Counting statistics of electrons and photons

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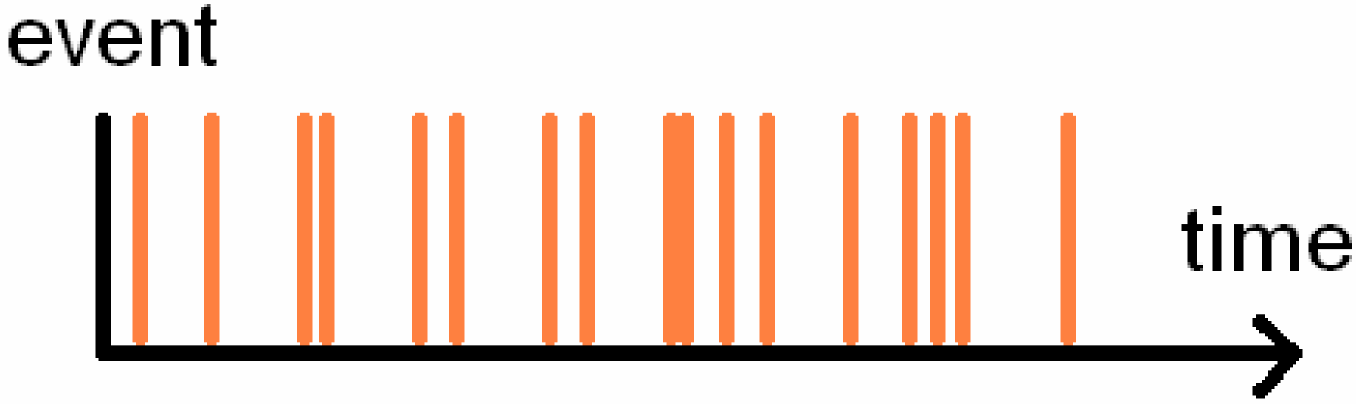
4th Windsor Summer School on Condensed Matter Theory
10 August 2007

Plan

- **Counting and probability theory**
  Distributions, moments, cumulants
- **Electrons**
  Levitov formula, simple applications
- **Photons**
  Glauber/Kelley/Kleiner formulas
  simple applications
- **Photons from electrons**

Counting and probability theory

Countable sequence of events

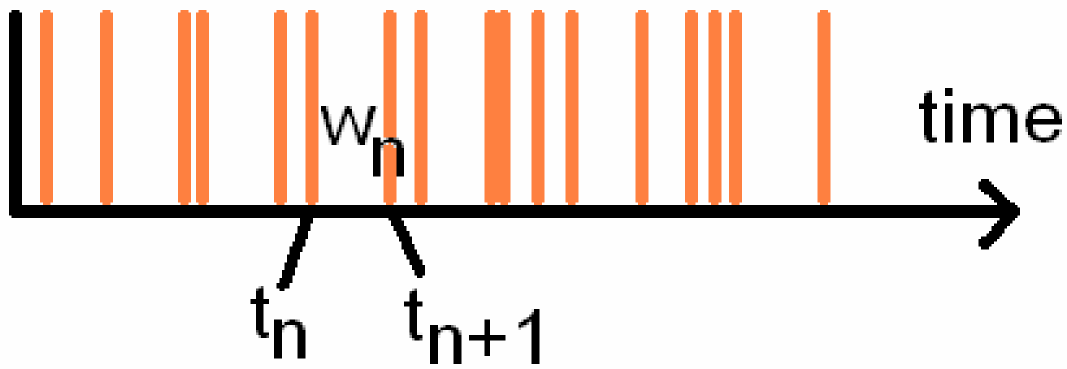


- People entering a room
- Cars crossing a traffic light

- Photons entering a detector
- Electrons passing through a mesoscopic device

How to characterize this sequence?

- **waiting time distribution** $\tilde{P}(w)$



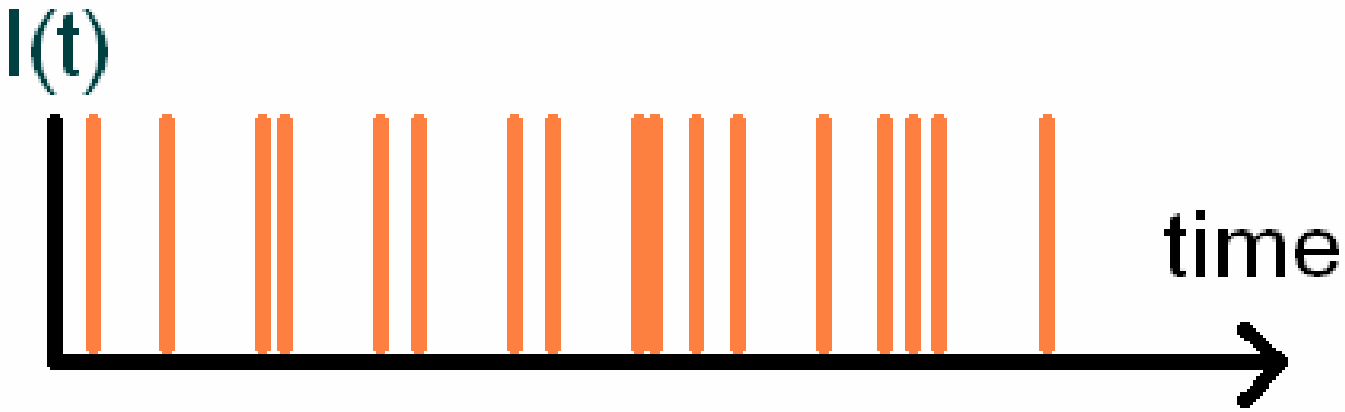
e.g. radioactive decay:
(Poisson process)

$$\tilde{P}(w) = \frac{1}{\langle w \rangle} \exp\left(-w / \langle w \rangle\right)$$

- ***how about correlations of w's?***

How to characterize this sequence?

