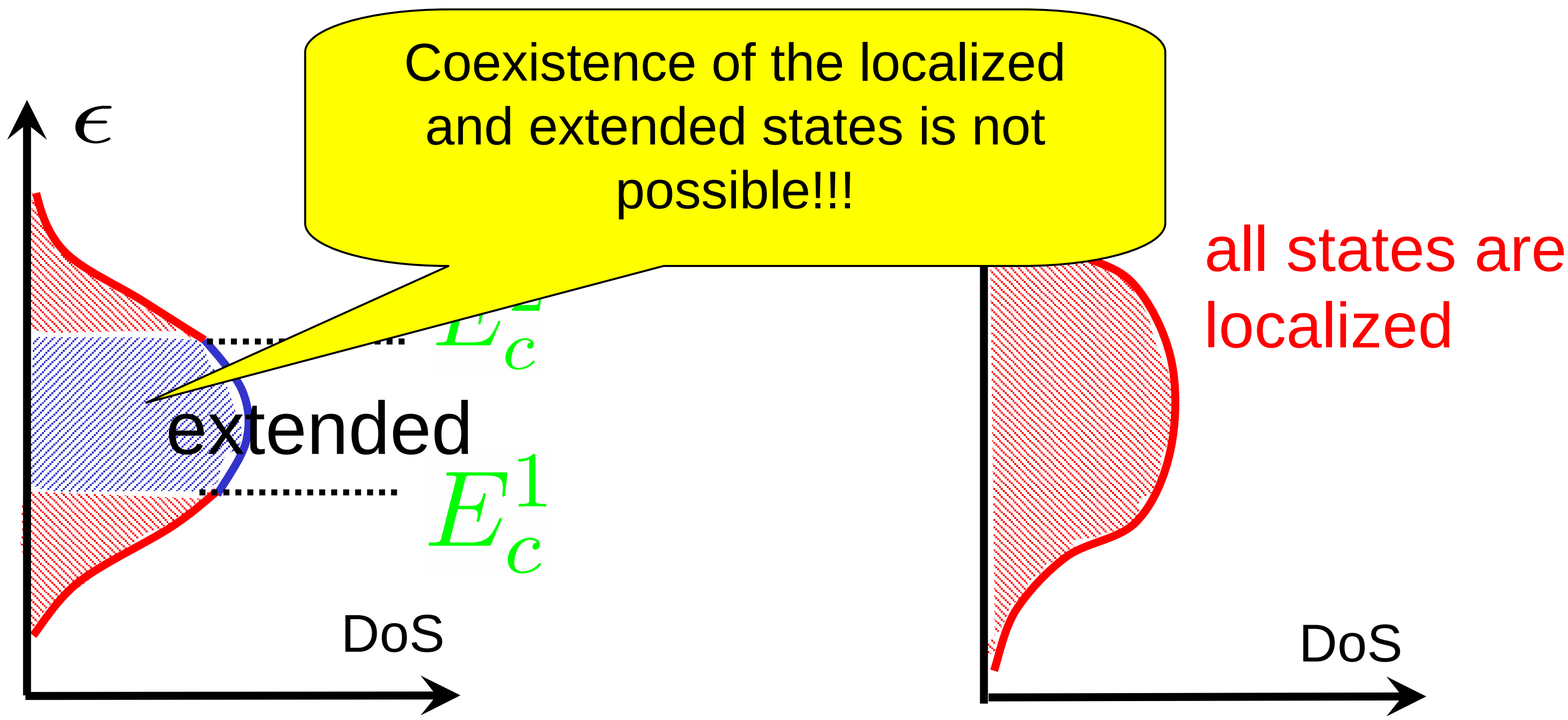
$I < I_c$



E_c - mobility edges (one particle)

Anderson Transition

$I > I_c$ $\qquad$ $I < I_c$



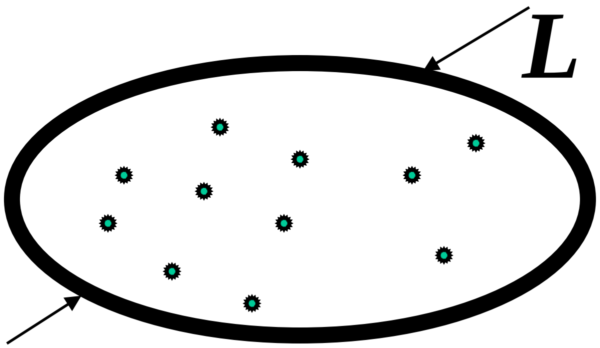
E_c - mobility edges (one particle)

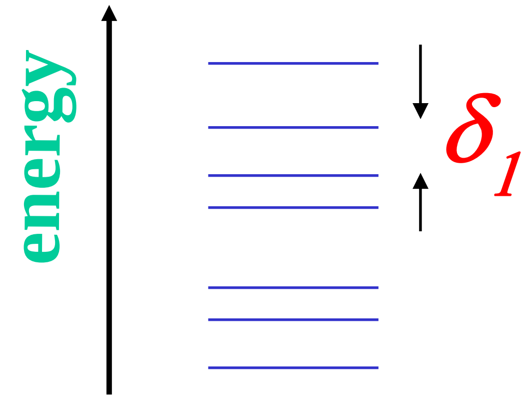
Quantum particle in a random potential (*Thouless, 1972*)

Energy scales

1. Mean level spacing

$$\delta_1 = 1/\nu \times L^d$$





L *is the system size;*

d *is the number of dimensions*

2. Thouless energy

$$E_T = hD/L^2$$

D *is the diffusion const*

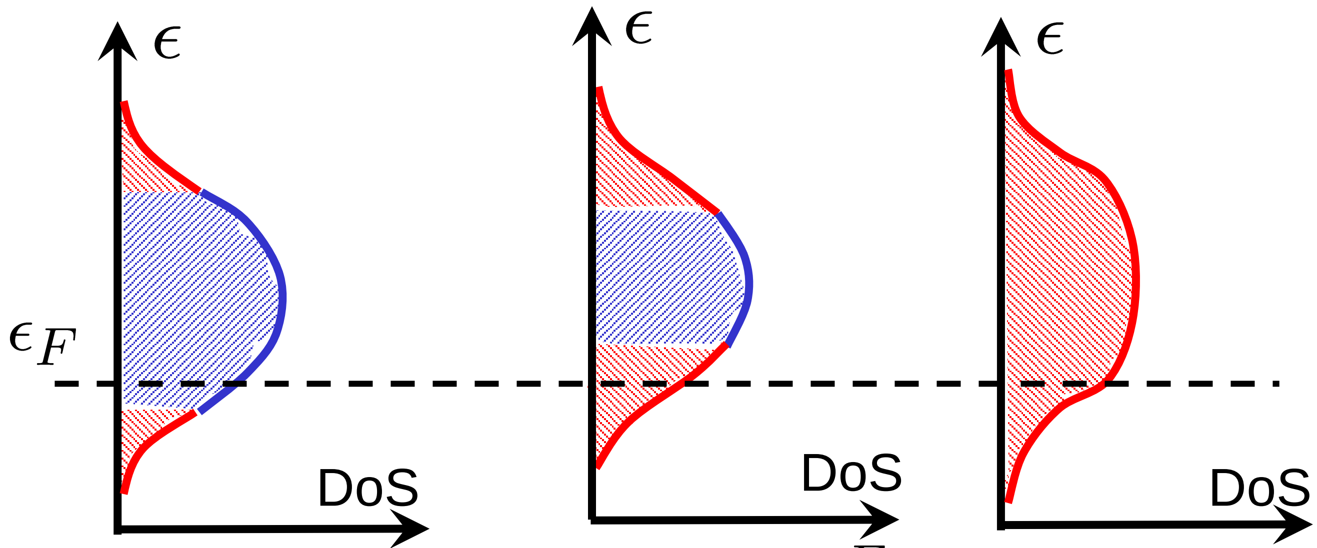
E_T *has a meaning of the inverse diffusion time of the traveling through the system or the escape rate (for open systems)*

$$g = E_T / \delta_1$$

dimensionless Thouless conductance

$$g = Gh/e^2$$

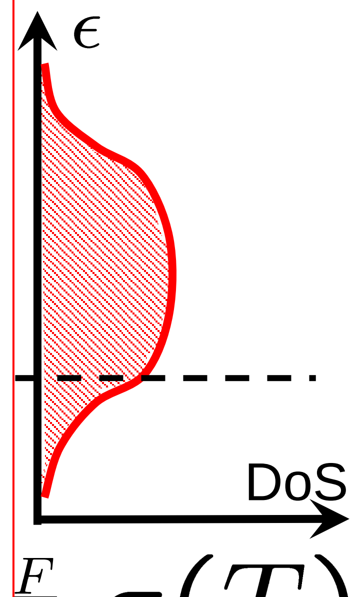
Temperature dependence of the conductivity of noninteracting electrons



$$\sigma(T \to 0) > 0 \qquad \sigma(T) \propto e^{-\frac{E_c - \epsilon_F}{T}} \qquad \sigma(T) = 0$$

