Many-Body Localization

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Quantum Transport and Dynamics in Nanostructures
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Outline:

1. Introduction to Anderson Localization
2. Phononless conductivity
3. Localization beyond real space
4. Spectral Statistics and Localization
5. Many – Body Localization
6. Disordered bosons in 1D
7. Metal - Insulator transition in electronic systems

1.Introduction

>50 years of Anderson Localization

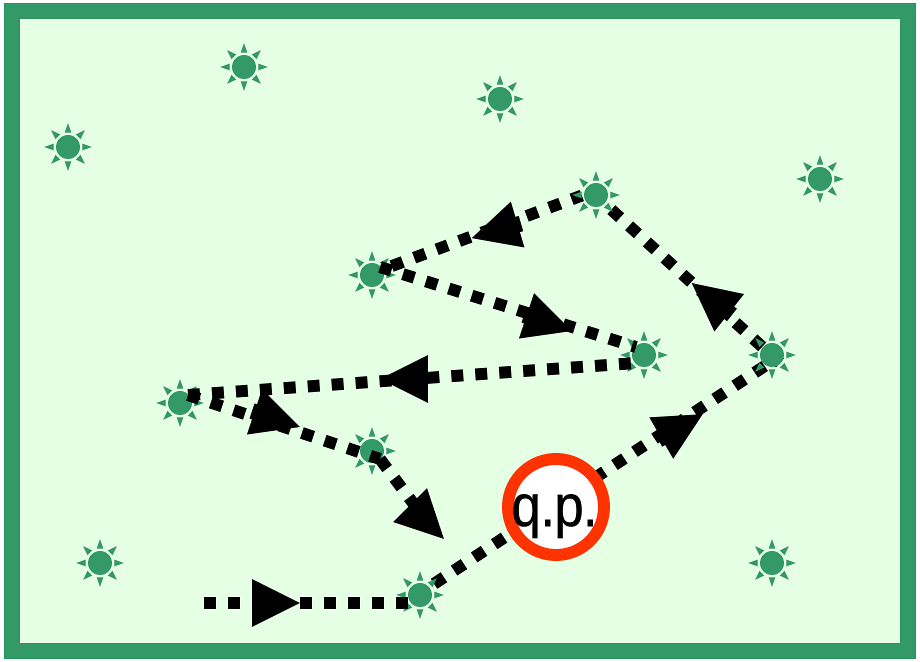
PHYSICAL REVIEW VOLUME 109, NUMBER 5 MARCH 1, 1958

Absence of Diffusion in Certain Random Lattices

P. W. ANDERSON
Bell Telephone Laboratories, Murray Hill, New Jersey
(Received October 10, 1957)

This paper presents a simple model for such processes as spin diffusion or conduction in the "impurity band." These processes involve transport in a lattice which is in some sense random, and in them diffusion is expected to take place via quantum jumps between localized sites. In this simple model the essential randomness is introduced by requiring the energy to vary randomly from site to site. It is shown that at low enough densities no diffusion at all can take place, and the criteria for transport to occur are given.

- One quantum particle
- Random potential (e.g., impurities) Elastic scattering



Einstein (1905):

$$\langle r^2 \rangle = Dt$$

diffusion constant

Random walk

⇓

always diffusion

as long as the system has no memory

Anderson(1958):

For quantum particles

⇓

not always!

It might be that

$$\langle r^2 \rangle \xrightarrow[t\to\infty]{} const$$

$$D = 0$$

Quantum interference ⇒ memory

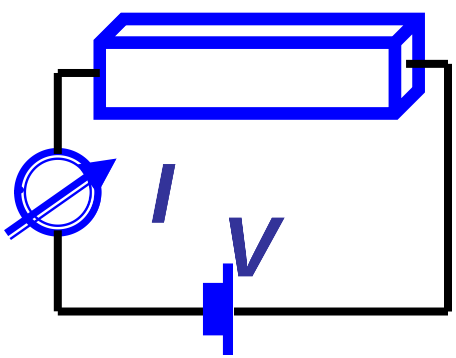
Einstein Relation (1905)

$$\sigma = e^2 D \nu \qquad \nu \equiv \frac{dn}{d\mu}$$

Conductivity

Diffusion Constant

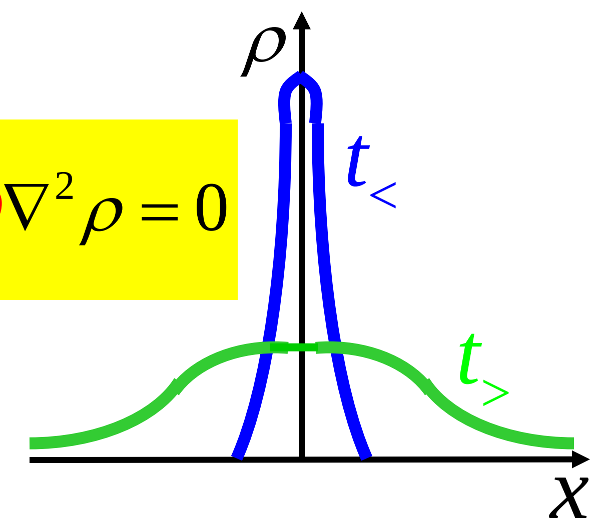
Density of states



$$G = \left(\frac{I}{V}\right)_{V=0} ;$$

$$\sigma = G\frac{L}{A}$$

$$\frac{\partial \rho}{\partial t} - D\nabla^2 \rho = 0$$



Einstein Relation (1905)

$$\sigma = e^2 D \nu \qquad \nu \equiv \frac{dn}{d\mu}$$

Conductivity

Diffusion Constant

Density of states

No diffusion – no conductivity

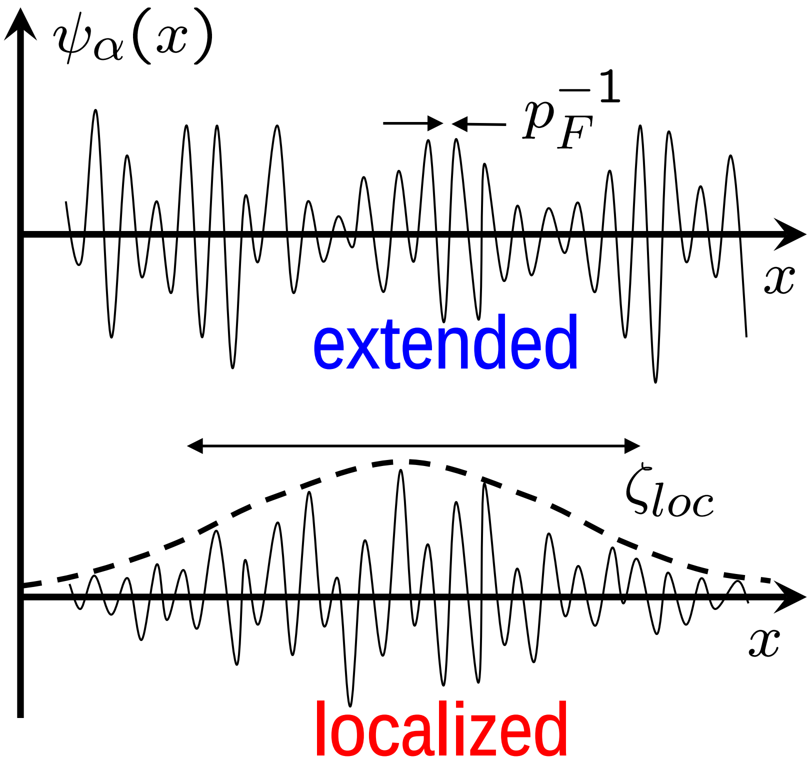
Localized states – insulator
Extended states – metal

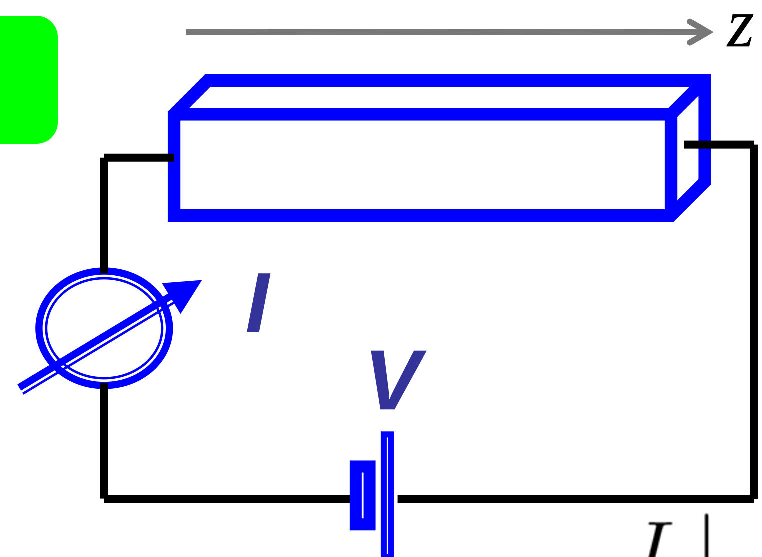
Metal – insulator transition

Localization of single-electron wave-functions:

$$\left[-\frac{\nabla^2}{2m} + U(\boldsymbol{r}) - \epsilon_F\right]\psi_\alpha(\boldsymbol{r}) = \xi_\alpha\psi_\alpha(\boldsymbol{r})$$