- **Correlation functions**



- **current I(t) ➔ correlation functions**

$$\langle I(t_1)I(t_2)I(t_3)\times\cdots\times I(t_n)\rangle$$

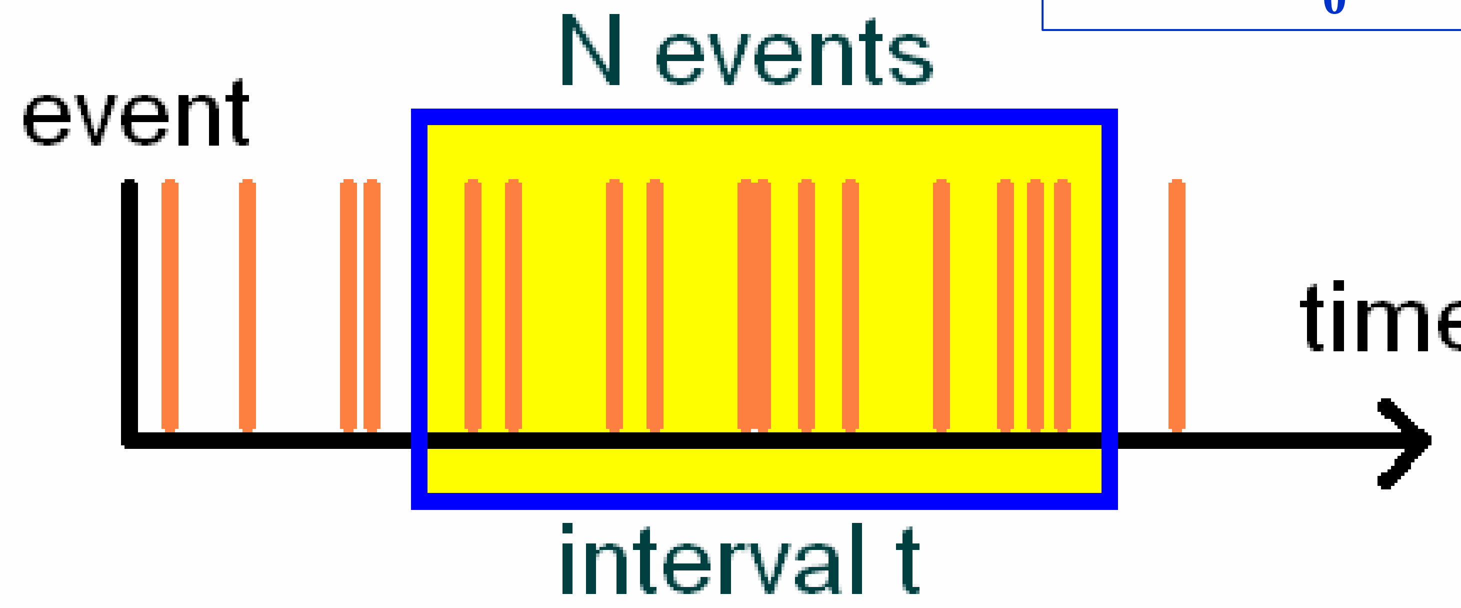
good qm starting point, but practically too general

- ***information overkill***

How to characterize this sequence?

- **counting statistics:**

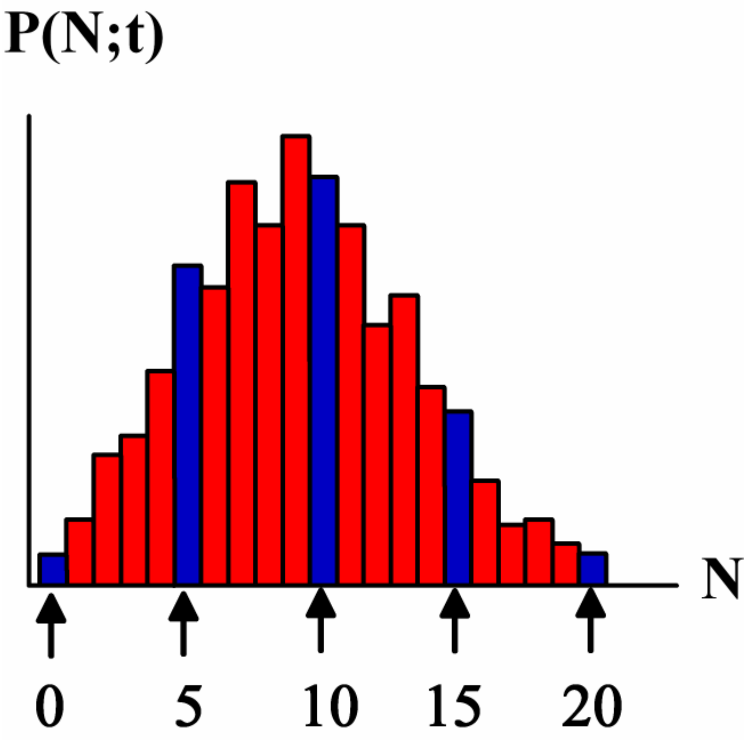
$$N(t) = \int_0^t I(t')\,dt'$$



- ***N is a discrete random number***

How to characterize this sequence?

- **counting statistics:**



- ***eg Poisson process:***

$$P(N) = \frac{\langle N \rangle^N}{N!} \exp(-\langle N \rangle)$$

$$\langle N \rangle = t / \langle w \rangle$$

- ***t*** **natural 'large' parameter**

Lessons from probability theory

Normalization $$\sum_N P(N) = 1$$

Average $$\langle N \rangle = \sum_N N\, P(N)$$

Moments $$M_n = \langle N^n \rangle = \sum_N N^n\, P(N)$$

Gen. fct. $$\Phi(\chi; t) = \langle \exp[i\chi\, N(t)] \rangle = \sum_{n=0}^{\infty} \frac{(i\chi)^n}{n!} M_n$$

Lessons from probability theory

Cumulant gen. fct $S(\chi;t) = \ln\Phi(\chi;t) = \sum_{n=1}^{\infty} \frac{(i\chi)^n}{n!} C_n$

Average $C_1 = M_1 = \langle N \rangle$

Variance $C_2 = M_2 - M_1^2 = \langle N^2 \rangle - \langle N \rangle^2 = \mathrm{var}\, N$

Skewness $C_3 = M_3 - 3M_2M_1 + 2M_1^3$

Physical interpretation

Average current

$$C_1 = \left\langle \int I(t')dt' \right\rangle \sim \langle I \rangle t$$

Fluctuations

$$\delta I(t) = I(t) - \langle I \rangle$$

Noise

$$C_2 = \left\langle \iint \delta I(t')\delta I(t'')dt'dt'' \right\rangle = \frac{1}{2} P_{noise}(0)t$$

$$P_{noise}(\omega) = 2\left\langle \int \delta I(0)\delta I(t)\exp(i\omega t)dt \right\rangle$$

Fano factor $F = \dfrac{C_2}{C_1}$

Physical interpretation

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Higher irreducible correlations

$$C_n = \left\langle\left\langle \iint \cdots \int I(t_1)I(t_2)\cdots I(t_n)dt_1 dt_2 \cdots dt_n \right\rangle\right\rangle$$

"zero-frequency" components of correlation functions

➔ **short-time characteristics not important**

e.g.: smear out signal: $\tilde{I}(t) = \int_{-\infty}^{\infty} dt'\, g(t-t') I(t')$