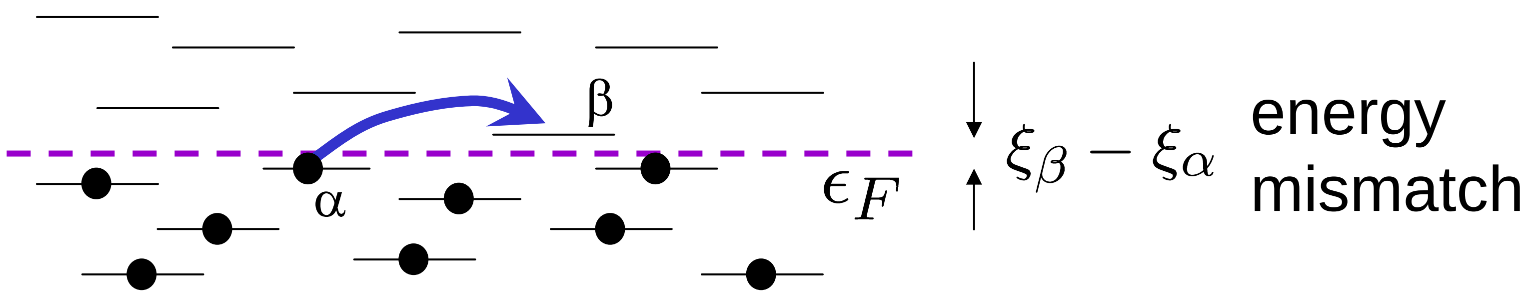
Temperature dependence of the conductivity of noninteracting electrons

Assume that all the states are localized



$$\sigma(T) = 0$$

Inelastic processes - transitions between localized states

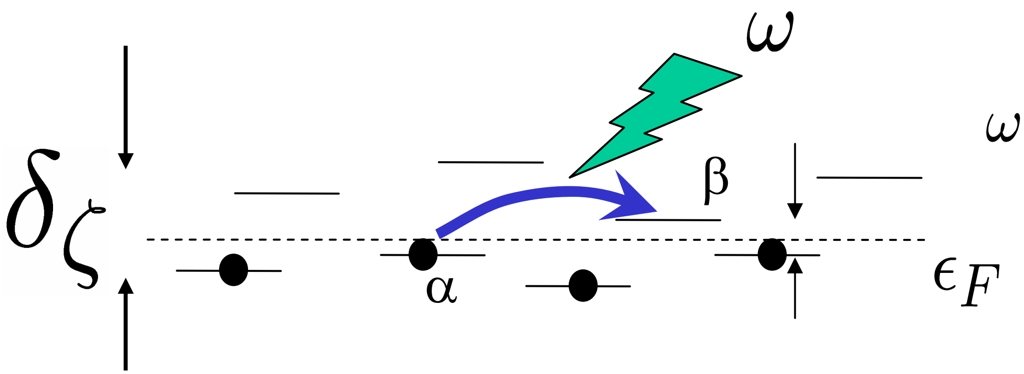


$$\sigma(T) \propto \Gamma_\alpha$$ (inelastic lifetime)$^{-1}$

$$T = 0 \quad \Rightarrow \quad \sigma = 0$$ (any mechanism)

$$T > 0 \quad \Rightarrow \quad \sigma = ?$$

Phonon-induced hopping



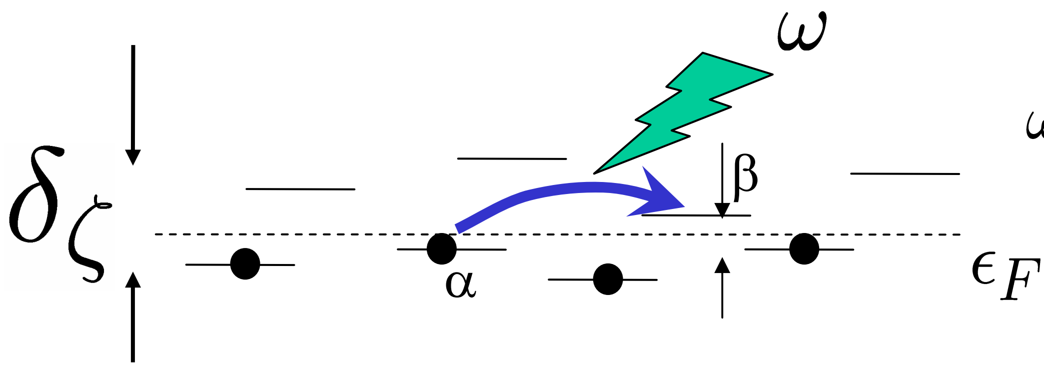
$$\omega = \xi_\beta - \xi_\alpha \ll \delta_\zeta$$

$$\delta_\zeta \equiv \frac{1}{\nu \zeta_{loc}{}^d}$$

localization spacing

energy difference can be matched by a phonon

Phonon-induced hopping



$$\omega = \xi_\beta - \xi_\alpha \ll \delta_\zeta$$

Variable Range Hopping
N.F. Mott (1968)

$$\sigma(T) \propto T^{\gamma} \exp\left[-\left(\frac{\delta_\zeta}{T}\right)^{\frac{1}{d+1}}\right]$$

Mechanism-dependent prefactor

Optimized phase volume

Any bath with a continuous spectrum of delocalized excitations down to $\boldsymbol{\omega = 0}$ will give the same exponential

Q: Can e-h pairs replace phonons and lead to phonon-less *Variable Range Hoping* ?

A#1: Sure

Easy steps:

1) Recall phonon-less AC conductivity:

Sir N.F. Mott (1970)

$$\sigma(\omega) \simeq \frac{e^2 \zeta_{loc}^{d-2}}{\hbar} \left(\frac{\hbar\omega}{\delta_\zeta} \right)^2 \ln^{d+1} \left| \frac{\delta_\zeta}{\hbar\omega} \right|$$

2) Calculate the Nyquist noise.

3) Use the electric noise instead of phonons.

4) Do self-consistency (whatever it means).

Q: Can e-h pairs replace phonons and lead to phonon-less *Variable Range Hoping* ?

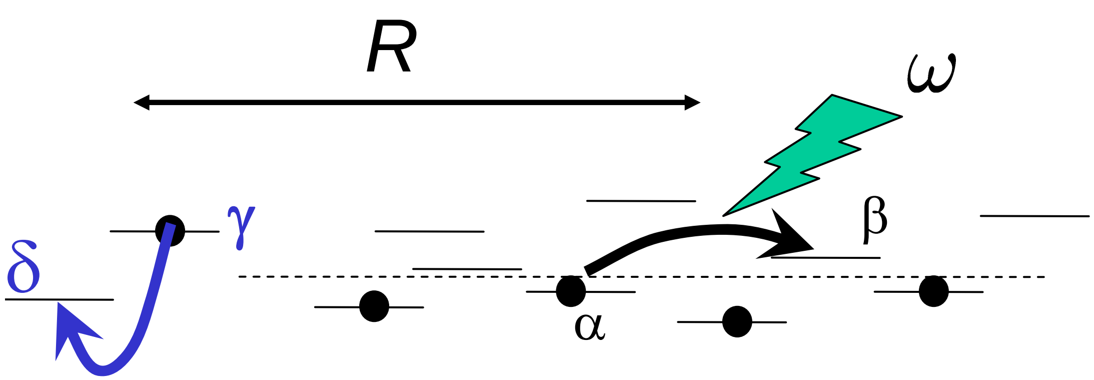
A#1: Sure

A#2: No way [L. Fleishman. P.W. Anderson (1980)]

(for Coulomb interaction in 3D – may be)

$$\sigma(\omega) \simeq \frac{e^2 \zeta_{loc}^{d-2}}{\hbar} \left(\frac{\hbar\omega}{\delta_\zeta}\right)^2 \ln^{d+1}\left|\frac{\delta_\zeta}{\hbar\omega}\right|$$

is contributed by rare resonances



$$\omega = \xi_\beta - \xi_\alpha = \xi_\gamma - \xi_\delta$$

ϵ_F

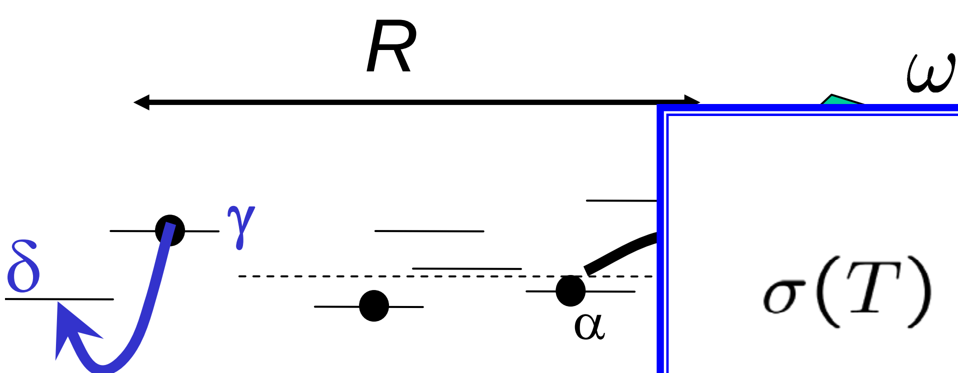
Q: Can e-h pairs replace phonons and lead to phonon-less *Variable Range Hoping* ?