Disorder





Conductance $G = \left.\frac{I}{V}\right|_{V\to 0}$

$$= \begin{cases} \sigma \dfrac{L_x L_y}{L_z} & \text{extended} \\ \propto \exp\left(\dfrac{-L_z}{\varsigma_{loc}}\right) & \text{localized} \end{cases}$$

Philip W. Anderson

The Nobel Prize in Physics 1977

Nobel Lecture

Nobel Lecture, December 8, 1977

Local Moments and Localized States

I was cited for work both. in the field of magnetism and in that of disordered systems, and I would like to describe here one development in each held which was specifically mentioned in that citation. The two theories I will discuss differed sharply in some ways. The theory of local moments in metals was, in a sense, easy: it was the condensation into a simple mathematical model of ideas which. were very much in the air at the time, and it had rapid and permanent acceptance because of its timeliness and its relative simplicity. What mathematical difficulty it contained has been almost fully- cleared up within the past few years.

Localization was a different matter: very few believed it at the time, and even fewer saw its importance; among those who failed to fully understand it at first was certainly its author. It has yet to receive adequate mathematical treatment, and one has to resort to the indignity of numerical simulations to settle even the simplest questions about it .

Experiment

Spin Diffusion

Feher, G., Phys. Rev. 114, 1219 (1959); Feher, G. & Gere, E. A., Phys. Rev. 114, 1245 (1959).

Light

Wiersma, D.S., Bartolini, P., Lagendijk, A. & Righini R. "Localization of light in a disordered medium", *Nature* 390, 671-673 (1997).

Scheffold, F., Lenke, R., Tweer, R. & Maret, G. "Localization or classical diffusion of light", *Nature* 398,206-270 (1999).

Schwartz, T., Bartal, G., Fishman, S. & Segev, M. "Transport and Anderson localization in disordered two dimensional photonic lattices". *Nature* 446, 52-55 (2007).

C.M. Aegerter, M.Störzer, S.Fiebig, W. Bührer, and G. Maret : JOSA A, 24, #10, A23, (2007)

Microwave

Dalichaouch, R., Armstrong, J.P., Schultz, S.,Platzman, P.M. & McCall, S.L. "Microwave localization by 2-dimensional random scattering". *Nature* 354, 53, (1991).

Chabanov, A.A., Stoytchev, M. & Genack, A.Z. Statistical signatures of photon localization. *Nature* 404, 850, (2000).

Pradhan, P., Sridar, S, "Correlations due to localization in quantum eigenfunctions od disordered microwave cavities", PRL 85, (2000)

Sound

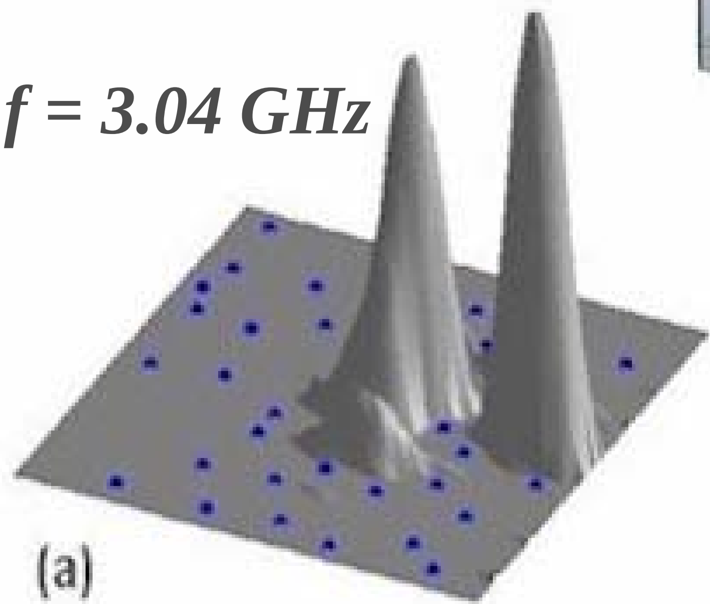
Weaver, R.L. Anderson localization of ultrasound. *Wave Motion* 12, 129-142 (1990).

VOLUME 85, NUMBER 11 PHYSICAL REVIEW LETTERS 11 SEPTEMBER 2000

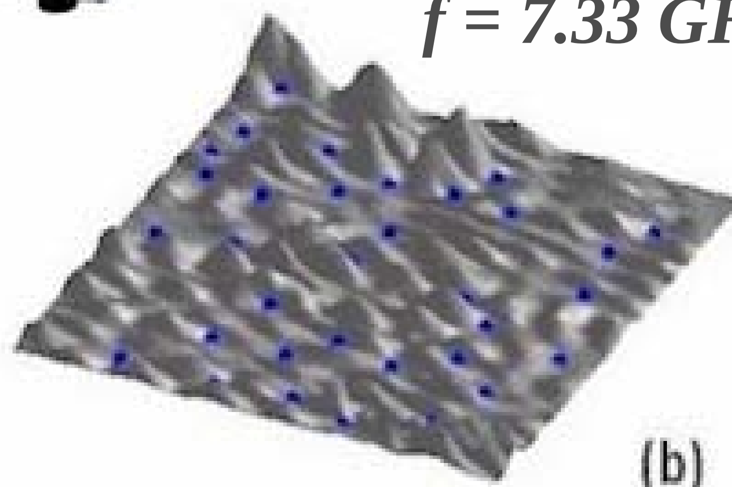
Correlations due to Localization in Quantum Eigenfunctions of Disordered Microwave Cavities

Prabhakar Pradhan and S. Sridhar
Department of Physics, Northeastern University, Boston, Massachusetts 02115
(Received 28 February 2000)

f = 3.04 GHz



f = 7.33 GHz

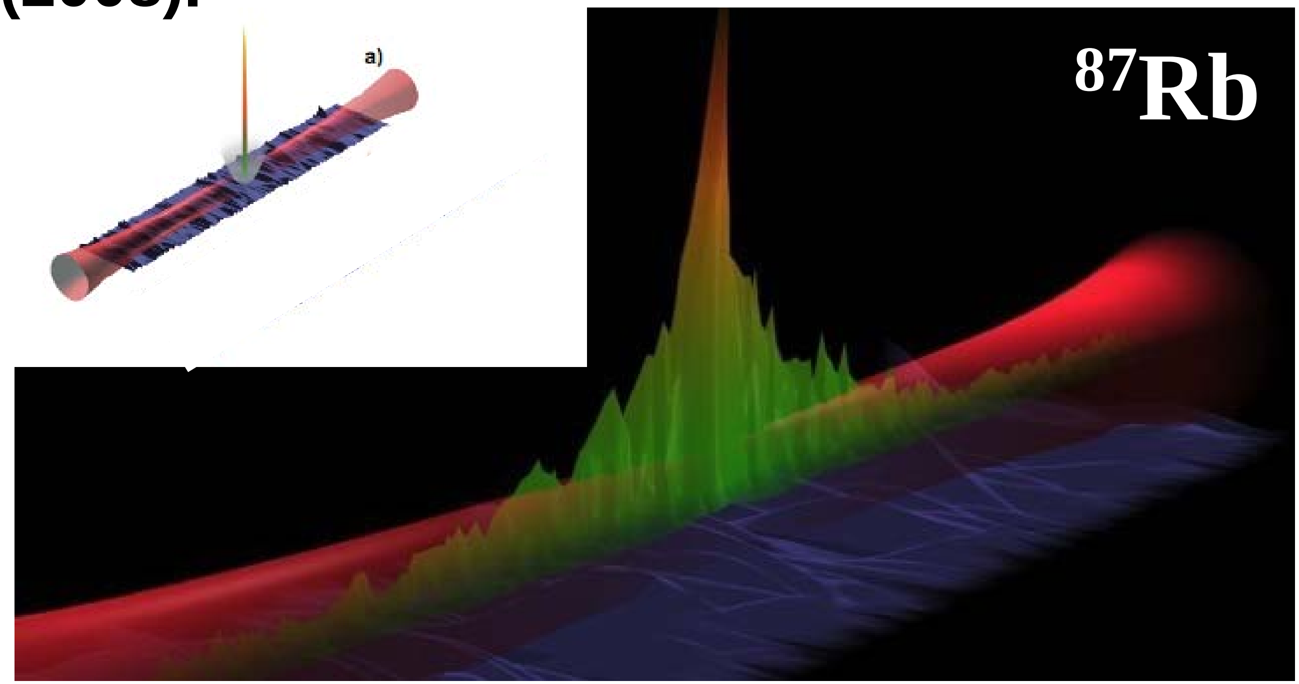


Localized State
Anderson Insulator

Extended State
Anderson Metal

Localization of cold atoms

Billy et al. “Direct observation of Anderson localization of matter waves in a controlled disorder”. Nature <u>453</u>, 891- 894 (2008).