Convolution ➔

$$\tilde{I}(\omega=0) = g(\omega=0) I(\omega=0) \underset{\text{normalization of } g(t)}{=} I(\omega=0)$$

➔ **correlaton fct's @** $\omega = 0$ **unchanged**

Physical constraints

$$\Phi(\chi;t) = \left\langle \exp[i\chi N(t)] \right\rangle$$

- discrete particles: *N* integer ➔ periodicity

$$\Phi(\chi + 2\pi; t) = \Phi(\chi; t)$$

- finite dwell time in mesoscopic device/ finite time of flight, and finite correlation times in source ➔ $\boxed{\Phi(\chi; r\,t) = [\Phi(\chi;t)]^r}$ $(t > t_c)$

Fct. Eq. of exp. $\boxed{\Phi(\chi;t) = \exp[t\,s(\chi)]}$

Large deviation statistics

Exp. form: $$\ln\Phi(\chi;t) = t\,s(\chi;t) = \sum_{n=1}^{\infty}\frac{(i\chi)^n}{n!}C_n$$

Law of large numbers $$C_1 = \langle N\rangle \sim t$$

Central limit theorem $$C_2 = \mathrm{var}\,N \sim t$$

Extends to higher cumulants: $$C_n = c_n t$$

Central limit theorem

Rescaled variable $\tilde{N} = \dfrac{N - \langle N \rangle}{\sqrt{t}}$

$$\tilde{C}_1 = \langle \tilde{N} \rangle = 0, \quad \tilde{C}_2 = \mathrm{var}\, \tilde{N} = c_2$$

$$\tilde{C}_n = c_n t^{1-n/2} \to 0 \quad (n \geq 3)$$

➔ $P(\tilde{N})$ **converges to a Gaussian**

Physical consequences

Exp. form: $$\ln \Phi(\chi;t) = t\, s(\chi;t) = \sum_{n=1}^{\infty} \frac{(i\chi)^n}{n!} C_n$$

Stationary current $$C_1 = \langle N \rangle \sim \langle I \rangle t$$

Stationary noise $$C_2 = \mathrm{var}\, N \sim \tfrac{1}{2} P_{noise} t$$

Extends to higher correl fct's: $$C_n = c_n t$$

Fano factor $$F = \frac{c_2}{c_1}$$

Electrons vs photons

- **different quantum statistics**

  Fermions vs bosons

- **different types of sources**

  Electronic or superconducting reservoirs vs quantum emitters

- **different types of scattering**

  multiple phasecoherent scattering; interactions vs potentially, an active or nonlinear medium

- **different types of detectors**

  el./sc. reservoir vs potentially, an active detector

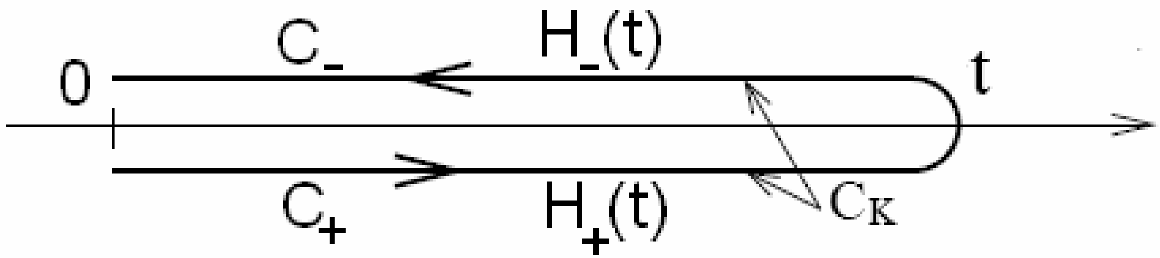
Common starting point: QM counting

- **Generating function**

$$\Phi(\chi) = \exp[S(\chi)] = \left\langle \exp[i\chi\hat{N}] \right\rangle \quad \text{where} \quad \hat{N} = \int_0^t \hat{I}(t')dt'$$

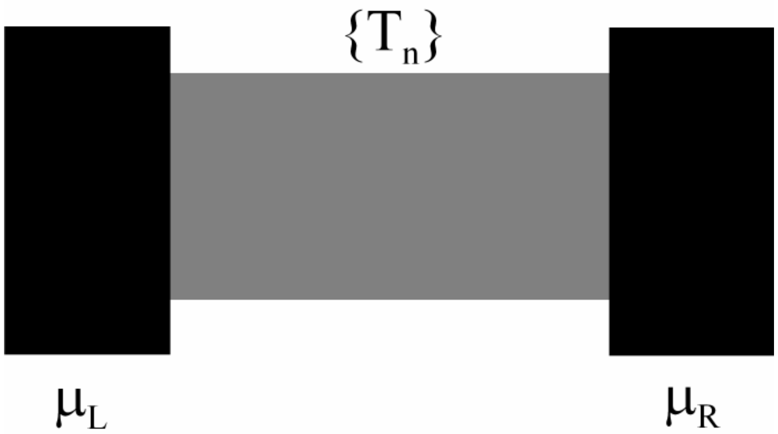
- ***Keldysh time ordering***

$$\Phi(\chi) = \left\langle \mathrm{T}_K \exp[-\frac{i}{2}\int_{C_K} \chi(t)\hat{I}(t)dt] \right\rangle$$



$$\chi(t) = \pm\chi \quad \text{for} \quad t \in C_\pm, \quad H_\pm = H \pm \chi\hat{I}$$

Electrons passing through a mesoscopic device



- source: electronic reservoir at chemical potential μ_L
- detector: reservoir at $\mu_R = \mu_L - eV$

- passage of a single electron in a single channel:

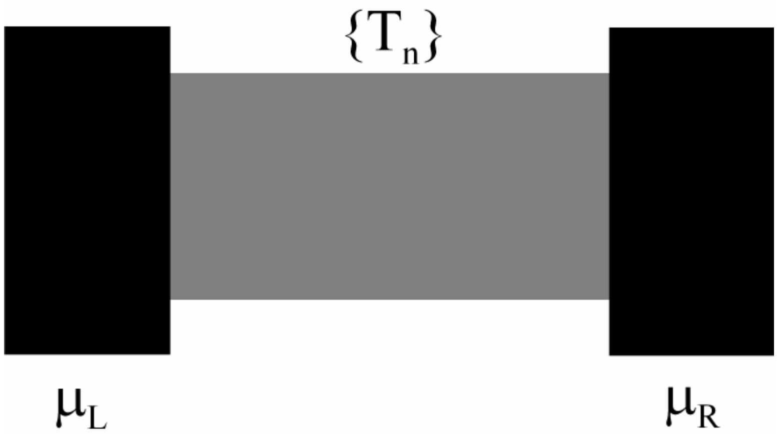
$$P(1) = T, P(0) = R = 1 - T$$

binomial distribution with transmission prob. *T*

$$\Phi(\chi) = 1 - T + T\exp(i\chi)$$

$$S(\chi) = \ln[1 + T(\exp(i\chi) - 1)]$$

Electrons passing through a mesoscopic device



- source: electronic reservoir at chemical potential μ_L
- detector: reservoir at $\mu_R = \mu_L - eV$