A#1: Sure

A#2: No way [L. Fleishman. P.W. Anderson (1980)]

(for Coulomb interaction in 2D – may be)

$R \rightarrow \infty$ ***Thus, the matrix element vanishes !!!***



$$\sigma(T) \propto 0 * \exp\left[-\left(\frac{\delta_\zeta}{T}\right)^{\frac{1}{d+1}}\right]$$

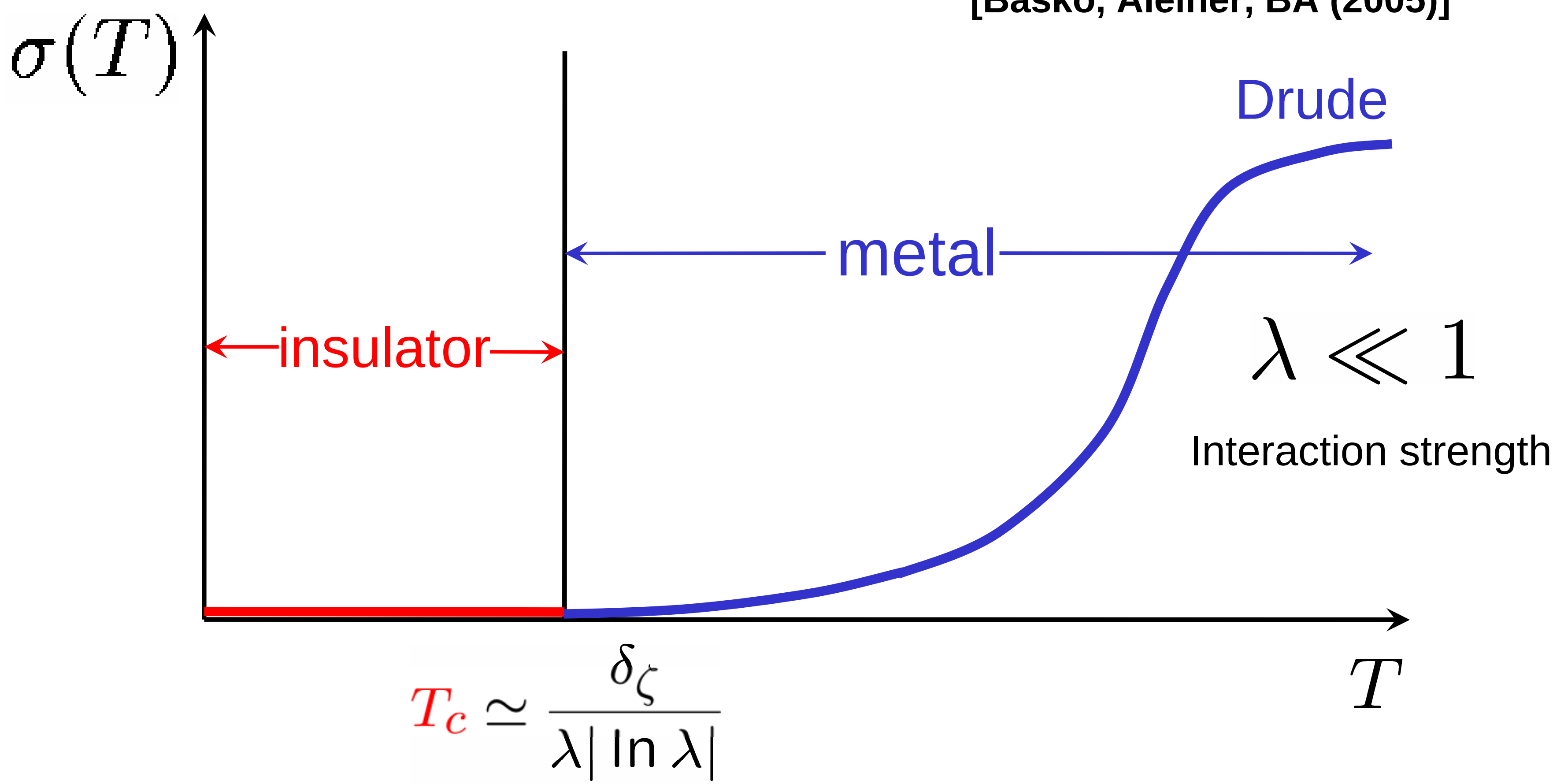
Q: Can we replace phonons with e-h pairs and obtain **phonon-less** *VRH?*

A#1: Sure [a person from the street (2005)]:

A#2: No way [L. Fleishman. P.W. Anderson (1980)]

A#3: Finite *T* ***Metal-Insulator Transition***

[Basko, Aleiner, BA (2005)]



Q: Can we replace phonons with e-h pairs and obtain **phonon-less** *VRH?*

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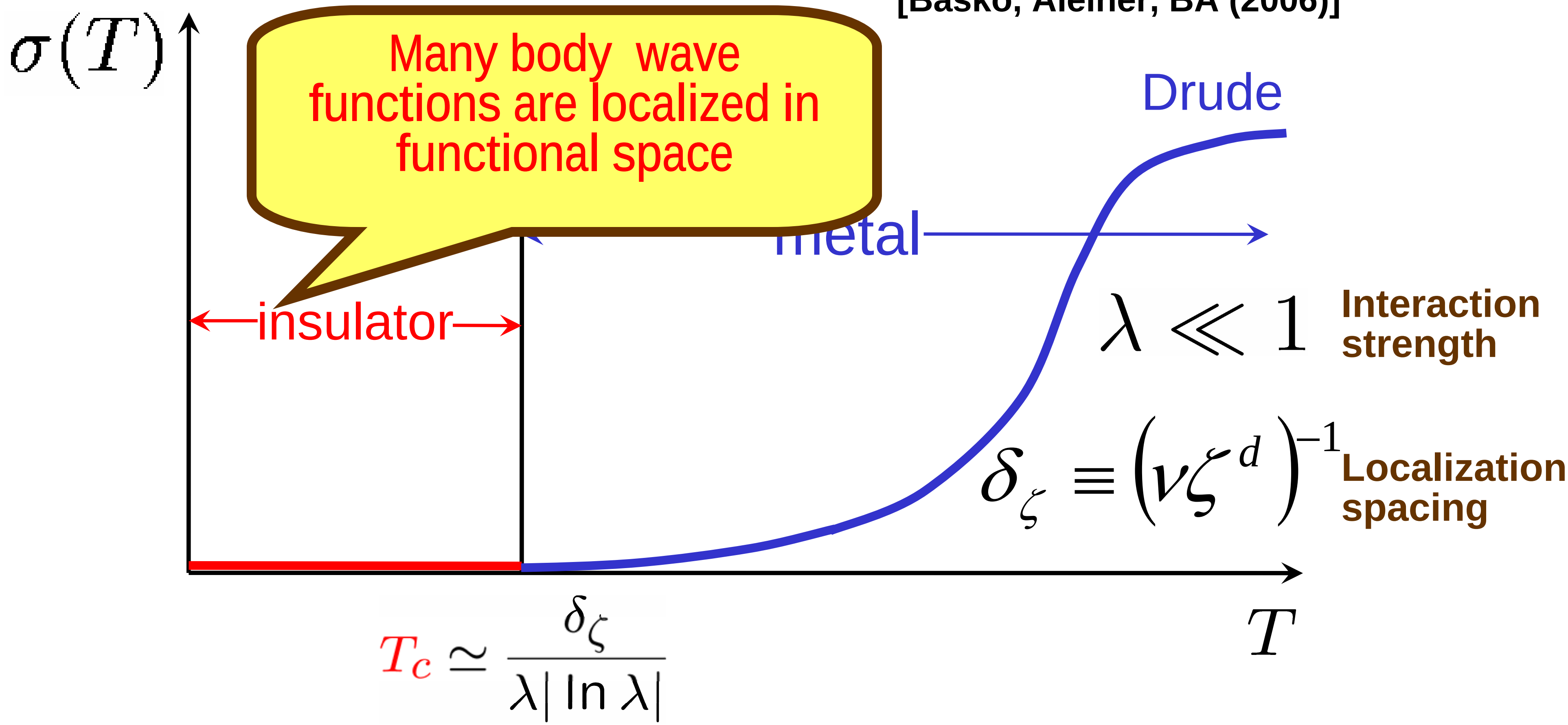
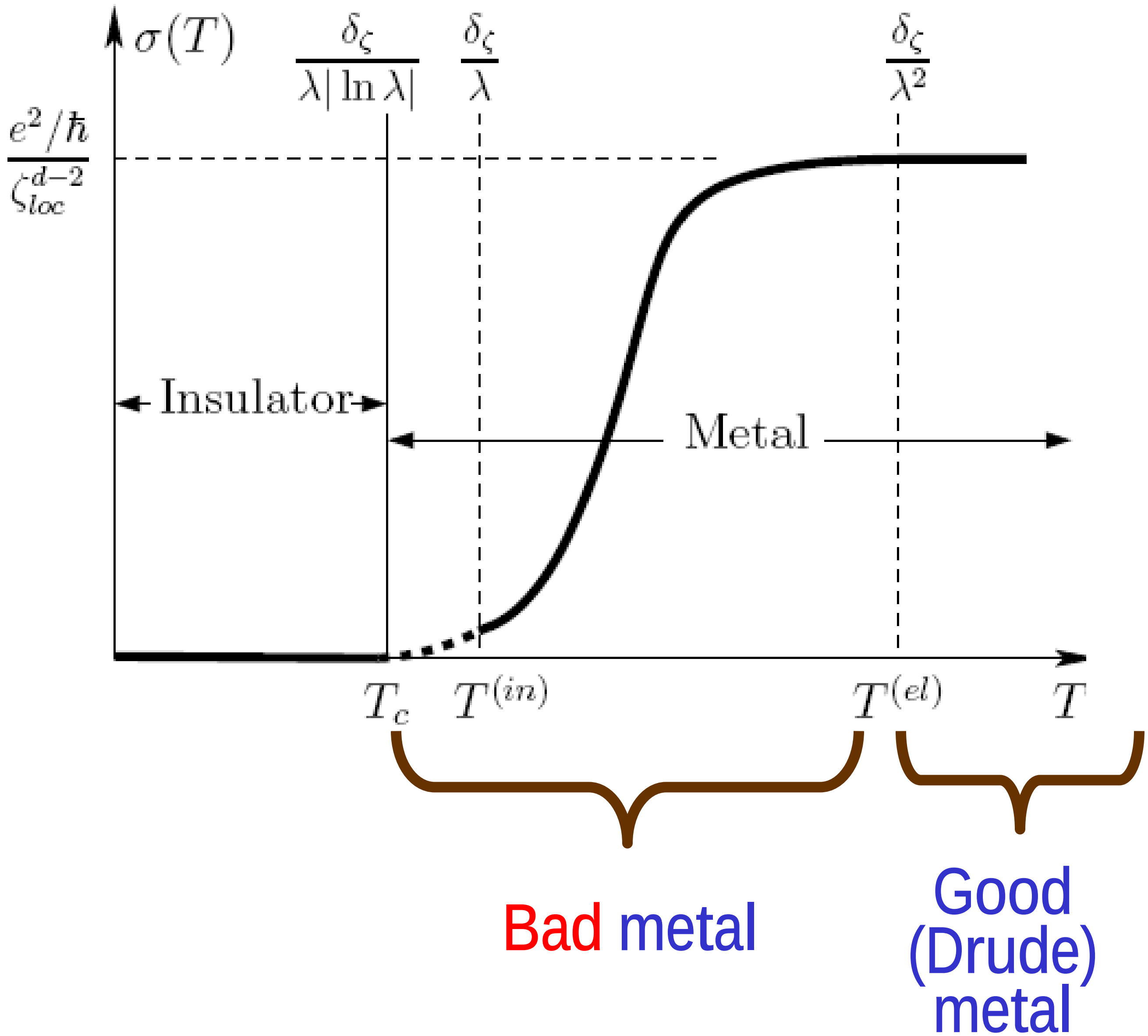


