Part II: Functional RG for Fermi systems

A natural way of dealing with many energy scales in interacting electron systems and a powerful source of new approximations.

• applicable to microscopic models (not only field theory)

• no adjustable parameters

• RG treatment of infrared singularities built in

1. Generating functionals
2. Exact flow equations
3. Truncations

1. Generating functionals

Interacting Fermi system with bare action

$$S[\psi, \bar{\psi}] = -(\bar{\psi}, C^{-1}\psi) + V[\psi, \bar{\psi}]$$

$\psi_K, \bar{\psi}_K$ Grassmann variables, K = quantum numbers + Matsubara frequency

C bare propagator, $V[\psi, \bar{\psi}]$ interaction

$$(\bar{\psi}, C^{-1}\psi) = \sum_K \bar{\psi}_K \, (C^{-1}\psi)_K \quad \text{with} \quad (C^{-1}\psi)_K = \sum_{K'} (C^{-1})_{KK'} \, \psi_{K'}$$

Spin-$\frac{1}{2}$ fermions with momentum $\mathbf{k}$ and spin orientation σ: $K = (k_0, \mathbf{k}, \sigma)$

Bare propagator in case of translation and spin-rotation invariance:

$$C(K) = \frac{1}{ik_0 - \xi_{\mathbf{k}}} \quad \text{(diagonal), where } \xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu$$

Two-particle interaction:

$$V[\psi, \bar{\psi}] = \frac{1}{4} \sum_{K_1, K_2} \sum_{K_1', K_2'} V(K_1', K_2'; K_1, K_2)\, \bar{\psi}_{K_1'} \psi_{K_1} \bar{\psi}_{K_2'} \psi_{K_2}$$

Generating functional for connected Green functions

$$\mathcal{G}[\eta,\bar\eta] = -\log\left\{\int \prod_K d\psi_K d\bar\psi_K \, e^{-S[\psi,\bar\psi]}\, e^{(\eta,\bar\psi)+(\psi,\bar\eta)}\right\}$$

Connected m-particle Green function

$$\begin{aligned} G_m(K_1',\ldots,K_m';K_1,\ldots,K_m) &= \\ -\underbrace{\langle \psi_{K_1'}\ldots\psi_{K_m'}\bar\psi_{K_m}\ldots\bar\psi_{K_1}\rangle_c}_{\text{connected average}} &= \frac{\partial^m}{\partial\eta_{K_1}\ldots\partial\eta_{K_m}}\frac{\partial^m}{\partial\bar\eta_{K_m'}\ldots\partial\bar\eta_{K_1'}}\mathcal{G}[\eta,\bar\eta]\Big|_{\eta=\bar\eta=0}\end{aligned}$$

Legendre transform of $\mathcal{G}[\eta,\bar\eta]$: effective action

$$\Upsilon[\psi,\bar\psi] = \mathcal{G}[\eta,\bar\eta] + (\bar\psi,\eta) - (\bar\eta,\psi) \quad \text{with} \quad \psi = \frac{\partial\mathcal{G}}{\partial\bar\eta} \quad \text{and} \quad \bar\psi = \frac{\partial\mathcal{G}}{\partial\eta}$$

generates one-particle irreducible (1PI) vertex functions Γ_m

$$\Gamma_1 = G^{-1} = C^{-1} - \Sigma$$

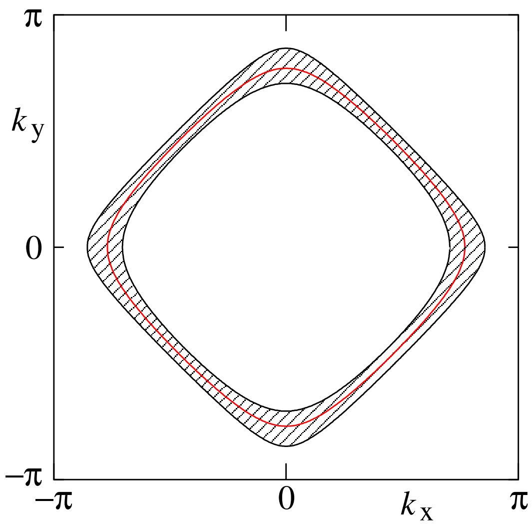
Reciprocity relations:

$$\frac{\partial\Upsilon}{\partial\psi} = \bar\eta \ , \quad \frac{\partial\Upsilon}{\partial\bar\psi} = \eta \ , \quad \frac{\partial^2\Upsilon}{\partial\psi\partial\bar\psi} = \left(\frac{\partial^2\mathcal{G}}{\partial\eta\partial\bar\eta}\right)^{-1}$$

2. Exact flow equations

Impose infrared cutoff at energy scale $\Lambda > 0$, e.g. a momentum cutoff

$$C^\Lambda(K) = \frac{\Theta^\Lambda(\mathbf{k})}{ik_0 - \xi_\mathbf{k}} \quad \text{with} \quad \Theta^\Lambda(\mathbf{k}) = \Theta(|\xi_\mathbf{k}| - \Lambda)$$



Momentum space region around the Fermi surface excluded by a sharp momentum cutoff in a 2D lattice model

Cutoff regularizes divergence of $C(K)$ in $k_0 = 0$, $\xi_\mathbf{k} = 0$ (Fermi surface)

Other choices: smooth cutoff, frequency cutoff,
mixed momentum-frequency cutoff $\Theta^\Lambda(\sqrt{\xi_\mathbf{k}^2 + k_0^2})$

Cutoff excludes "soft modes" below scale Λ from functional integral.

Λ-dependent functionals $\mathcal{G}^{\Lambda}[\eta, \bar{\eta}]$ and $\Upsilon^{\Lambda}[\psi, \bar{\psi}]$.

Functionals $\mathcal{G}$ and Υ recovered for $\Lambda \to 0$.

Flow equation for Υ^Λ: Wetterich '93, Morris '94, Salmhofer + Honerkamp '01

$$\partial_\Lambda \Upsilon^\Lambda[\psi,\bar\psi] = (\bar\psi, \dot Q^\Lambda \psi) - \mathrm{tr}\left[\dot Q^\Lambda \left(\frac{\partial^2 \Upsilon^\Lambda}{\partial\psi\partial\bar\psi}\right)^{-1}\right] \qquad \begin{array}{l} Q^\Lambda = (C^\Lambda)^{-1} \\ \dot Q^\Lambda = \partial_\Lambda Q^\Lambda \end{array}$$

(derivation later)

Expansion in fields:

$$\frac{\partial^2 \Upsilon^\Lambda}{\partial\psi_K \partial\bar\psi_{K'}} = (G^\Lambda)^{-1}_{K,K'} + \tilde\Upsilon^\Lambda_{K,K'}[\psi,\bar\psi] \quad \Rightarrow$$