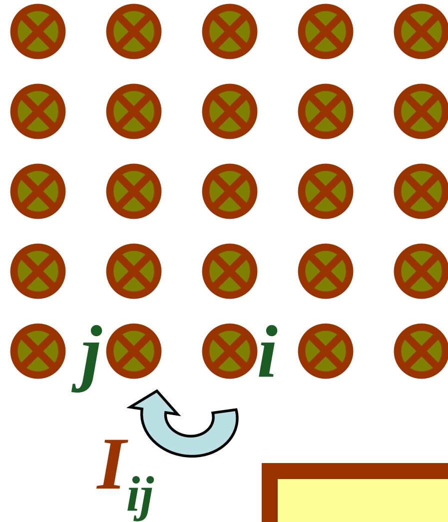


Roati et al. “Anderson localization of a non-interacting Bose-Einstein condensate“. Nature <u>453</u>, 895-898 (2008).

Q: What about electrons ?

A: Yes,… but electrons interact with each other

Anderson Model



- *Lattice - tight binding model*
- *Onsite energies* ε_i *- random*
- *Hopping matrix elements* I_{ij}

$-W < \varepsilon_i < W$

uniformly distributed

$$I_{ij} = \begin{cases} I & i \text{ and } j \text{ are nearest neighbors} \\ 0 & \text{otherwise} \end{cases}$$

Anderson Transition

$$I_c = f(d) * W$$

$I < I_c$

Insulator

All eigenstates are localized

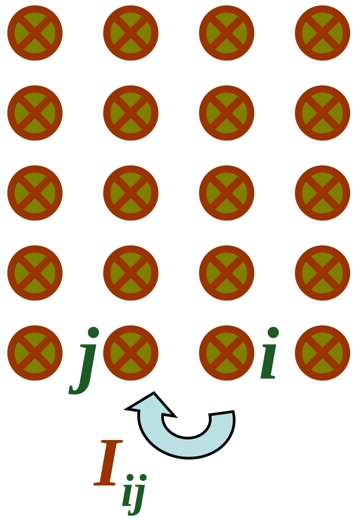
Localization length ξ

$I > I_c$

Metal

There appear states extended all over the whole system

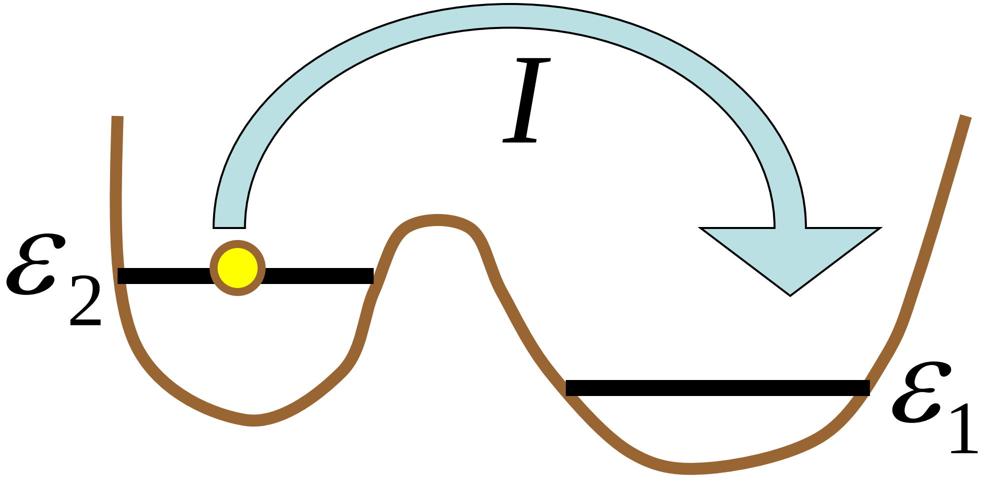
Q: Why arbitrary weak hopping I is not sufficient for the existence of the diffusion ?



Einstein (1905): Marcovian (no memory) process → diffusion

Quantum mechanics is not marcovian
There is memory in quantum propagation!

Why?



Hamiltonian

$$\hat{H} = \begin{pmatrix} \varepsilon_1 & I \\ I & \varepsilon_2 \end{pmatrix} \xrightarrow{\text{diagonalize}} \hat{H} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

$$\boxed{E_2 - E_1 = \sqrt{(\varepsilon_2 - \varepsilon_1)^2 + I^2}}$$

$$\hat{H} = \begin{pmatrix} \varepsilon_1 & I \\ I & \varepsilon_2 \end{pmatrix} \xrightarrow{\text{diagonalize}} \hat{H} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

$$E_2 - E_1 = \sqrt{(\varepsilon_2 - \varepsilon_1)^2 + I^2} \approx \begin{cases} \varepsilon_2 - \varepsilon_1 & \varepsilon_2 - \varepsilon_1 >> I \\ I & \varepsilon_2 - \varepsilon_1 << I \end{cases}$$

von Neumann & Wigner "noncrossing rule"

Level repulsion

v. Neumann J. & Wigner E. 1929 Phys. Zeit. v.30, p.467

What about the eigenfunctions ?

$$\hat{H} = \begin{pmatrix} \varepsilon_1 & I \\ I & \varepsilon_2 \end{pmatrix}$$

$$E_2 - E_1 = \sqrt{(\varepsilon_2 - \varepsilon_1)^2 + I^2} \approx \begin{matrix} \varepsilon_2 - \varepsilon_1 & \varepsilon_2 - \varepsilon_1 >> I \\ I & \varepsilon_2 - \varepsilon_1 << I \end{matrix}$$

What about the eigenfunctions ?

$$\phi_1, \varepsilon_1; \phi_2, \varepsilon_2 \quad \Leftarrow \quad \psi_1, E_1; \psi_2, E_2$$

$$\varepsilon_2 - \varepsilon_1 >> I$$

$$\psi_{1,2} = \varphi_{1,2} + O\left(\frac{I}{\varepsilon_2 - \varepsilon_1}\right)\varphi_{2,1}$$

Off-resonance

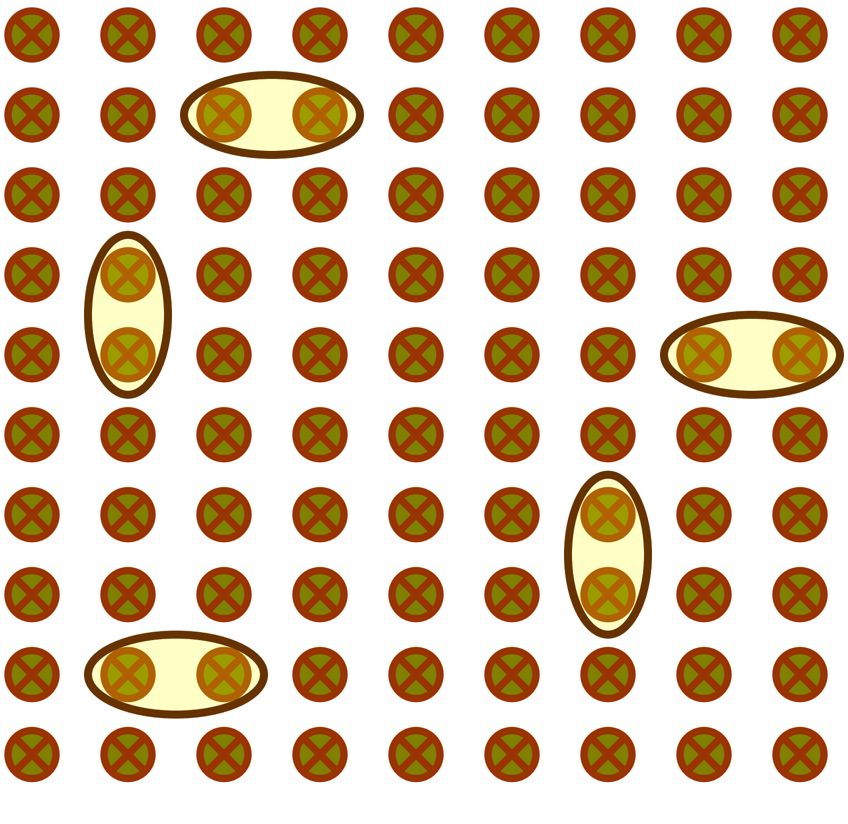
Eigenfunctions are close to the original on-site wave functions