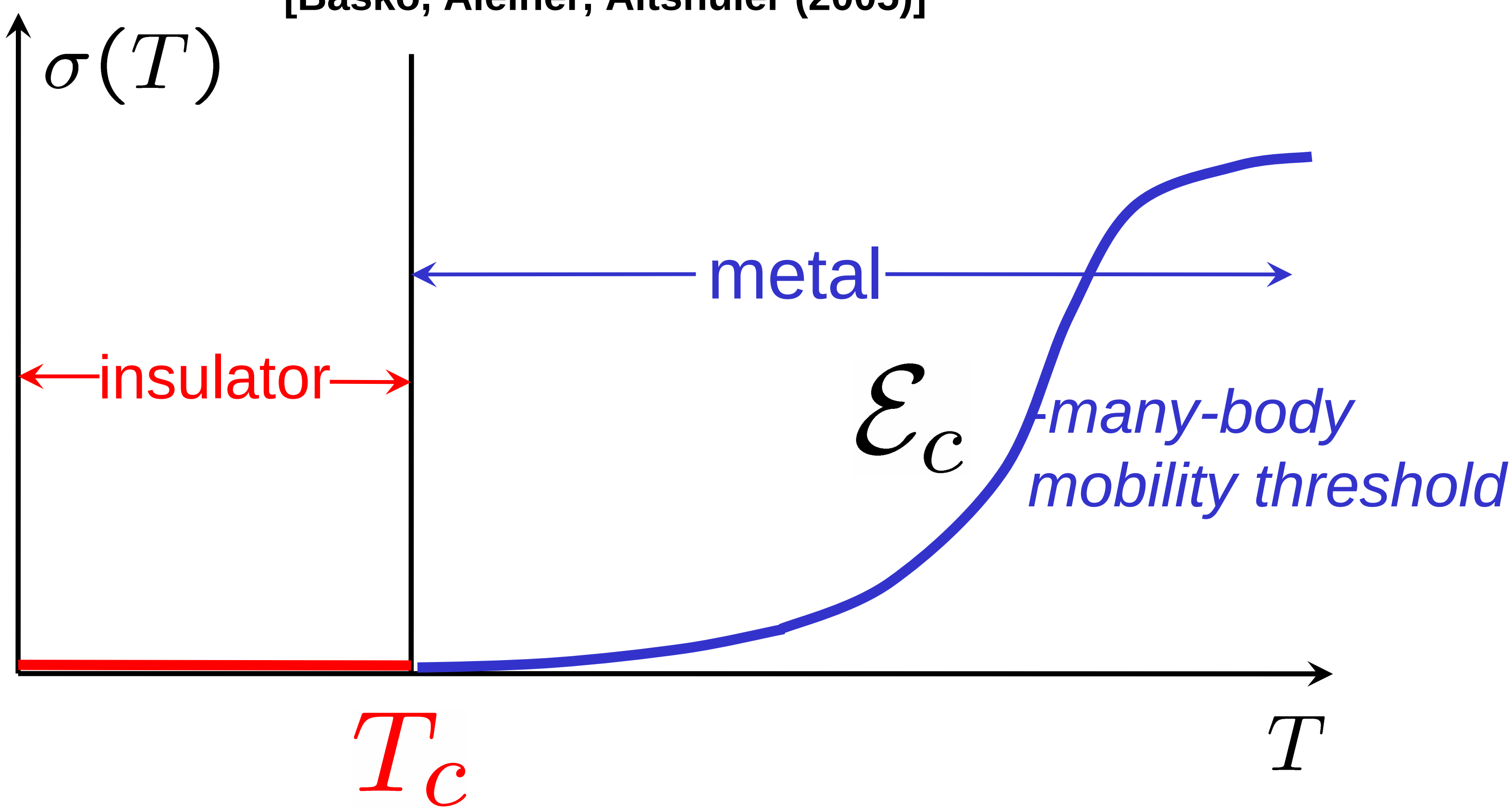


$\sigma(T)$
$\frac{\delta_\zeta}{\lambda|\ln\lambda|}$
$\frac{\delta_\zeta}{\lambda}$
$\frac{\delta_\zeta}{\lambda^2}$
$\frac{e^2/\hbar}{\zeta_{loc}^{d-2}}$
Insulator
Metal
T_c
$T^{(in)}$
$T^{(el)}$
T
Bad metal
Good (Drude) metal

Many-body mobility threshold

$$\left[\hat{H}_1 + \hat{H}_{int}\right]\Psi_\alpha = \mathcal{E}_\alpha \Psi_\alpha$$

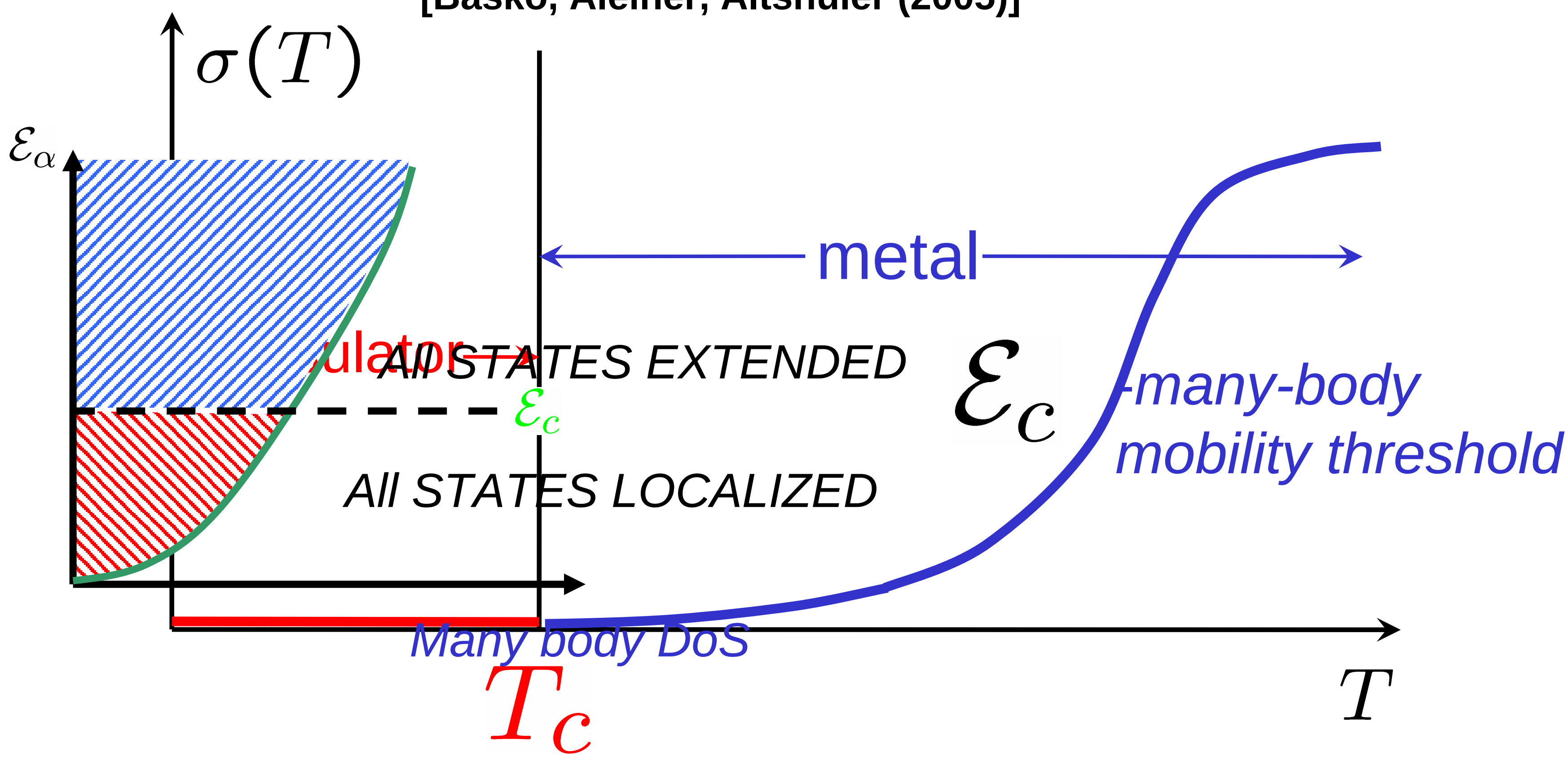
[Basko, Aleiner, Altshuler (2005)]