- passage of successive el's, attempt freq. (eV/h)

$$S(\chi) = \frac{eV}{h} t \ln[1 + T(\exp(i\chi) - 1)]$$

- many channels ➔ Levitov formula

$$S(\chi) = \sum_n \frac{eV}{h} t \ln[1 + T_n(\exp(i\chi) - 1)]$$

Levitov formula

$$S(\chi) = \sum_n \frac{eV}{h} t \ln[1 + T_n(\exp(i\chi) - 1)]$$

$$= t\frac{eV}{h}(i\chi)\sum_n T_n$$

• Landauer conductance

$$+ t\frac{eV}{h}\frac{(i\chi)^2}{2}\sum_n T_n(1 - T_n)$$

• Büttiker shot noise

$$+ t\frac{eV}{h}\frac{(i\chi)^3}{6}\sum_n T_n(1 - T_n)(2 - T_n)$$

• third cumulant

$$+ \ldots$$

• noise Fano factor

$$F = \frac{C_2}{C_1} = \frac{\sum_n T_n(1 - T_n)}{\sum_n T_n}$$

Tunnel junction: T<<1

- linearize

$$S(\chi) = \frac{eV}{h} t \ln[1 + T(\exp(i\chi) - 1)] \approx \frac{eV}{h} t T(\exp(i\chi) - 1)$$

➔ ***Poisson distribution***

$$P(N) = \frac{\langle N \rangle^N}{N!} \exp(-\langle N \rangle), \quad \langle N \rangle = \frac{eT}{h} Vt$$

(corresponding to uncorrelated events)

$$\boxed{\Phi(\chi) = \exp[(e^{i\chi} - 1)\langle N \rangle]}$$

- **noise Fano factor** $F = 1$

- **in general** $C_n / C_1 = 1$

Quantum dot

- **Levitov formula**

$$S(\chi) = \sum_n \frac{eV}{h} t \ln[1 + T_n(\exp(i\chi) - 1)])$$

- **transmission eigenvalues (RMT),#chan. M>>1**

$$P(T) = \frac{1}{\pi\sqrt{T(1-T)}} \quad \Rightarrow \quad C_1 = \frac{M}{2}\frac{eVt}{h}, \quad G = \frac{M}{2}\frac{e}{h}$$

$$C_2 = \frac{M}{8}\frac{eVt}{h}, \quad F = \frac{1}{4}$$

$$C_3 = \frac{3M}{16}\frac{eVt}{h}, \quad \frac{C_3}{C_1} = \frac{3}{8}$$

Diffusive quantum wire

- el. mean free path *l*<<*L*: transmission eval's

$$P(T) = \frac{l}{2L}\frac{1}{T\sqrt{(1-T)}}\Theta(T-4\exp(-2L/l))$$

$$\Rightarrow\ S(\chi) = \frac{Ml}{L}\frac{eV}{h}t\frac{1}{4}\mathrm{acosh}^2(2e^{i\chi}-1)$$

$$\Rightarrow\ G = \frac{Ml}{L}\frac{e}{h}, \quad F = \frac{1}{3}, \quad \frac{C_3}{C_1} = \frac{1}{15}$$

Ohm's law

Beenakker&Buttiker/Nagaev

Lee et al 1996

Diffusive quantum wire

•el. mean free path *l*<<*L*: transmission eval's

$$P(T) = \frac{l}{2L}\frac{1}{T\sqrt{(1-T)}}\Theta(T-4\exp(-2L/l))$$

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$$\Rightarrow \quad G = \frac{Ml}{L}\frac{e}{h}, \quad F = \frac{1}{3}, \quad \frac{C_3}{C_1} = \frac{1}{15}$$

• Superconducting terminals:

$$S(\chi) = \frac{Ml}{L}\frac{eV}{h}t\frac{1}{8}\text{acosh}^2(2e^{i2\chi}-1)$$

➔ same conductance, doubled charge

Finite temperatures

At finite temperature:

charge transmitted both ways (thermal fluctuations) **(Lesovik/Levitov)**

$$S(\chi) = \frac{t}{h}\int dE \ln[1 + Tf_L(1 - f_R)(e^{i\chi} - 1) + Tf_R(1 - f_L)(e^{-i\chi} - 1)]$$

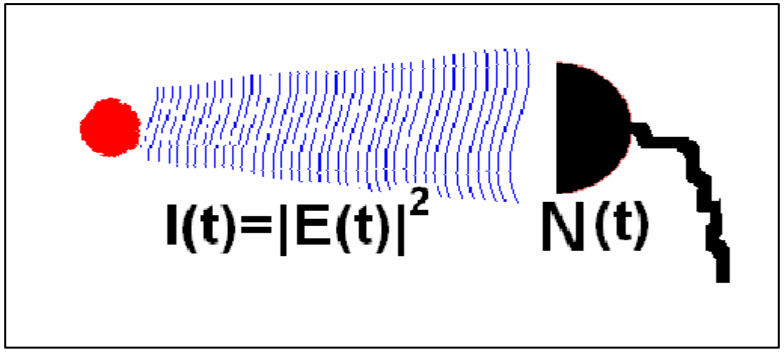
Fermi functions $f_{L/R} = \left[1 + \exp[\beta_{L/R}(E - \mu_{L/R})]\right]^{-1}$

e.g. no bias, nondegenerate gas (high temperatures): ***P*(*N*)=*P*(-*N*)**

$$C_1 = 0, \quad C_2 = P_{NJ} t / 2, \;\; P_{NJ} = 4G / \beta$$

$$C_3 = 0, \quad C_4 \neq 0$$

Counting photons



$$\Phi(\chi)=\left\langle \mathrm{T}_K \exp[-\frac{i}{2}\int_{C_K}\chi(t)\hat{I}(t)dt]\right\rangle$$