$$\varepsilon_2 - \varepsilon_1 << I$$

$$\psi_{1,2} \approx \varphi_{1,2} \pm \varphi_{2,1}$$

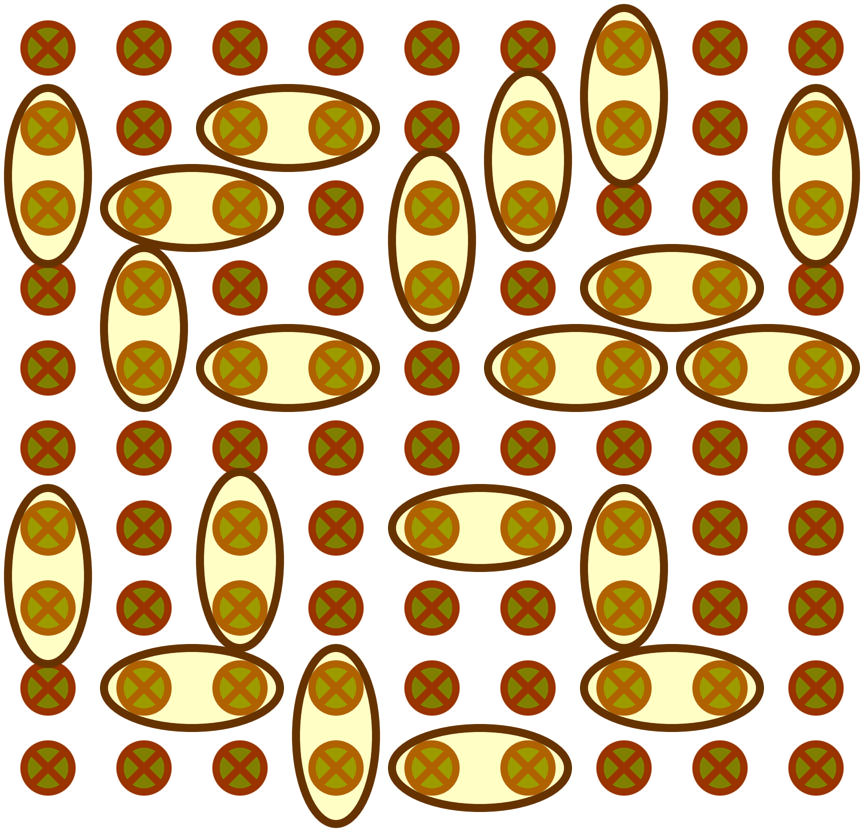
Resonance

In both eigenstates the probability is equally shared between the sites



Anderson insulator

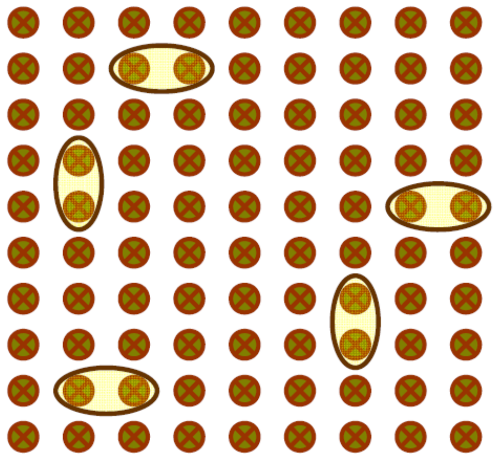
Few isolated resonances



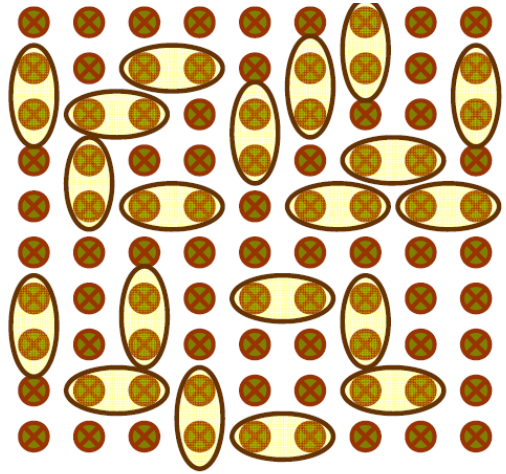
Anderson metal

There are many resonances and they overlap

Transition: **Typically each site is in the resonance with some other one**



Anderson insulator
Few isolated resonances



Anderson metal
There are many resonances and they overlap

Condition for Localization:

$$I < \frac{\text{energy mismatch}}{\text{\# of n.neighbors}}$$

$$\text{energy mismatch} = \left|\varepsilon_i - \varepsilon_j\right|_{typ} = W$$

$$\text{\# of nearest neighbors} = 2d$$

Transition: Typically each site is in the resonance with some other one

A bit more precise:

$$\boxed{\frac{I_c}{W} \simeq \left(\frac{1}{2d}\right)\left(\frac{1}{\ln d}\right)}$$

Logarithm is due to the resonances, which are not nearest neighbors

Condition for Localization:

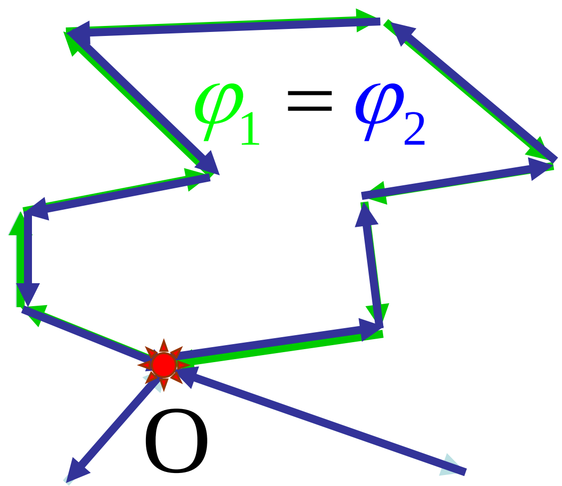
$$\boxed{\frac{I_c}{W} \simeq \left(\frac{1}{2d}\right)\left(\frac{1}{\ln d}\right)}$$

Q: Is it correct?

A1: For low dimensions – **NO**. $I_c = \infty$ for $d = 1, 2$
All states are localized. Reason – loop trajectories

$$\varphi = \oint \vec{p}\, d\vec{r}$$

Phase accumulated when traveling along the loop



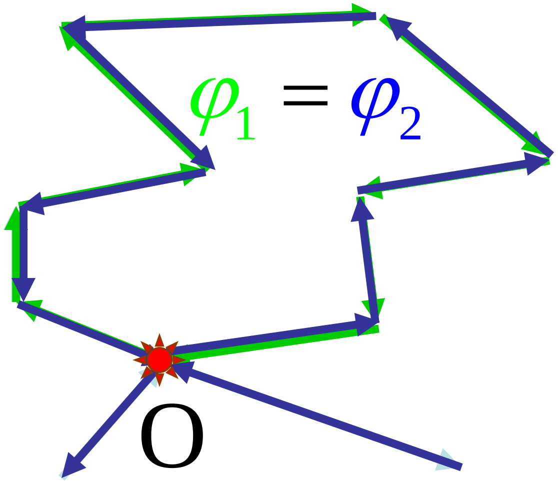
The particle can go around the loop in two directions

Memory!

For d=1,2 all states are localized.

$$\varphi = \oint \vec{p} d\vec{r}$$

Phase accumulated when traveling along the loop



The particle can go around the loop in two directions

Memory!

Weak Localization:

The localization length ς can be large

Inelastic processes lead to dephasing, which is characterized by the dephasing length L_φ

If $\varsigma >> L_\varphi$, then only small corrections to a conventional metallic behavior

Condition for Localization:

$$\boxed{\frac{I_c}{W} \simeq \left(\frac{1}{2d}\right)\left(\frac{1}{\ln d}\right)}$$

Q: Is it correct?