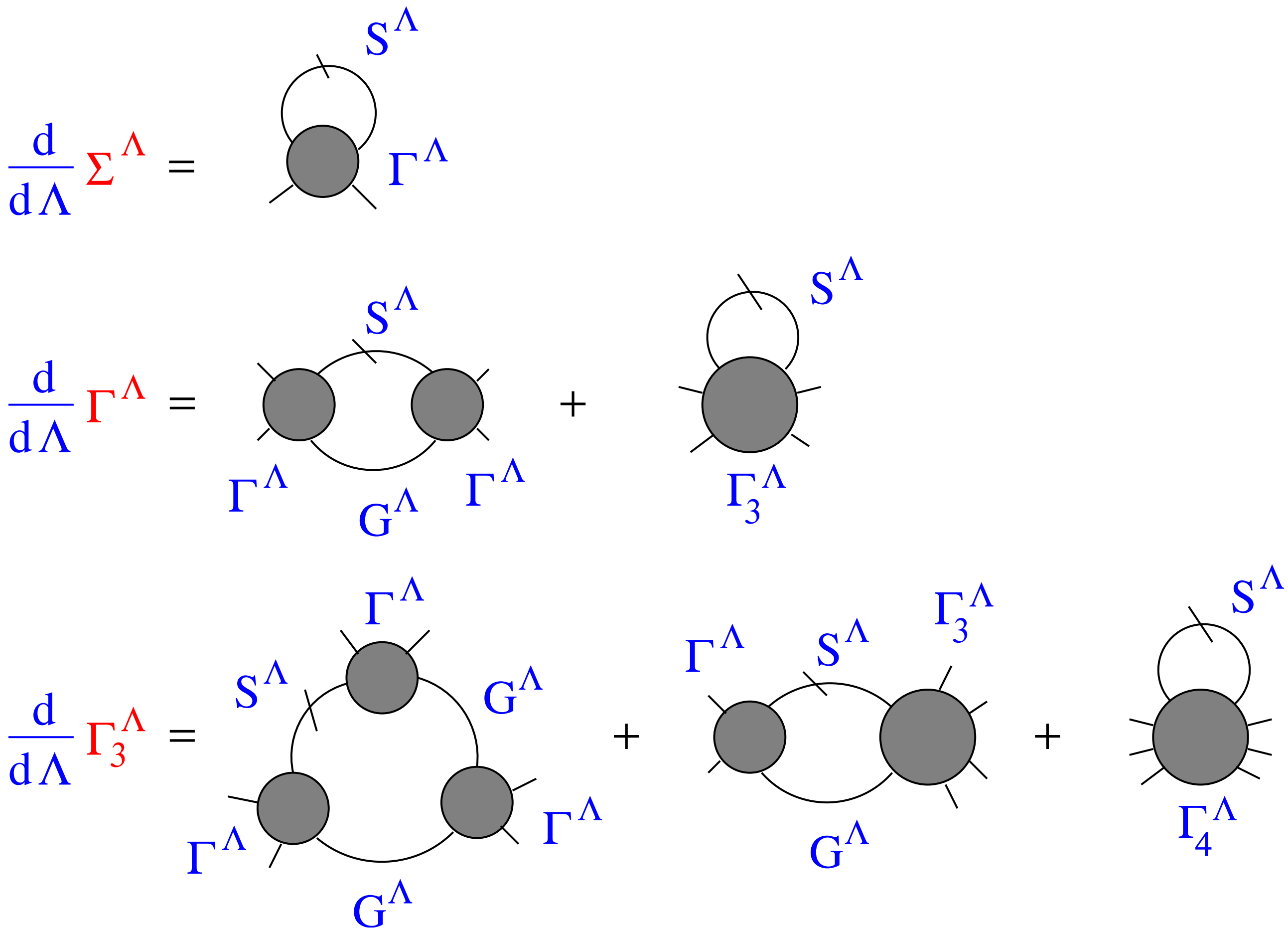
$$\left(\frac{\partial^2 \Upsilon^\Lambda}{\partial\psi\partial\bar\psi}\right)^{-1} = \left(1 + G^\Lambda \tilde\Upsilon^\Lambda\right)^{-1} G^\Lambda = \left[1 - G^\Lambda\tilde\Upsilon^\Lambda + (G^\Lambda\tilde\Upsilon^\Lambda)^2 - \dots\right] G^\Lambda \quad \Rightarrow$$

$$\partial_\Lambda \Upsilon^\Lambda = -\mathrm{tr}\left[\dot Q^\Lambda G^\Lambda\right] + (\bar\psi, \dot Q^\Lambda \psi) + \mathrm{tr}\left[S^\Lambda(\tilde\Upsilon^\Lambda - \tilde\Upsilon^\Lambda G^\Lambda \tilde\Upsilon^\Lambda + \dots)\right]$$

where $S^\Lambda = G^\Lambda \dot Q^\Lambda G^\Lambda$ (*"single scale propagator"*)

Expand $\Upsilon^\Lambda[\psi,\bar\psi]$ in powers of ψ and $\bar\psi$, compare coefficients $\Rightarrow$

Flow equations for self-energy $\Sigma^\Lambda = Q^\Lambda - \Gamma_1^\Lambda$, two-particle vertex $\Gamma^\Lambda = \Gamma_2^\Lambda$, and many-particle vertices Γ_3^Λ, Γ_4^Λ, etc.