Many-body mobility threshold

$$\left[\hat{H}_1 + \hat{H}_{int}\right]\Psi_\alpha = \mathcal{E}_\alpha \Psi_\alpha$$

[Basko, Aleiner, Altshuler (2005)]



"All states are localized "

means that

Probability to find an extended state:

$$\mathcal{P}_{ext} \propto \exp\left(-\#\frac{\mathcal{V}}{\mathcal{V}_{loc}(\mathcal{E})}\right)$$

$$\lim_{\mathcal{E}\to\mathcal{E}_c-0} \mathcal{V}_{loc}(\mathcal{E}) = \infty$$

System volume

Main energy scale

$$\delta_\zeta \equiv \frac{1}{\nu \zeta^d}$$

Energy spacing between the states localized nearby

ζ **localization length**

ν **one-electron density of states**

Need a model with small parameters

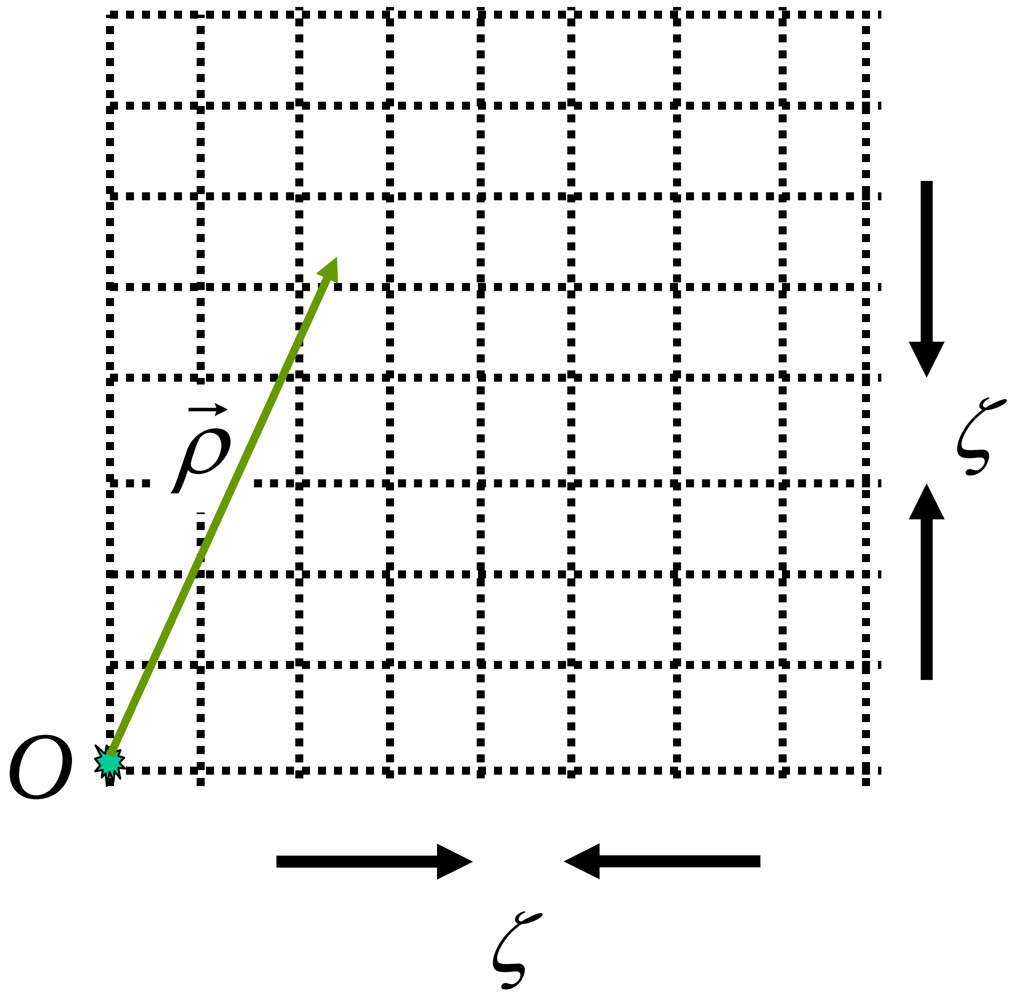
We have to take into account that

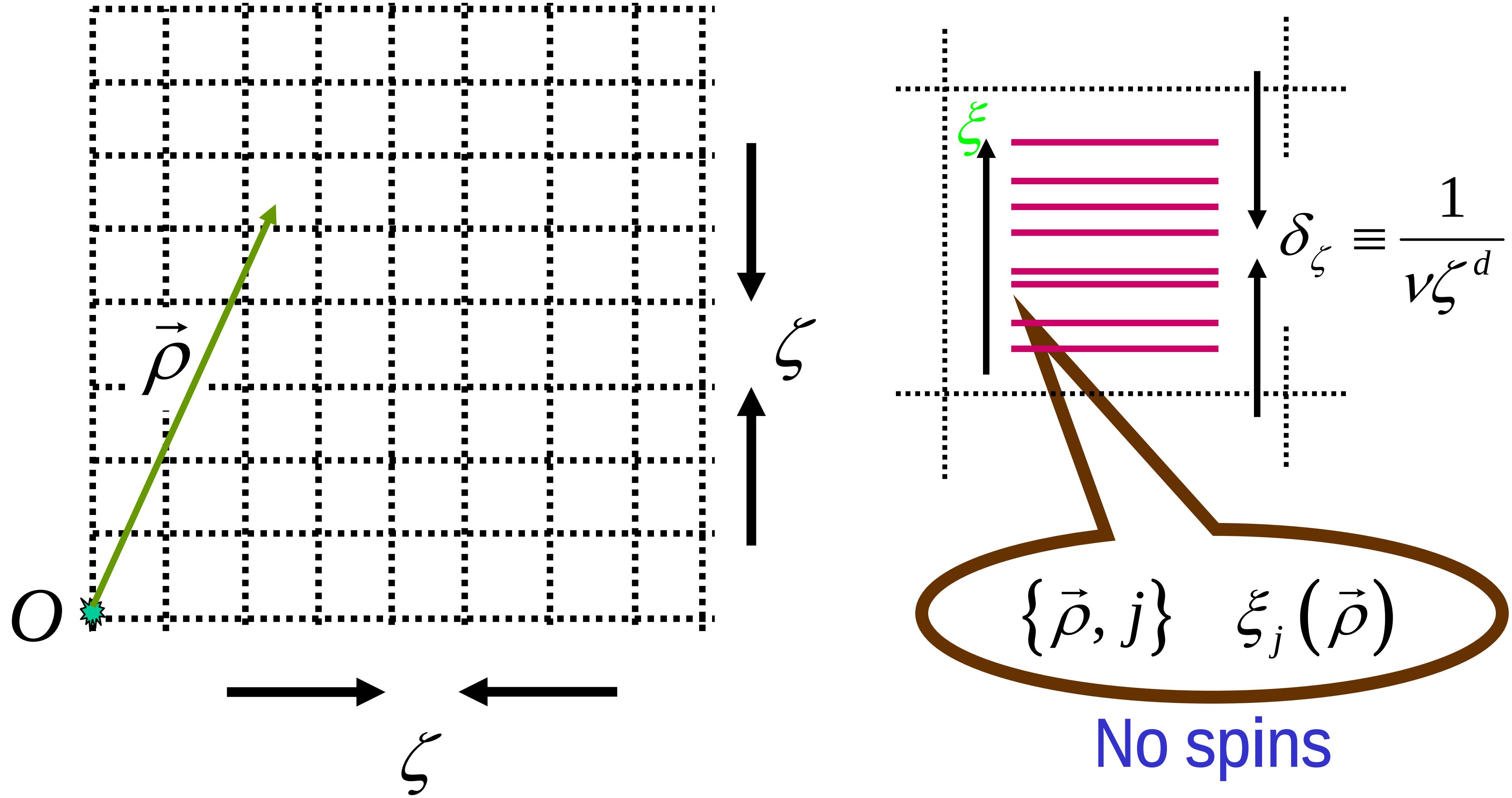
1. A one-electron wave function decays exponentially as a function of the distance from its center.