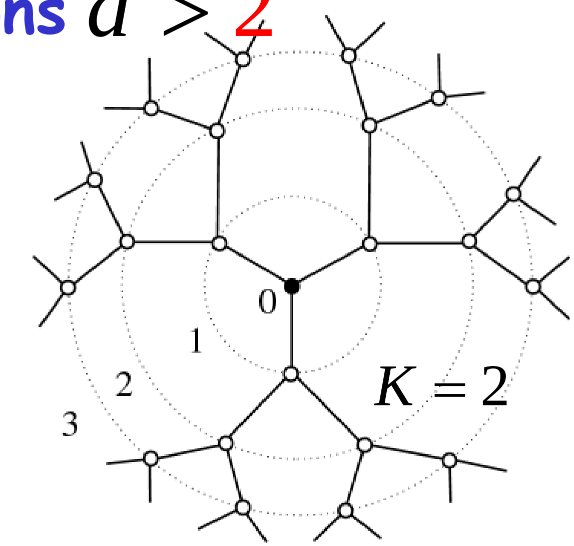
A1: For low dimensions – **NO**. $I_c = \infty$ for $d = 1, 2$
All states are localized. Reason – loop trajectories

A2: Works better for larger dimensions $d > 2$

A3: Is exact on the Cayley tree

$$I_c = \frac{W}{K \ln K},$$

K is the branching number



J. Phys. C: Solid State Phys., Vol. 6, 1973. Printed in Great Britain. © 1973

Anderson Model on a Cayley tree

A selfconsistent theory of localization

R Abou-Chacra†, P W Anderson‡§ and D J Thouless†

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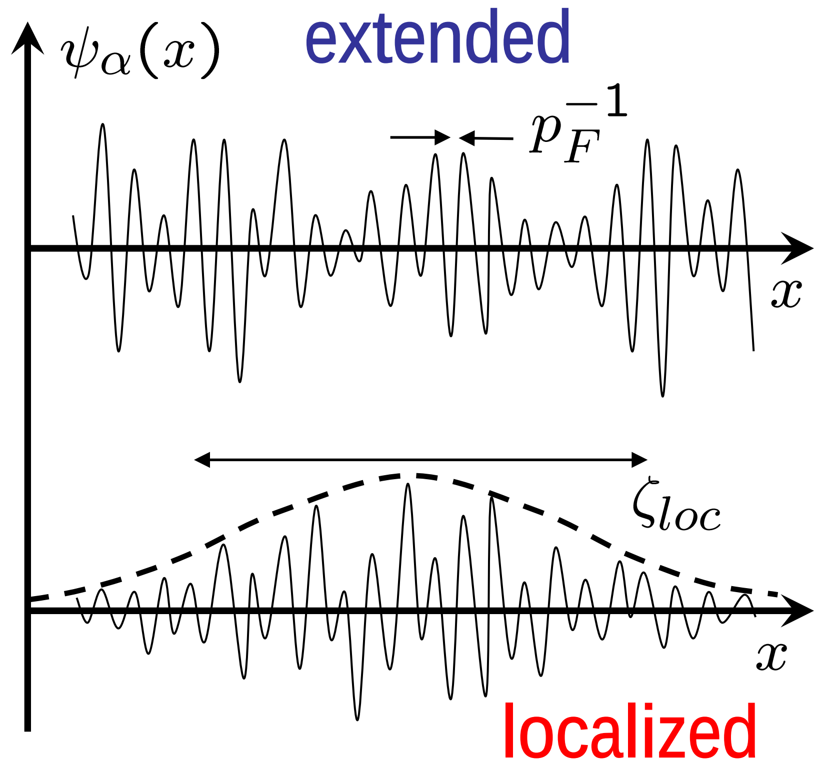
‡ Cavendish Laboratory, Cambridge, England and Bell Laboratories, Murray Hill, New Jersey, 07974, USA

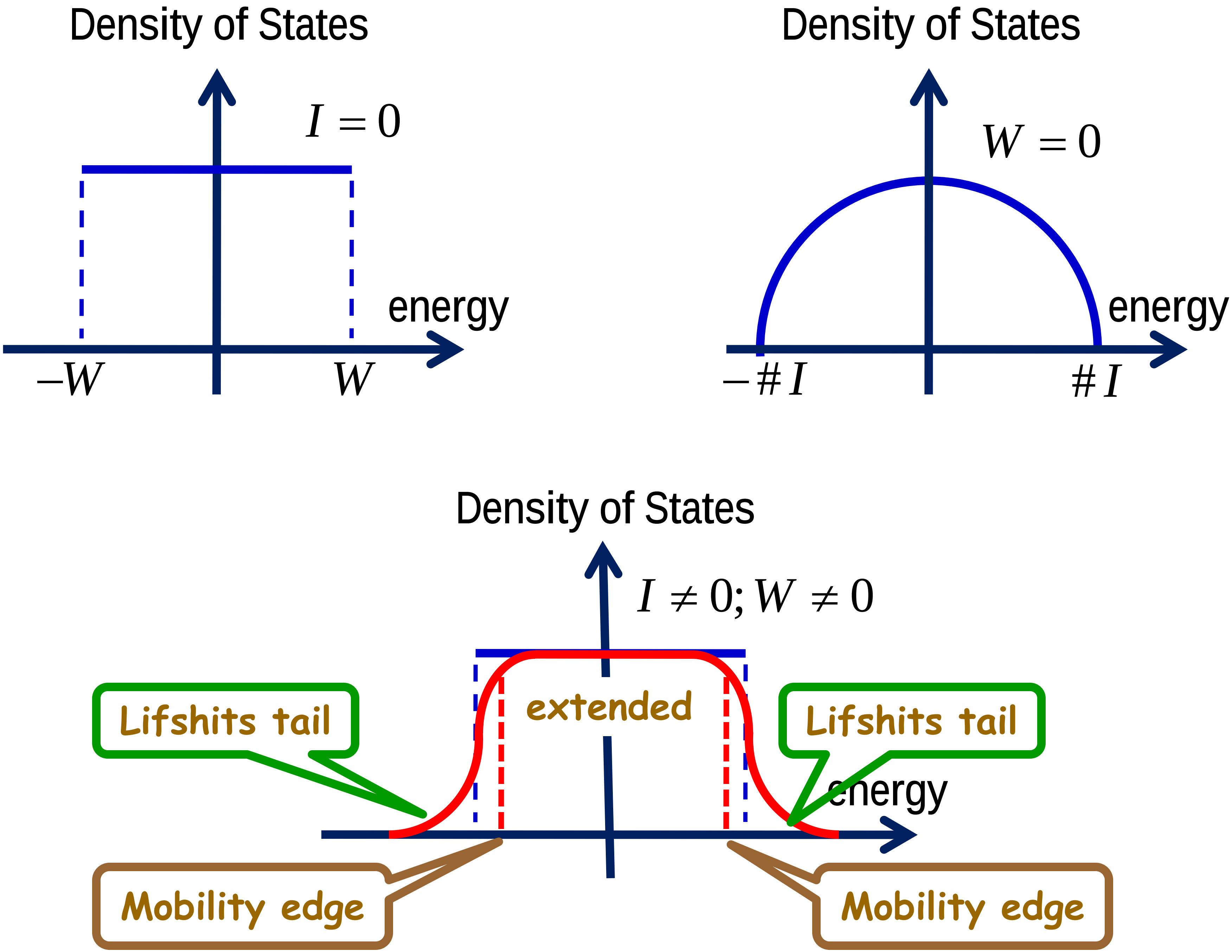
Received 12 January 1973

Abstract. A new basis has been found for the theory of localization of electrons in disordered systems. The method is based on a selfconsistent solution of the equation for the self energy in second order perturbation theory, whose solution may be purely real almost everywhere (localized states) or complex everywhere (nonlocalized states). The equations used are exact for a Bethe lattice. The selfconsistency condition gives a nonlinear integral equation in two variables for the probability distribution of the real and imaginary parts of the self energy. A simple approximation for the stability limit of localized states gives Anderson's 'upper limit approximation'. Exact solution of the stability problem in a special case gives results very close to Anderson's best estimate. A general and simple formula for the stability limit is derived; this formula should be valid for smooth distribution of site energies away from the band edge. Results of Monte Carlo calculations of the selfconsistency problem are described which confirm and go beyond the analytical results. The relation of this theory to the old Anderson theory is examined, and it is concluded that the present theory is similar but better.

Eigenfunctions



Q: Does anything interesting happen with the spectrum ?