Hierarchy of 1-loop diagrams; all one-particle irreducible

Initial conditions:

Σ^{Λ_0} = bare single-particle potential (if any)

Γ^{Λ_0} = antisymmetrized bare two-particle interaction

Derivation of flow equation:

$$e^{-\mathcal{G}^\Lambda[\eta,\bar\eta]} = \int \prod_K d\psi_K d\bar\psi_K \, e^{(\bar\psi, Q^\Lambda \psi)} \, e^{-V[\psi,\bar\psi]} \, e^{(\eta,\bar\psi)+(\psi,\bar\eta)}$$

Take Λ-derivative on both sides $\Rightarrow$

$$\begin{aligned} -(\partial_\Lambda \mathcal{G}^\Lambda) \, e^{-\mathcal{G}^\Lambda} &= \int \prod_K d\psi_K d\bar\psi_K \, (\bar\psi, \dot{Q}^\Lambda \psi) \, e^{(\bar\psi, Q^\Lambda \psi)} \, e^{-V[\psi,\bar\psi]} \, e^{(\eta,\bar\psi)+(\psi,\bar\eta)} \\ &= -(\partial_\eta, \dot{Q}^\Lambda \partial_{\bar\eta}) \, e^{-\mathcal{G}^\Lambda[\eta,\bar\eta]} \end{aligned}$$

$\Rightarrow$ Flow equation for $\mathcal{G}^\Lambda$

$$\partial_\Lambda \mathcal{G}^\Lambda[\eta,\bar\eta] = \left(\frac{\partial \mathcal{G}^\Lambda}{\partial \eta}, \dot{Q}^\Lambda \frac{\partial \mathcal{G}^\Lambda}{\partial \bar\eta} \right) - \mathrm{tr} \left(\dot{Q}^\Lambda \, \frac{\partial^2 \mathcal{G}^\Lambda}{\partial \eta \partial \bar\eta} \right)$$

Legendre transform

$$\Upsilon^\Lambda[\psi,\bar\psi] = \mathcal{G}^\Lambda[\eta^\Lambda,\bar\eta^\Lambda] + (\bar\psi,\eta^\Lambda) - (\bar\eta^\Lambda,\psi)$$

Note that η^Λ and $\bar\eta^\Lambda$ are Λ-dependent functions of ψ and $\bar\psi$.

$$\partial_\Lambda \Upsilon^\Lambda[\psi,\bar\psi] = \frac{d}{d\Lambda}\mathcal{G}^\Lambda[\eta^\Lambda,\bar\eta^\Lambda] + (\bar\psi,\partial_\Lambda\eta^\Lambda) - (\partial_\Lambda\bar\eta^\Lambda,\psi)$$

The total derivative acts also on the Λ-dependence of η^Λ and $\bar\eta^\Lambda$.

$$\frac{\partial \mathcal{G}^\Lambda}{\partial \bar\eta} = \psi\,,\ \frac{\partial \mathcal{G}^\Lambda}{\partial \eta} = \bar\psi \quad \Rightarrow \quad \partial_\Lambda \Upsilon^\Lambda[\psi,\bar\psi] = \partial_\Lambda \mathcal{G}^\Lambda[\eta^\Lambda,\bar\eta^\Lambda]$$

Insert flow equation for $\mathcal{G}^\Lambda$ and use

$$\partial_\eta \mathcal{G}^\Lambda = \bar\psi\,,\ \partial_{\bar\eta}\mathcal{G}^\Lambda = \psi\,,\ \frac{\partial^2\mathcal{G}^\Lambda}{\partial\eta\partial\bar\eta} = \left(\frac{\partial^2\Upsilon^\Lambda}{\partial\psi\partial\bar\psi}\right)^{-1}$$