Mandel/Glauber/Kelley/Kleiner

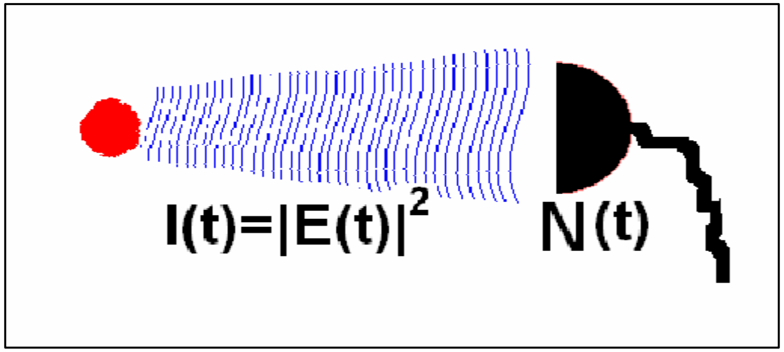
$$P(N)=\left\langle \mathrm{T}_K \frac{W^N}{N!}e^{-W}\right\rangle,\ W=\int_0^\infty d\omega\,\alpha(\omega)\int\int_0^t dt_+dt_-E(t_-)E(t_+)e^{i\omega(t_+-t_-)}$$

Neglect backaction;

$$E_{free}(t)\propto\int_0^\infty d\omega[a^+(\omega)e^{i\omega t}+a(\omega)e^{-i\omega t}]$$

rotating wave approximation: neglect $e^{i(\omega+\omega')t}$

Counting photons



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Mandel/Glauber/Kelley/Kleiner

$$P(N) = \left\langle \mathrm{T}_K \frac{W^N}{N!} e^{-W} \right\rangle, \; W = \int_0^\infty d\omega\, \alpha(\omega) \int \int_0^t dt_+ dt_- E(t_-)E(t_+) e^{i\omega(t_+ - t_-)}$$

Time ordering ➔ normal ordering of creation/ann. ops

$$S(\chi) = \frac{t\delta\omega}{2\pi} \ln \left\langle : e^{a^+ a(\exp(i\chi)-1)} : \right\rangle$$

Simple examples

- ***Black body radiation***: partition of *N* bosons among $\nu = t\,\delta\omega / 2\pi$ states in the frequency interval $\delta\omega$

➔ Negative binomial dist. $P(N) \propto \binom{N+\nu-1}{N}\left(\frac{\nu}{\langle N \rangle}+1\right)^{-N}$

F>1

- ***Laser***: coherent state $|\alpha\rangle, \quad a|\alpha\rangle = \alpha|\alpha\rangle$

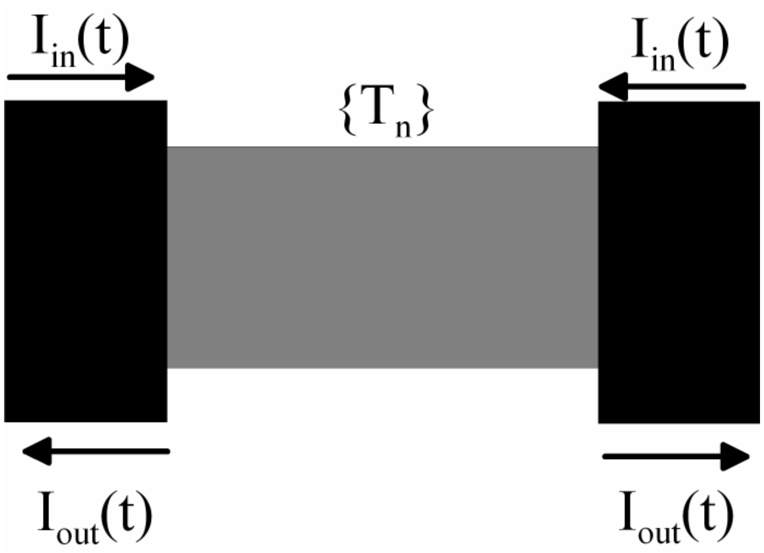
➔ Poisson distribution, F=1

- ***Photons produced by classical current I(t):***
minimal coupling $(a^{+}+a)I$, interaction repr.

➔ displacement op. of vacuum into coherent stat $|\alpha(t)\rangle$

➔ Poisson distribution, F=1

Phase-coherent current



- **Minimal coupling**
- **Embed into mesoscopic circuit:**

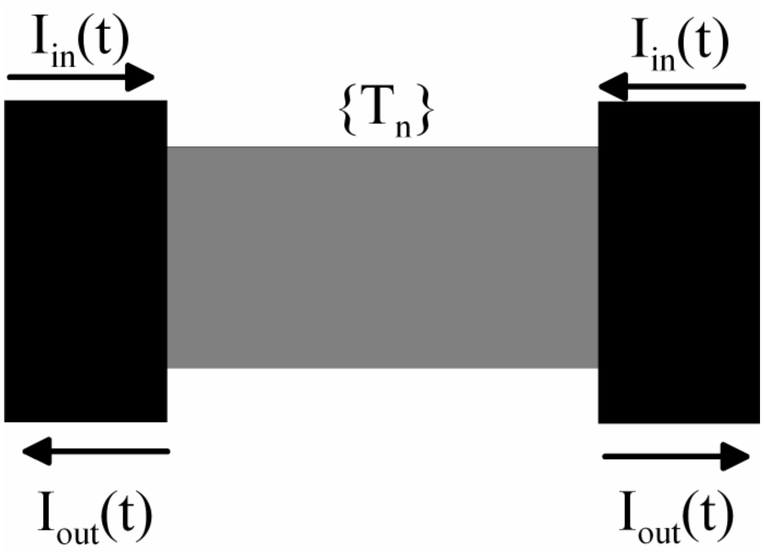
$$P(N) = \left\langle \mathrm{T}_K \frac{W^N}{N!} e^{-W} \right\rangle, \quad W = \int_0^\infty d\omega\, \alpha(\omega) \int \int_0^t dt_+ dt_- E(t_-) E(t_+) e^{i\omega(t_+ - t_-)}$$

$$E(t) = E_{free}(t) + \int_{-\infty}^{t} dt'\, I(t')\, g(t-t')$$

(Propagator g)

[Beenakker & HS PRL 86, 700 (2001)]

Phase-coherent current



- Eliminate free field:

➔ Keldysh order of I

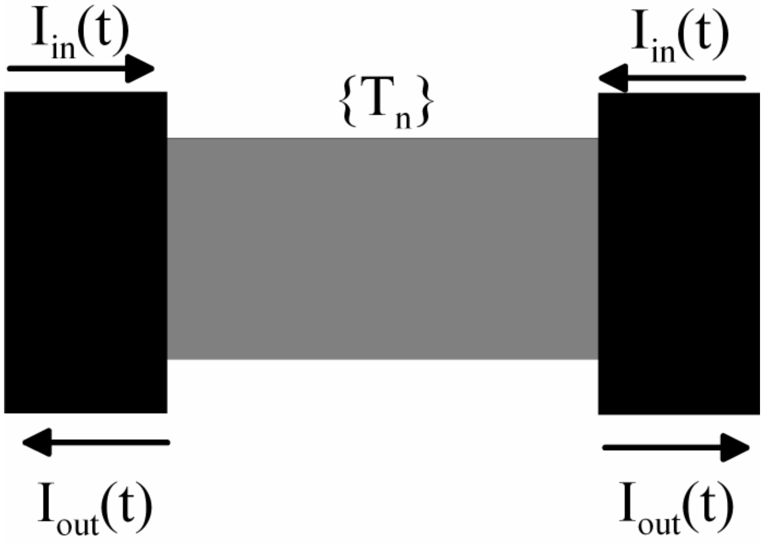
$$P(N) = \left\langle \mathrm{T}_K \frac{W^N}{N!} e^{-W} \right\rangle,$$

$$W = \int_0^\infty d\omega\, \alpha(\omega) \int_0^t \int_0^t dt' dt'' e^{i\omega(t''-t')} \int_{-\infty}^{\infty} \int dt_+ dt_- g(t'-t_-) g(t''-t_+) I(t_-) I(t_+)$$

- FT? Time ordering…

[Beenakker & HS PRL 86, 700 (2001)]

Phase-coherent current



- **time ordering ➔ operator ordering:**

$$I(t) = I_{in} + I_{out},$$

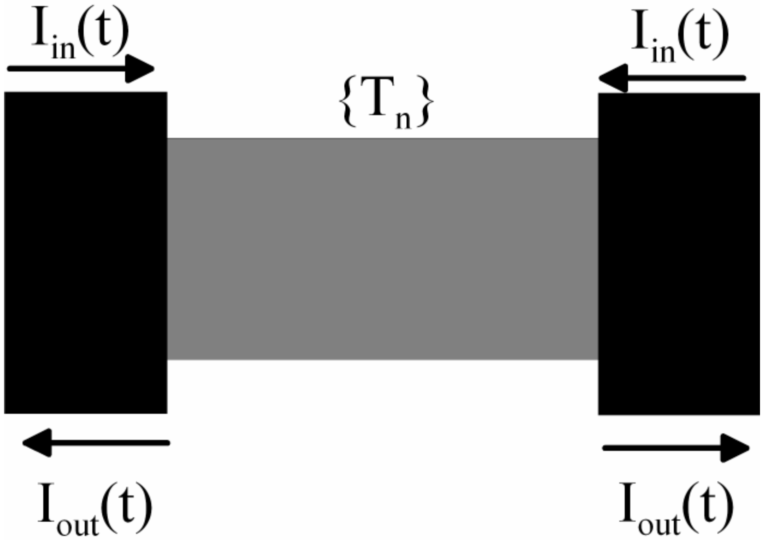
$$I_{in}(t_-), I_{out}(t_-), I_{out}(t_+), I_{in}(t_+)$$

$$P(N) = \left\langle O \frac{W^N}{N!} e^{-W} \right\rangle,$$

$$W = \int_0^\infty d\omega\, \alpha(\omega) \int \int_0^t dt' dt'' e^{i\omega(t''-t')} \int \int_{-\infty}^\infty dt_+ dt_- g(t'-t_-) g(t''-t_+) I(t_-) I(t_+)$$

[Beenakker & HS PRL 86, 700 (2001)]

Phase-coherent current



- **Fourier transformation**

$$I_{in}(\omega) = \int dE\, c_{in}^{\ +}(E)c_{in}(E+\omega)$$

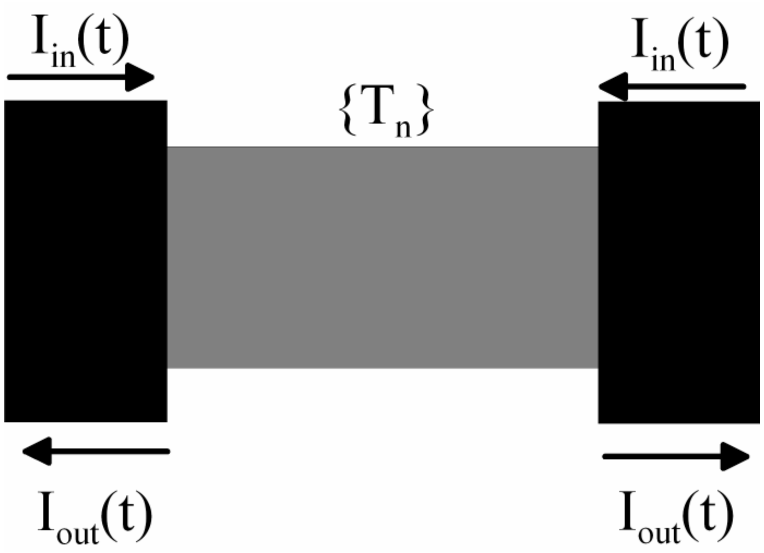
$$\Phi(\chi) = \left\langle O \exp\left[(e^{i\chi}-1)\int_0^\infty d\omega\, \gamma(\omega) I^+(\omega) I(\omega)\right]\right\rangle$$

where ordering *O*: $I_{in}^+(\omega), I_{out}^+(\omega), I_{out}(\omega), I_{in}(\omega)$

and discrete freq $\omega_p = 2\pi\, p / t$

[Beenakker & HS PRL 86, 700 (2001)]

Phase-coherent current



- **Expectation values**

$$\left\langle c_i^{+}(E) c_j(E') \right\rangle =$$

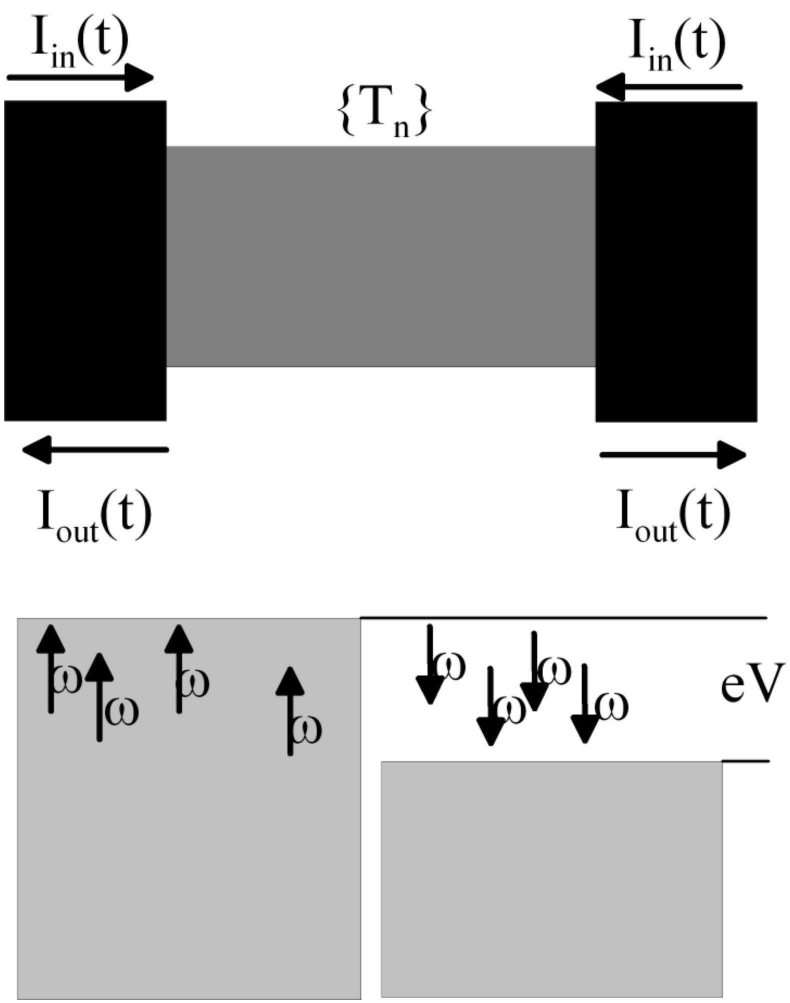
$$\delta_{ij} \delta(E - E') f(E - \mu_i)$$