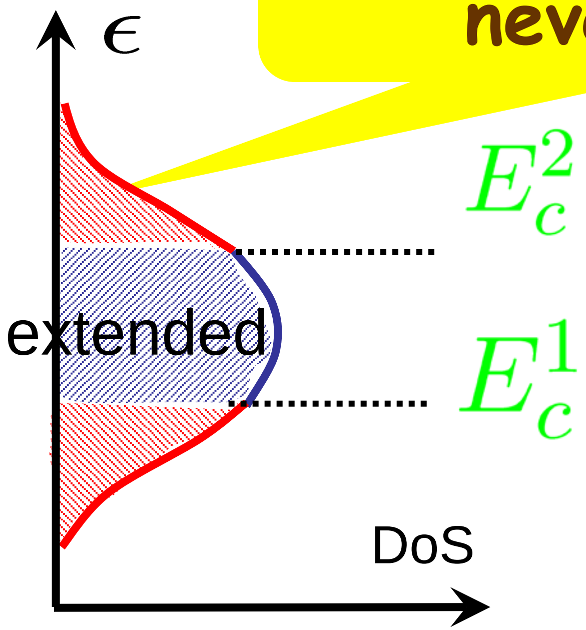

Density of States
$I = 0$
energy
$-W$
W
Density of States
$W = 0$
energy
$-\# I$
$\# I$
Density of States
$I \neq 0; W \neq 0$
extended
Lifshits tail
Lifshits tail
energy
Mobility edge
Mobility edge

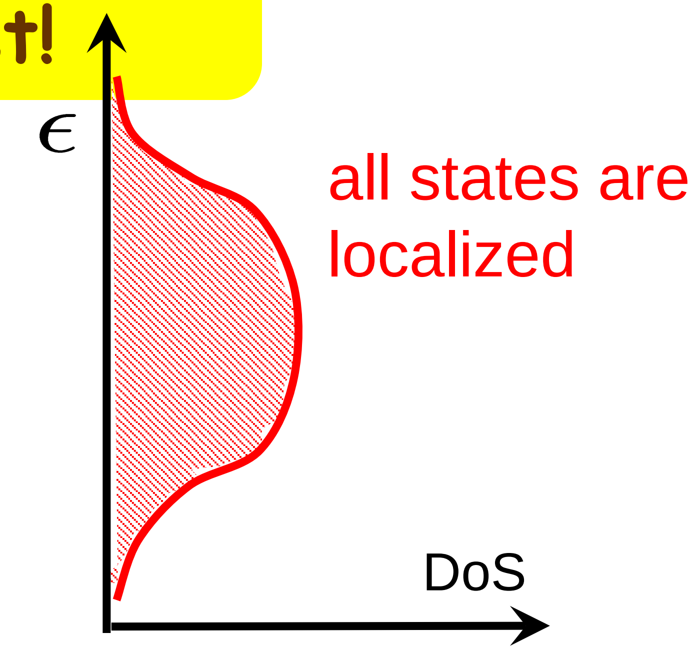
Anderson Transition

$I > I_c$

$I < I_c$

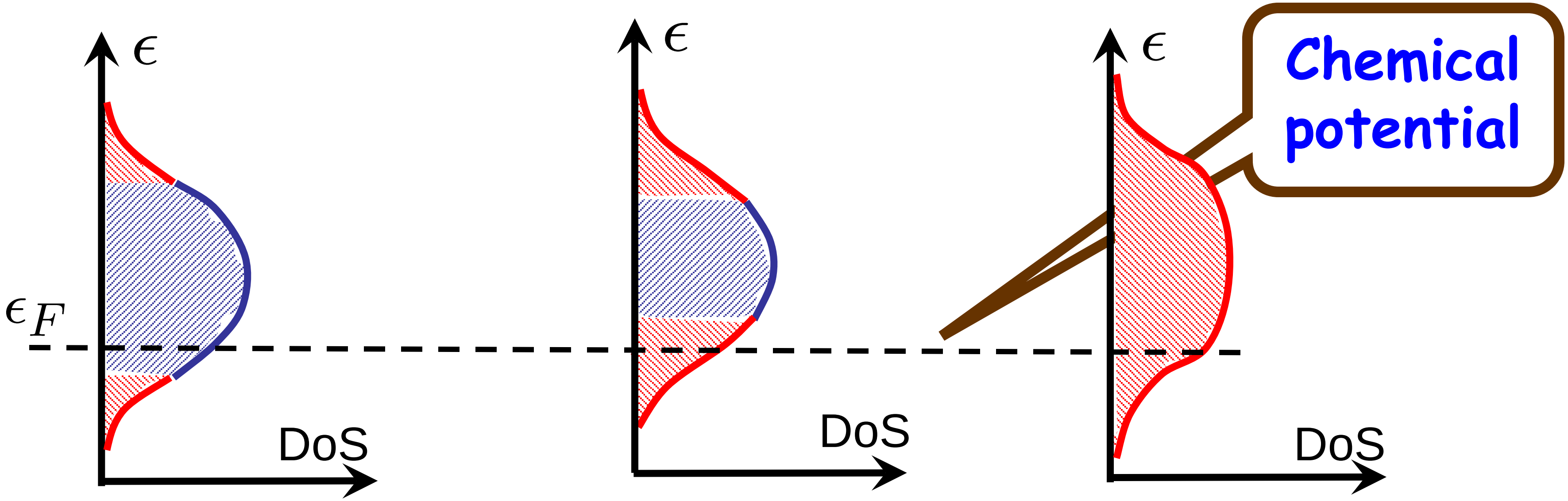
localized and extended never coexist!





E_c - mobility edges (one particle)

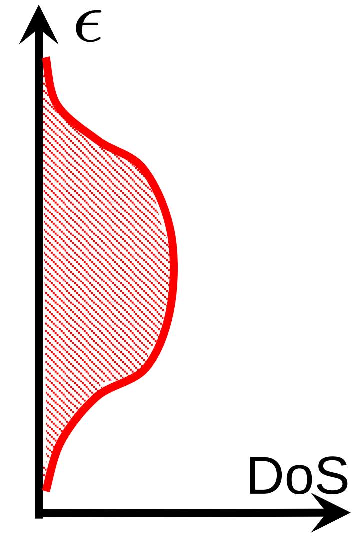
Temperature dependence of the conductivity
one-electron picture



$$\sigma(T \to 0) > 0 \qquad \sigma(T) \propto e^{-\frac{E_c - \varepsilon_F}{T}} \qquad \sigma(T) = 0 \;\; \forall T$$

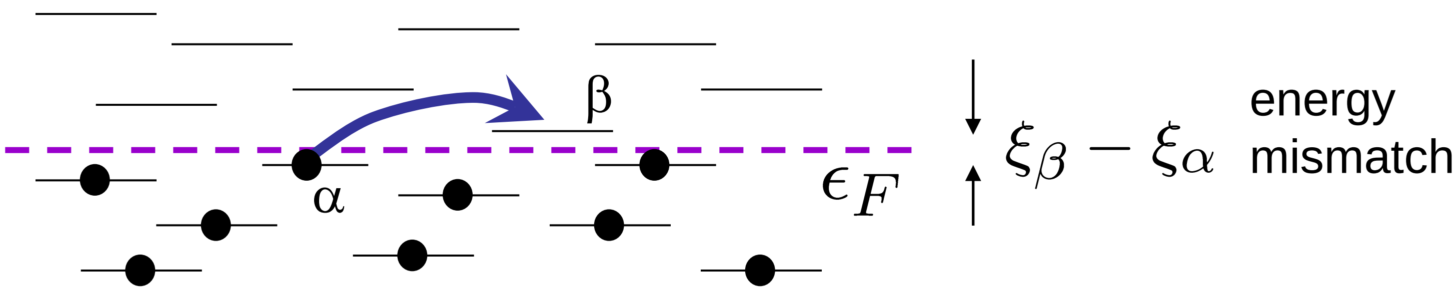
Temperature dependence of the conductivity
one-electron picture

Assume that all the states are localized; e.g. $d = 1,2$



$$\sigma(T) = 0 \quad \forall T$$

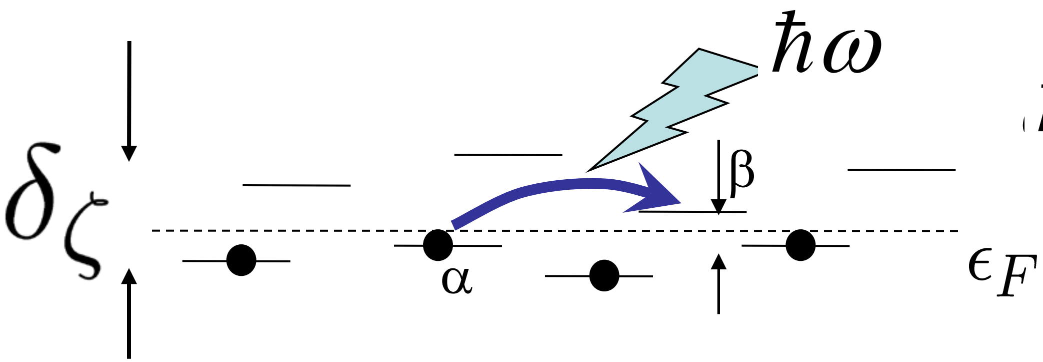
Inelastic processes
transitions between localized states



$$T = 0 \quad \Rightarrow \quad \sigma = 0$$

(any mechanism)

Phonon-assisted hopping



$$\hbar\omega = \varepsilon_\alpha - \varepsilon_\beta$$

$$\sigma(T=0) = 0$$

Variable Range Hopping
N.F. Mott (1968)

$$\sigma(T) \propto T^\gamma \exp\left[-\left(\frac{\delta_\zeta}{T}\right)^{\frac{1}{d+1}}\right]$$

Mechanism-dependent prefactor

Optimized phase volume

Any bath with a continuous spectrum of delocalized excitations down to $\omega = 0$ will give the same exponential

Lecture1.