$\Rightarrow$ Flow equation for Υ^Λ

$$\partial_\Lambda \Upsilon^\Lambda[\psi,\bar\psi] = (\bar\psi,\dot Q^\Lambda\psi) - \mathrm{tr}\left[\dot Q^\Lambda \left(\frac{\partial^2\Upsilon^\Lambda}{\partial\psi\partial\bar\psi}\right)^{-1}\right]$$

Alternative functional RG versions:

- Polchinski flow equations
- Wick ordered flow equations

3. Truncations

Infinite hierarchy of flow equations usually unsolvable.

Two types of approximation:

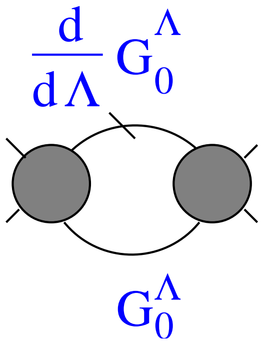
- Truncation of hierarchy at finite order
- Simplified parametrization of effective interactions

Truncations can be justified for weak coupling or small phase space.

Simple truncations in one-particle irreducible fRG:

- Set $\Gamma_3^\Lambda = 0$, neglect self-energy feedback in flow of Γ^Λ :

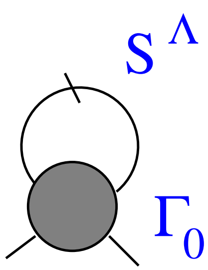
$$\frac{d}{d\Lambda}\Gamma^\Lambda =$$



Unbiased stability analysis
at weak coupling;
d-wave superconductivity
in 2D Hubbard model

- Compute flow of self-energy with bare interaction (neglecting flow of Γ^Λ):

$$\frac{d}{d\Lambda}\Sigma^\Lambda =$$



Captures properties
of isolated impurities
in 1D Luttinger liquid

Power counting:

Which interaction terms are important at low energy?

Conventional power counting procedure:
rescale momenta, frequencies and fields after mode elimination
such that quadratic part of action remains invariant;
see how interaction terms scale.

Consider 1D chiral Fermi system with
linear dispersion $\xi_k = v\, k$ at $T = 0$

Effective action

$$S^\Lambda = \int dk_0 \int_{-\Lambda}^{\Lambda} dk\, (ik_0 - vk)\, \bar{\psi}_{k_0,k} \psi_{k_0,k} - V^\Lambda[\psi, \bar{\psi}]$$

Mode elimination reduces Λ: $\Lambda' = \Lambda/s$, $s > 1$

Rescale momentum and frequency: $k = k'/s$, $k_0 = k'_0/s \quad \Rightarrow \quad |k'| \leq \Lambda$

$$dk_0\, dk\, (ik_0 - vk) = [dk'_0\, dk'\, (ik'_0 - vk')]/s^3$$

Compensate by rescaling fields $\psi = s^{3/2}\psi'$, $\bar\psi = s^{3/2}\bar\psi'$

Now see scaling of interaction terms:

2-particle interaction: $g \int \prod_{j=1}^{3} \underbrace{dk_{j0}\, dk_j}_{s^{-2}} \underbrace{\bar\psi\, \bar\psi\, \psi\, \psi}_{s^{6}}$ invariant, "marginal"

k-dependence of g: $g(k) = g(0) + \underbrace{\sum_j \gamma_j k_j}_{\text{extra } s^{-1}} + \ldots$ "irrelevant"

3-particle interaction: $(s^{-2})^5\, (s^{3/2})^6 = s^{-1}$ irrelevant **if** $g_3(0)$ finite

Usually $g_3(0)$ of order Λ^{-1} ! Not irrelevant !

Power counting in $d > 1$ cannot be done (easily) by scaling, since quadratic term *cannot be restored by homogeneous scaling of momenta!*

Better look directly at behavior of Feynman diagrams.

Interactions generally "less relevant" in $d > 1$
due to stronger phase space restrictions.