$$\Phi(\chi) = \left\langle O \exp\left[(e^{i\chi} - 1) \int_0^\infty d\omega\, \gamma(\omega) I^{+}(\omega) I(\omega) \right] \right\rangle$$

becomes a determinant

[Beenakker & HS PRL 86, 700 (2001)]

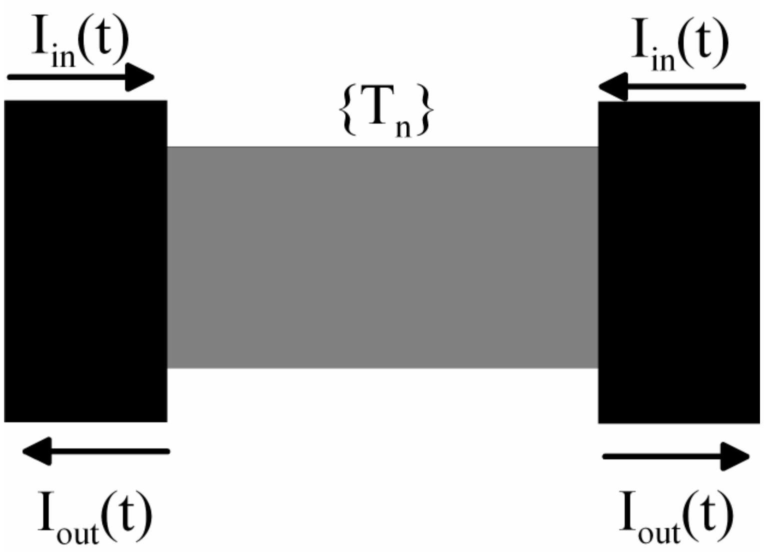
Phase-coherent current



- **Narrow band detection, $\delta\omega$ <<eV**
- **If ω<<eV:**

[Beenakker & HS PRL 86, 700 (2001)]

Phase-coherent current



- **Narrow band detection, $\delta\omega$ <<eV**
- **If ω<<eV:**

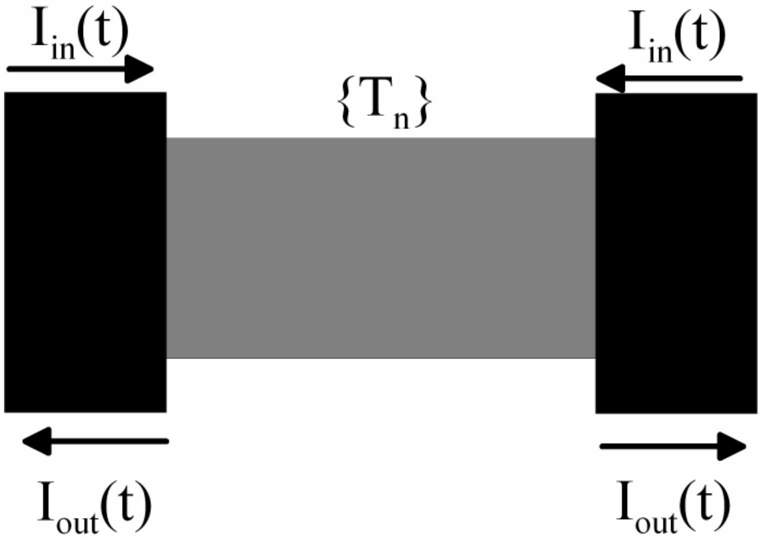
$$\langle N \rangle = \frac{\delta\omega\, t}{2\pi} n(\omega), \quad \mathrm{var}\, N = \langle N \rangle + \frac{\delta\omega\, t}{2\pi}[n(\omega)]^2$$

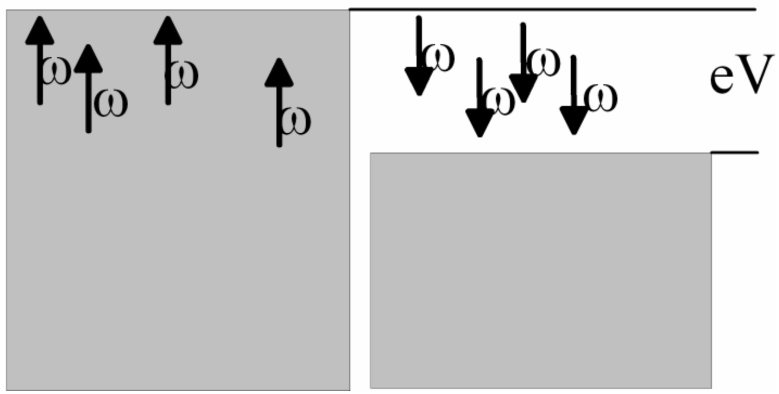
$$n(\omega) = \gamma \int_{\omega}^{V} dE\, \mathrm{Tr}\, \tau_E^{+} \tau_E \left(1 - \tau_{E-\omega}^{+} \tau_{E-\omega}\right)$$

driven by el shot noise F>1, superpoiss. statistics

[Beenakker & HS PRL 86, 700 (2001)]