2. Phononless conductivity in Anderson insulators with e-e interaction

Common belief: Anderson Insulator weak e-e interactions ⇨ Phonon assisted hopping transport

Can hopping conductivity exist **without phonons**?

Given:

1. All one-electron states are localized
2. Electrons interact with each other
3. The system is closed (no phonons)
4. Temperature is low but finite

Find: DC conductivity $\sigma(T,\omega=0)$ (zero or finite?)

Q: Can e-h pairs lead to phonon-less variable range hopping in the same way as phonons do ?

A#1: Sure

1. Recall phonon-less AC conductivity:
   Sir N.F. Mott (1970)

$$\sigma(\omega) = \frac{e^2 \zeta_{loc}^{d-2}}{\hbar} \left(\frac{\hbar\omega}{\delta_\zeta}\right)^2 \ln^{d+1}\left|\frac{\delta_\zeta}{\hbar\omega}\right|$$

2. Fluctuation Dissipation Theorem:
   there should be Johnson-Nyquist noise

3. Use this noise as a bath instead of phonons

4. Self-consistency (whatever it means)

Q: Can e-h pairs lead to phonon-less variable range hopping in the same way as phonons do ?

A#1: Sure

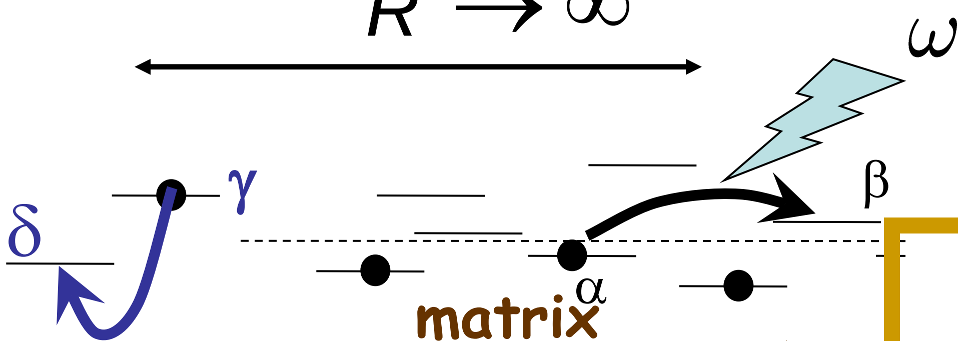
A#2: No way **(L. Fleishman. P.W. Anderson (1980))**

Except maybe Coulomb interaction in 3D

$$\sigma(\omega) \simeq \frac{e^2 \zeta_{loc}^{d-2}}{\hbar} \left(\frac{\hbar\omega}{\delta_\zeta}\right)^2 \ln^{d+1}\left|\frac{\delta_\zeta}{\hbar\omega}\right|$$

is contributed by rare resonances

$R \to \infty$



$$\omega = \xi_\beta - \xi_\alpha = \xi_\gamma - \xi_\delta$$

$R \to \infty \quad \Rightarrow$ matrix element vanishes

$$\sigma(T) \propto 0 \; \exp\left[-\left(\frac{\delta_\zeta}{T}\right)^{\frac{1}{d+1}}\right]$$

No phonons $\xrightarrow{???}$ No transport $\forall T$

Problem:

- If the localization length exceeds L_φ, then – metal.
- In a metal e-e interaction leads to a finite L_φ

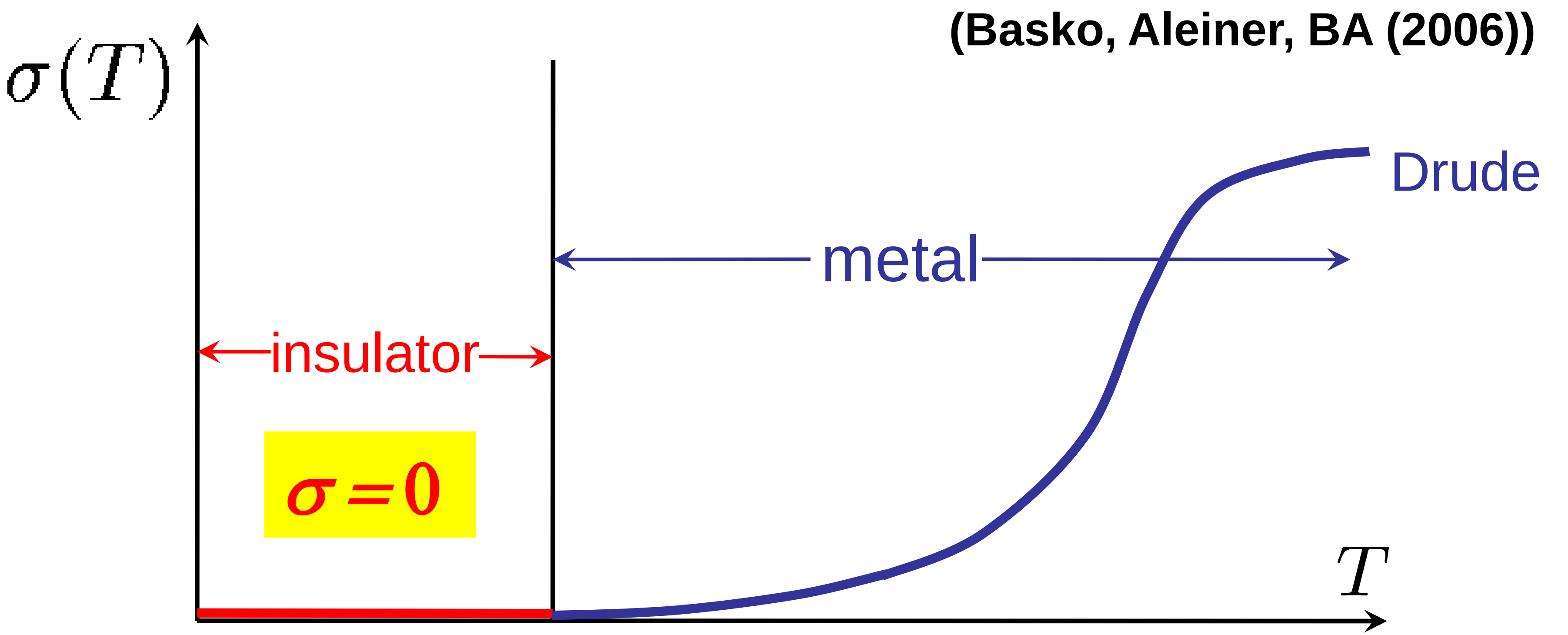
$\Rightarrow$ At high enough temperatures conductivity should be **finite** even **without phonons**

Q: Can e-h pairs lead to phonon-less variable range hopping in the same way as phonons do ?