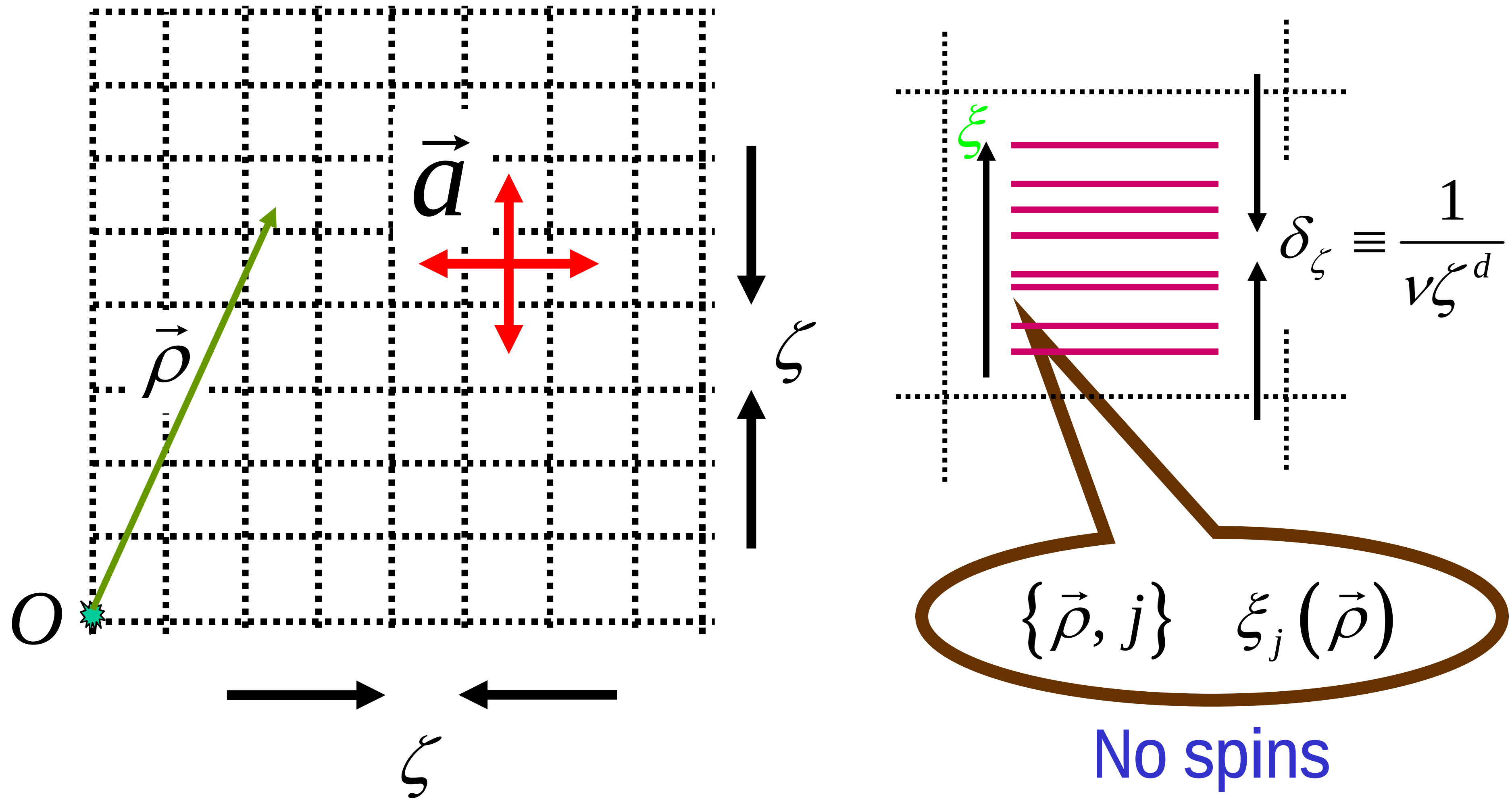
2. There is level repulsion for the states localized nearby

3. Matrix elements of the interaction decay (probably as a power law) when differences between the energies of involved quasiparticles is increased.

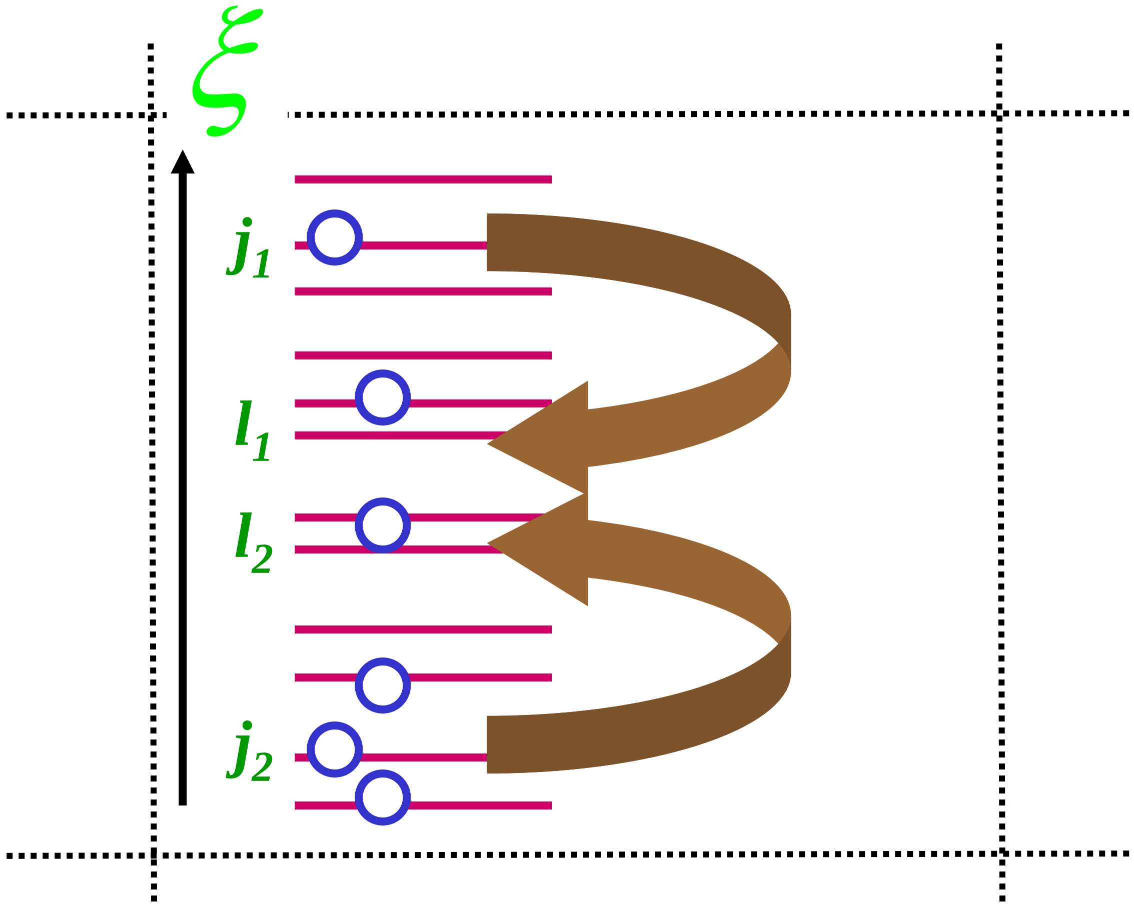
4. These matrix elements have random sign.

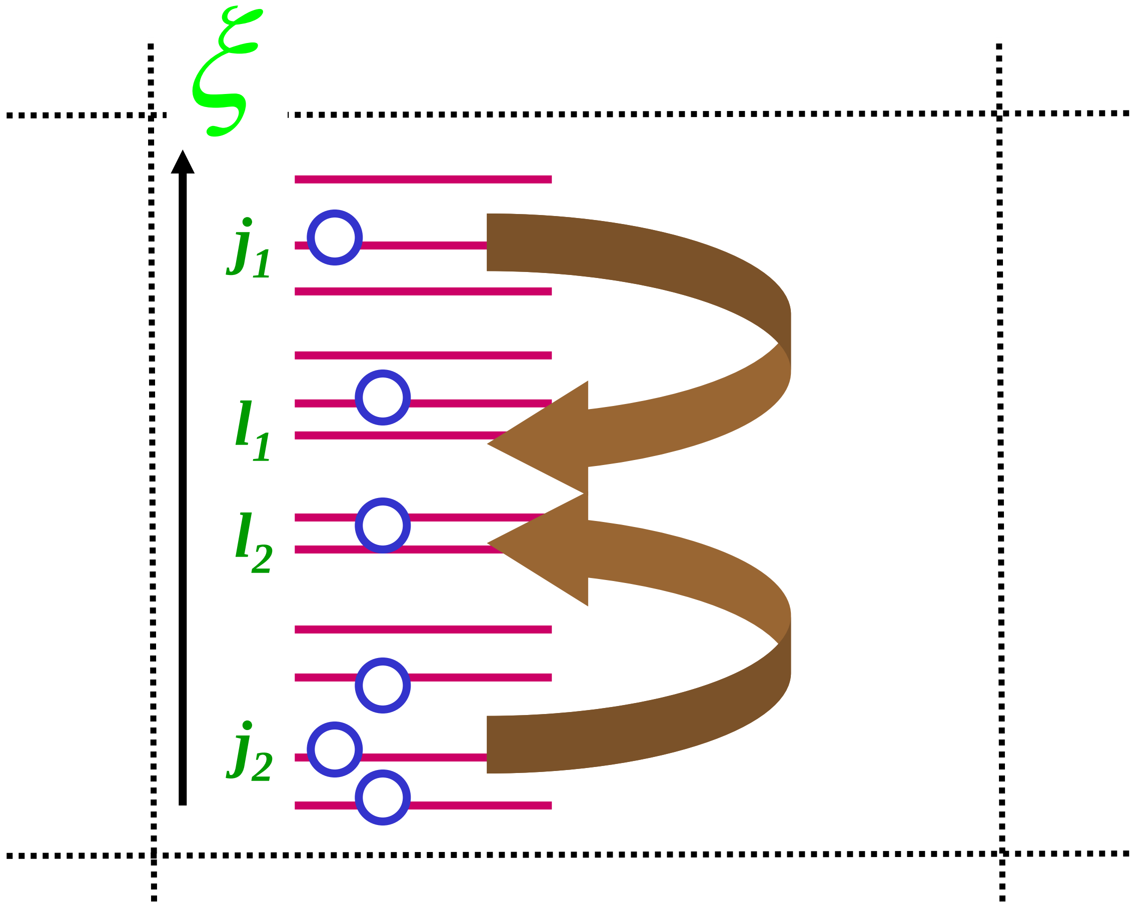

$\vec{\rho}$
O
ζ
ζ


O
$\vec{\rho}$
ζ
ζ
ξ
$\delta_\zeta \equiv \frac{1}{\nu \zeta^d}$
$\{\vec{\rho}, j\}$ $\xi_j(\vec{\rho})$
No spins



$$\hat{H}_0 = \sum_{\vec{\rho},l} \hat{c}_l^\dagger(\vec{\rho}) \left[\xi_l(\vec{\rho})\hat{c}_l(\vec{\rho}) + I\delta_\zeta \sum_{\vec{a},m} \hat{c}_m(\vec{\rho}+\vec{a}) \right]$$