Phase-coherent current





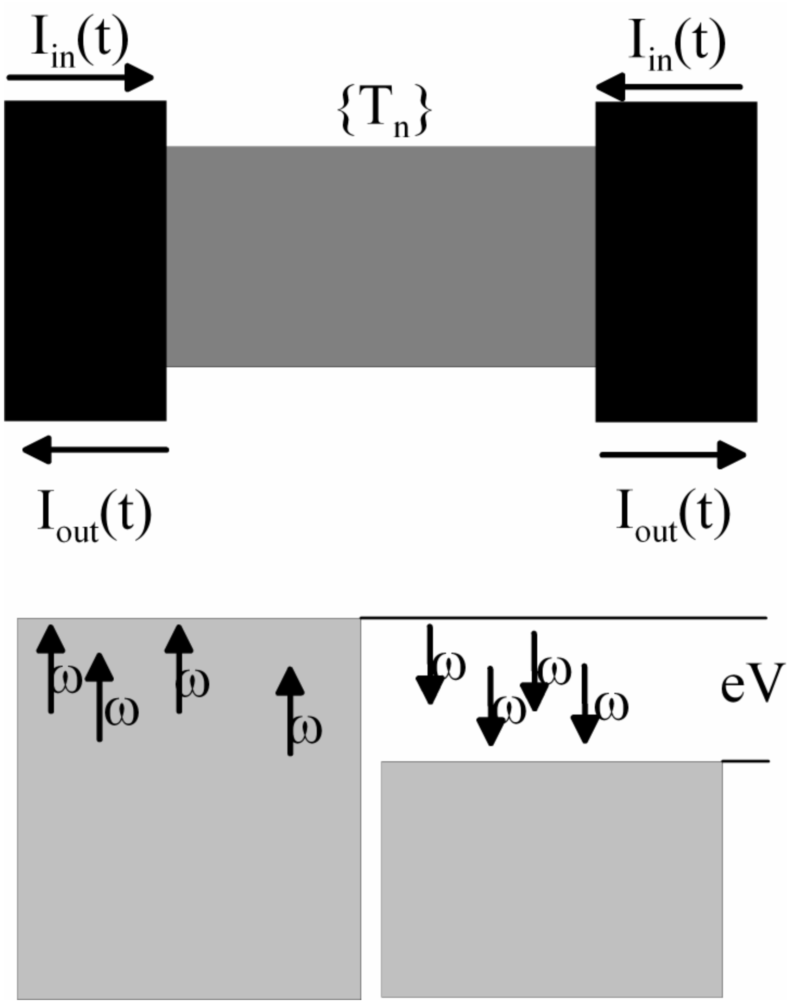
- **Narrow band detection, $\delta\omega$ <<eV**
- **If ω<<eV: Many equivalent electronic transitions**
- **Higher cumulants:**

➔ negative binomial (as for black body)

$$P(N) \propto \binom{N+\nu-1}{N}\left(\frac{\nu}{\langle N\rangle}+1\right)^{-N}$$

[Beenakker & HS PRL 86, 700 (2001)]

Phase-coherent current



- **Narrow band detection, δω <<eV**
- **However, if ω~eV: energy restrictions**

[Beenakker & HS PRL 93, 096801 (2004)]

Phase-coherent current

e.g. single channel, T:

$$\Phi(\chi) = \exp\left[\frac{t\,\delta\omega}{2\pi} \frac{(1+x)\ln(1+x) - x}{x} \right]$$

$$x = (e^{i\chi} - 1)\delta\omega\,\gamma\,T(1-T)$$

$$\langle N \rangle = \frac{\delta\omega\,t}{2\pi}\gamma\delta\omega\frac{T(1-T)}{2},$$

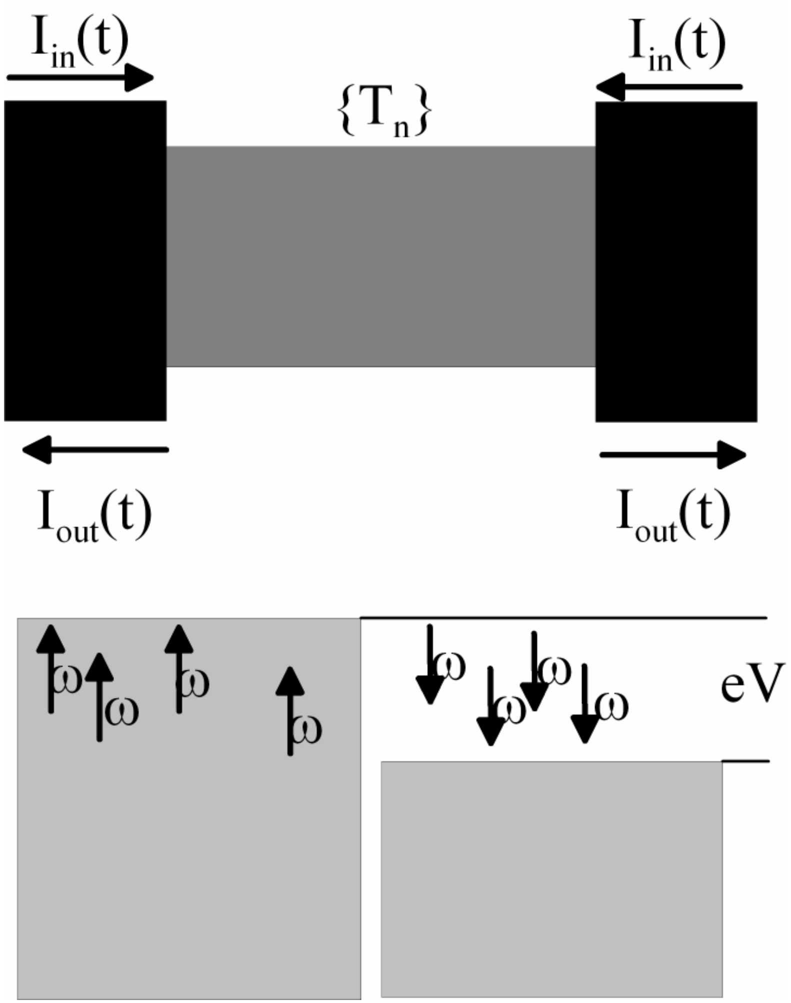
$$\mathrm{var}\,N = \langle N \rangle - \frac{\delta\omega\,t}{2\pi}\left(\gamma\delta\omega\right)^2\frac{T^2(1-T)^2}{3}$$

➔ **F<1: subpoissonian statistics**

➔ **"*nonclassical*" light**

[PRL 93, 096801 (2004)]

Phase-coherent current



- ***photons inherit electron statistics***
- ***Quantum emitter***
- ***Here: probed for large t (bunching/antibunching: small t)***

[PRL 93, 096801 (2004)]

Summary

- Counting statistics systematically organizes correlation functions

- Delivers information on:
  - Source (qm? superconducting? …)
  - Detector (el reservoir vs photodetector; passive versus active)
  - Path in between
  - QM statistics of the particles themselves