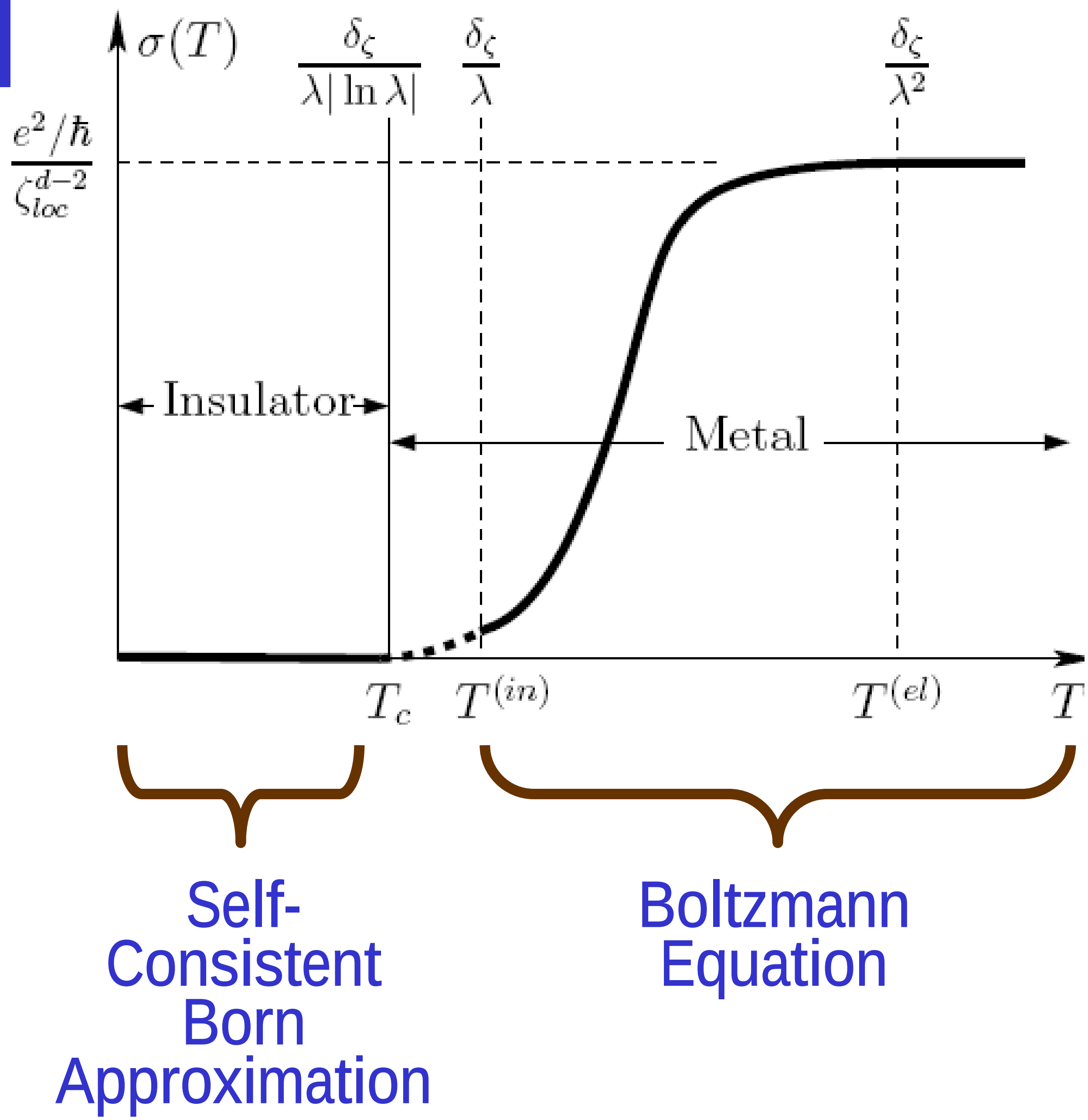

ξ
j_1
l_1
l_2
j_2



$$\hat{V}_{\text{int}} = \frac{1}{2} \sum_{\vec{\rho};l_1,l_2;j_1,j_2} V_{l_1,l_2}^{j_1,j_2}(\vec{\rho})\,\hat{c}_{l_1}^{\dagger}(\vec{\rho})\,\hat{c}_{l_2}^{\dagger}(\vec{\rho})\,\hat{c}_{j_1}(\vec{\rho})\,\hat{c}_{j_2}(\vec{\rho})$$

Interaction only within the same cell;
no diagonal matrix elements

Technique



$\sigma(T)$
$\frac{\delta_\zeta}{\lambda|\ln\lambda|}$
$\frac{\delta_\zeta}{\lambda}$
$\frac{\delta_\zeta}{\lambda^2}$
$\frac{e^2/\hbar}{\zeta_{loc}^{d-2}}$
Insulator
Metal
T_c
$T^{(in)}$
$T^{(el)}$
T
Self-Consistent Born Approximation
Boltzmann Equation

PHYSICAL REVIEW VOLUME 109, NUMBER 5 MARCH 1, 1958

Absence of Diffusion in Certain Random Lattices

P. W. Anderson
Bell Telephone Laboratories, Murray Hill, New Jersey
(Received October 10, 1957)

This paper presents a simple model for such processes as spin diffusion or conduction in the "impurity band." These processes involve transport in a lattice which is in some sense random, and in them diffusion is expected to take place via quantum jumps between localized sites. In this simple model the essential randomness is introduced by requiring the energy to vary randomly from site to site. It is shown that at low enough densities no diffusion at all can take place, and the criteria for transport to occur are given.

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A selfconsistent theory of localization

R Abou-Chacra†, P W Anderson‡§ and D J Thouless†
† Department of Mathematical Physics, University of Birmingham, Birmingham, B15 2TT
‡ Cavendish Laboratory, Cambridge, England and Bell Laboratories, Murray Hill, New Jersey, 07974, USA

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Idea of the calculation:

1. Start with some infinitesimal width η (***Im*** part of the self-energy due to a bath) of each one-electron eigenstate
2. Consider ***Im*** part of the self-energy Γ in the presence of tunneling and ***e-e*** interaction.
3. Calculate the probability distribution function $P(\Gamma)$
4. Consider the limit: $$\lim_{\eta\to 0, V\to\infty} P(\Gamma) \equiv P_0(\Gamma)$$

V is the volume of the system

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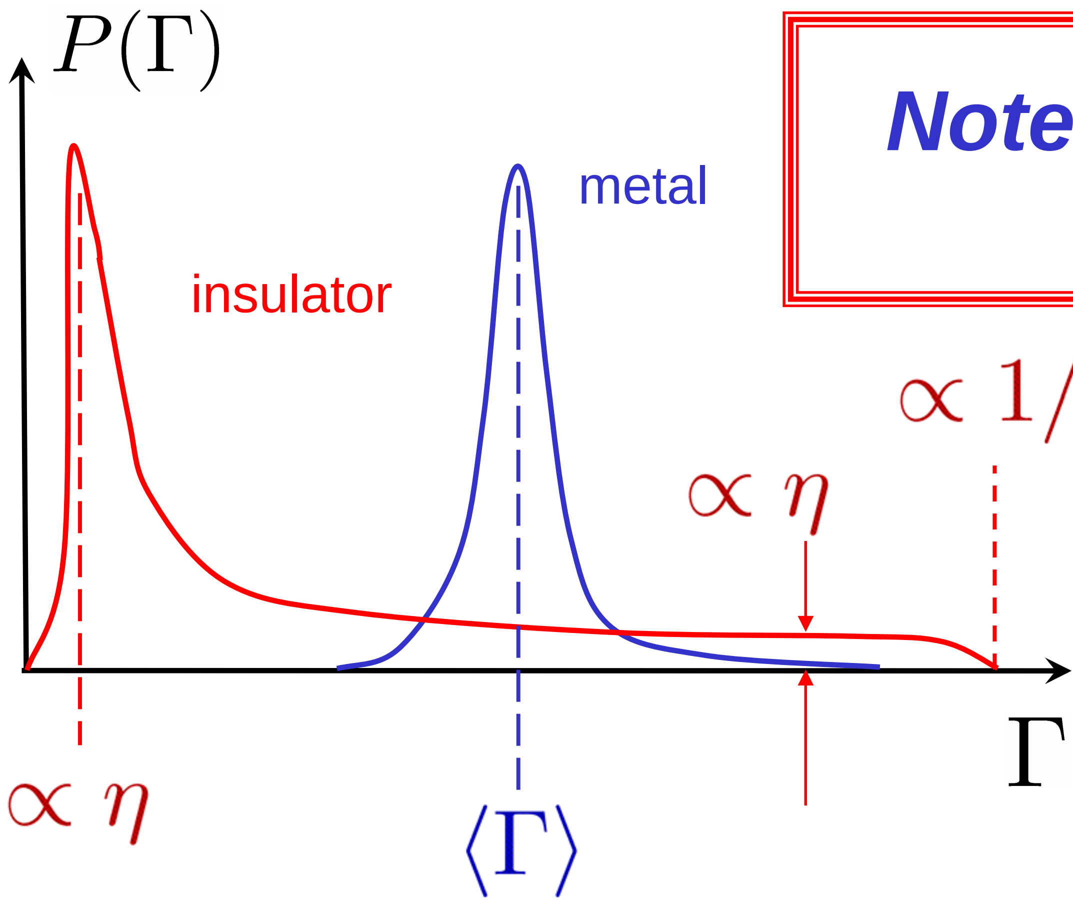
$$P_0(\Gamma) \begin{cases} = \delta(\Gamma) & \text{- insulator} \\ \neq 0 \; for \, \Gamma \neq 0 & \text{- metal} \end{cases}$$

Probability Distribution



Note: $\langle\Gamma\rangle = \langle\Gamma\rangle$

Look for:

$$\lim_{\eta\to+0}\lim_{\mathcal{V}\to\infty} P(\Gamma>0) = \begin{cases} >0; & metal \\ 0; & insulator \end{cases}$$

Stability of the insulating phase: NO spontaneous generation of broadening

$$\Gamma_\alpha(\varepsilon) = 0$$

is always a solution

$$\varepsilon \to \varepsilon + i\eta$$

linear stability analysis

$$\frac{\Gamma}{(\varepsilon - \xi_\alpha)^2 + \Gamma^2} \to \pi\delta(\varepsilon - \xi_\alpha) + \frac{\Gamma}{(\varepsilon - \xi_\alpha)^2}$$

After ***n*** iterations of the equations of the Self Consistent Born Approximation

$$P_n(\Gamma) \propto \frac{\eta}{\Gamma^{3/2}} \left(const \frac{\lambda T}{\delta_\zeta} \ln \frac{1}{\lambda} \right)^n$$

first $n \to \infty$

then $\eta \to 0$

$(\ldots) < 1$ – insulator is stable !

Stability of the metallic phase: Finite broadening is self-consistent

- $$P(\Gamma) = \frac{1}{\sqrt{2\pi\langle\delta\Gamma^2\rangle}} \exp\left[-\frac{(\Gamma - \langle\Gamma\rangle)^2}{2\langle\delta\Gamma^2\rangle}\right]$$

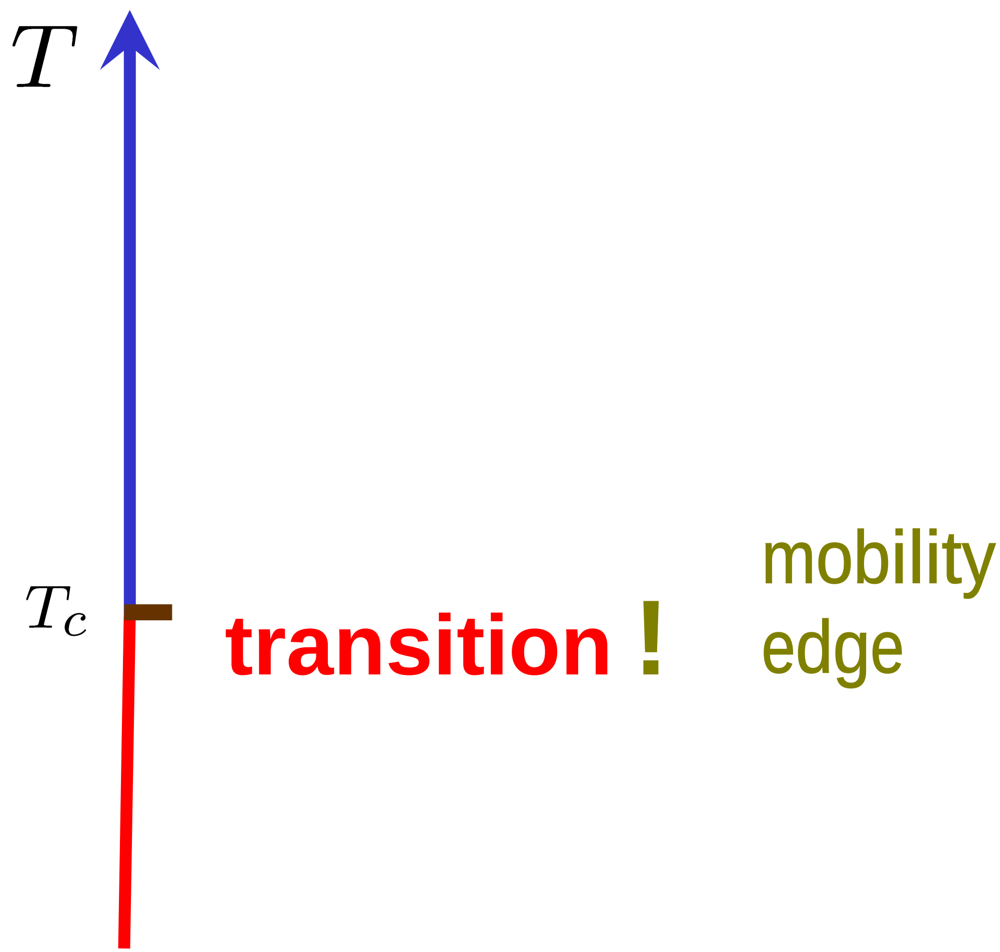
  $\sqrt{\langle\delta\Gamma^2\rangle} \ll \langle\Gamma\rangle$ as long as $T \gg \frac{\delta_\zeta}{\lambda}$

- $\langle\Gamma\rangle \ll \delta_\zeta$ (levels well resolved)

- quantum kinetic equation for transitions between localized states

  $\sigma(T) \propto \lambda^2 T^\alpha$ (model-dependent)

Many-body mobility edge



Many-body mobility edge

T

Large E (high T): extended states

(good metal)

ergodic states

δ_ζ/λ^2

Fermi Golden Rule hopping (bad metal)

nonergodic states

δ_ζ/λ

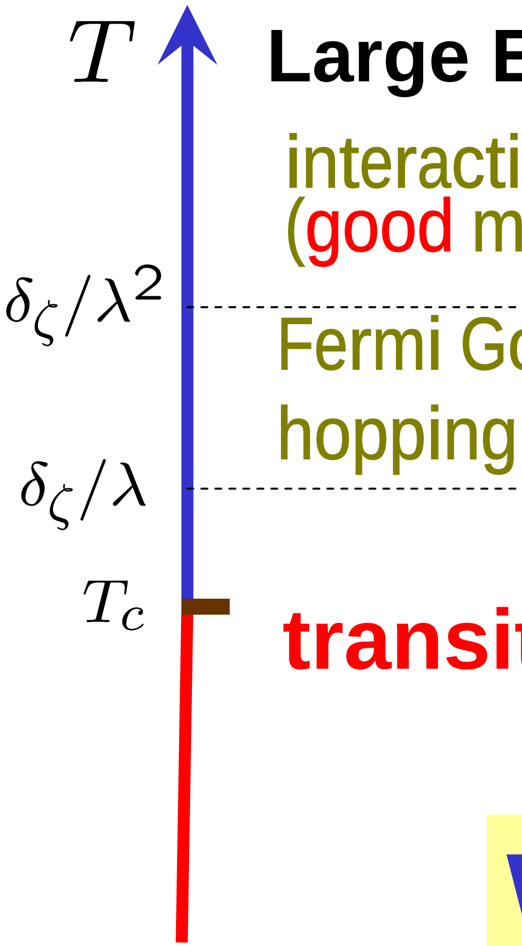
mobility edge

T_c

transition !

Such a state occupies infinitely many sites of the Anderson model but still negligible fraction of the total number of sites

Many-body mobility edge



Large E_k) high T: extended states

interaction ! dephasing ! cutoff of WL
(good metal)

Fermi Golden Rule
hopping (bad metal)

transition ! mobility edge

Why no activation?

Many-body mobility edge

T

Large E_k) high T: **extended states**

interaction **!** dephasing **!** cutoff of WL
(good metal)

δ_ζ/λ^2

Fermi Golden Rule
hopping (bad metal)

δ_ζ/λ

T_c

transition !

mobility
edge

$E, E_c \propto$ *volume*

No activation:

$$E_c \propto \frac{T_c^2}{\delta_\zeta \zeta^d} \qquad E \propto \frac{T^2}{\delta_\zeta \zeta^d} \qquad \times \textit{volume}$$

$$\exp\left(\frac{E(T)-E_c}{T}\right) \xrightarrow[\textit{volume} \to \infty]{} 0$$

Conclusions & Some speculations

Conductivity exactly vanishes at finite temperature. Finite temperature phase transition without any apparent symmetry change!
Is it an ordinary thermodynamic phase transition or low temperature phase is a glass?

We considered weak interaction.
What about strong electron-electron interactions?
Melting of a pined Wigner crystal?

What if we now turn on phonons?
Cascades.
Is conventional hopping conductivity picture ever correct?