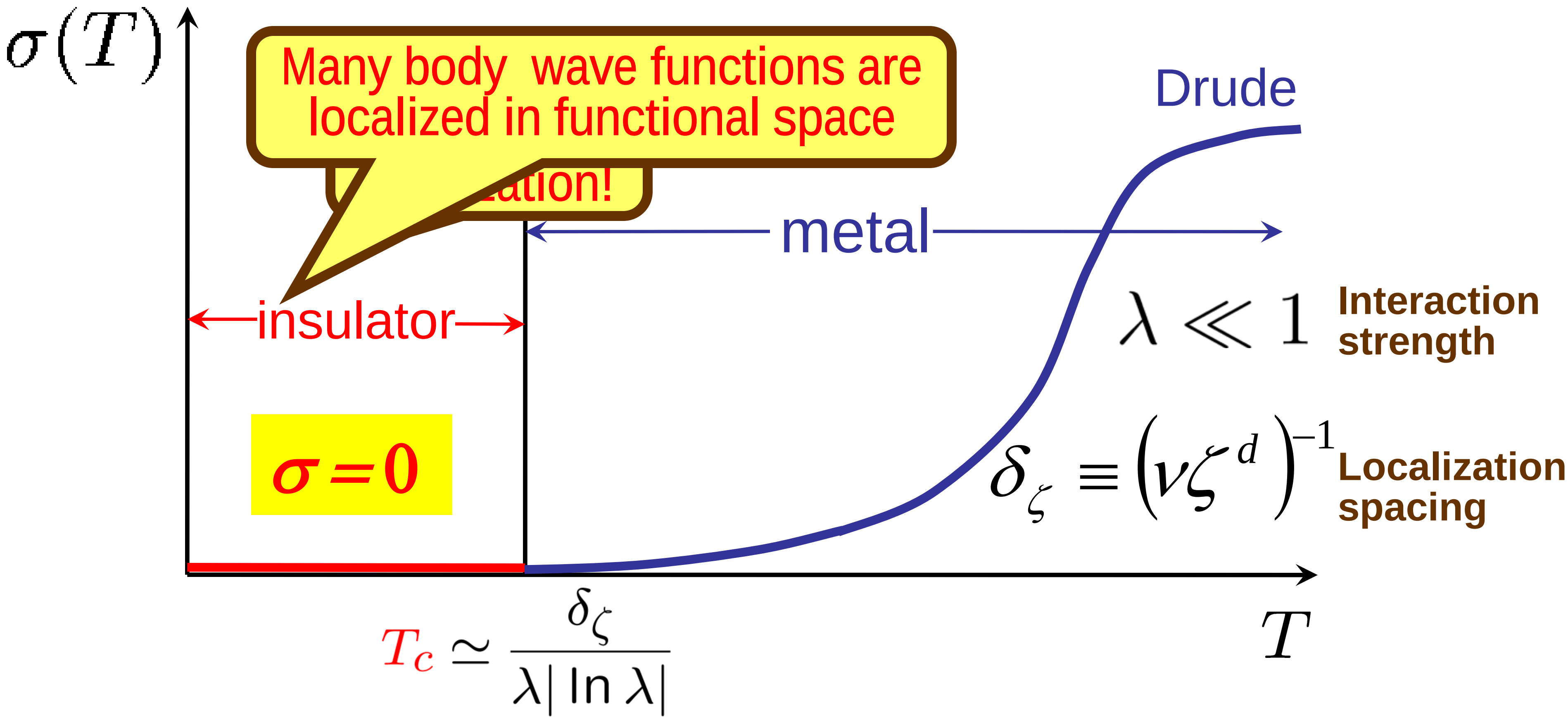
A#1: Sure

A#2: No way **(L. Fleishman. P.W. Anderson (1980))**

A#3: Finite temperature Metal-Insulator Transition



Finite temperature Metal-Insulator Transition



Definitions:

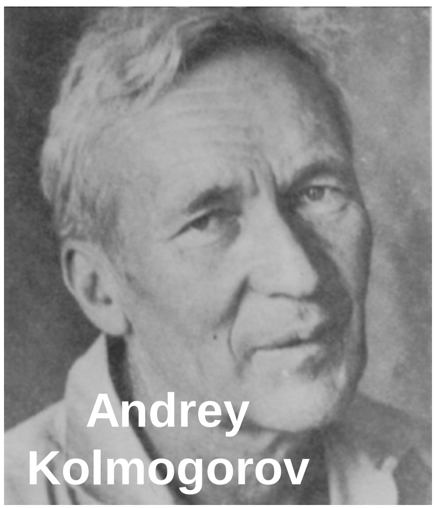
Insulator $\sigma = 0$ **Metal** $\sigma \neq 0$

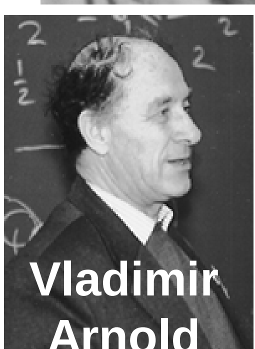
not $d\sigma/dT < 0$ **not** $d\sigma/dT > 0$

3. Localization beyond real space

Kolmogorov – Arnold – Moser (KAM) theory

A.N. Kolmogorov,
Dokl. Akad. Nauk SSSR, 1954.
Proc. 1954 Int. Congress of Mathematics, North-Holland, 1957





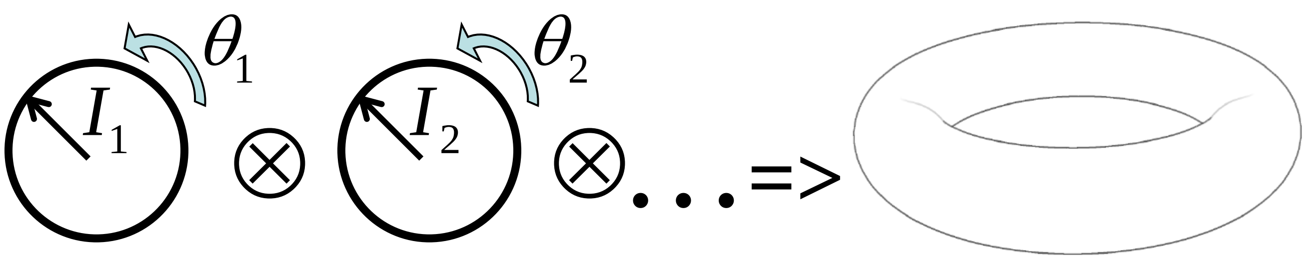
$$\hbar = 0$$

Integrable classical Hamiltonian $\hat{H}_0$, ***d>1***:

Separation of variables: *d* sets of action-angle variables

$$I_1, \theta_1 = 2\pi\omega_1 t; \ldots, I_2, \theta_2 = 2\pi\omega_2 t; ..$$

Quasiperiodic motion:
set of the frequencies, $\omega_1, \omega_2, .., \omega_d$ **which are in general incommensurate. Actions** I_i **are integrals of motion** $\partial I_i / \partial t = 0$



tori

Integrable dynamics:

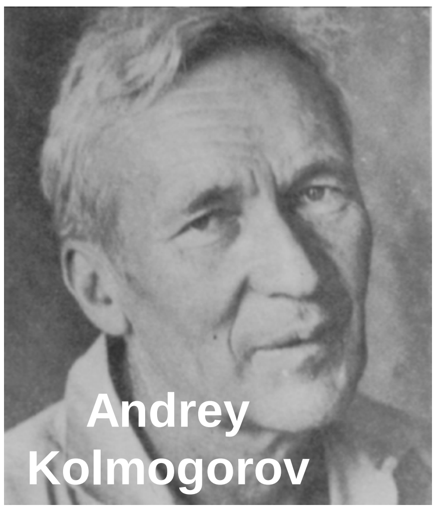
Each classical trajectory is quasiperiodic and confined to a particular torus, which is determined by a set of the integrals of motion

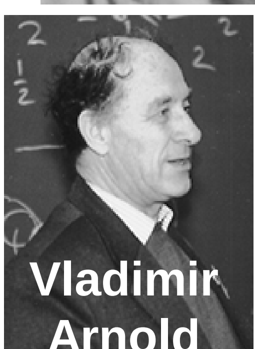
space	Number of dimensions
real space	d
phase space: (x,p)	$2d$
energy shell	$2d-1$
tori	d

Each torus has measure zero on the energy shell !

Kolmogorov – Arnold – Moser (KAM) theory

A.N. Kolmogorov,
Dokl. Akad. Nauk SSSR, 1954.
Proc. 1954 Int. Congress of Mathematics, North-Holland, 1957



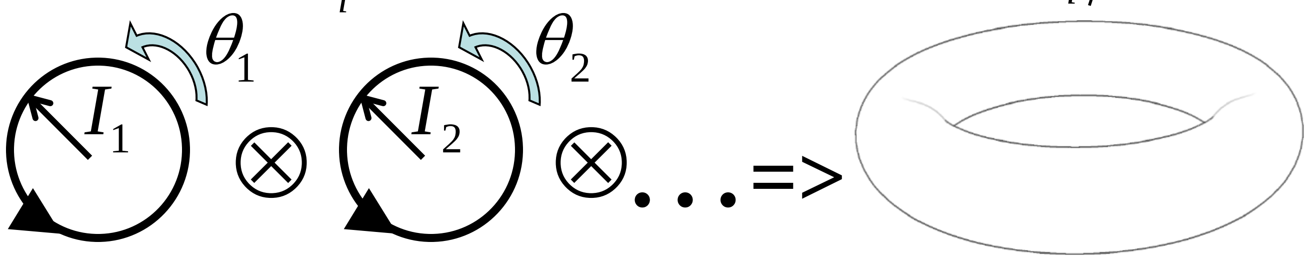


Integrable classical Hamiltonian $\hat{H}_0$, *d>1*:

Separation of variables: ***d*** sets of action-angle variables $I_1, \theta_1 = 2\pi\omega_1 t; .., I_2, \theta_2 = 2\pi\omega_2 t; ..$

Quasiperiodic motion: set of the frequencies, $\omega_1, \omega_2, .., \omega_d$ which are in general incommensurate

Actions I_i are integrals of motion $\partial I_i / \partial t = 0$



Q: Will an arbitrary weak perturbation $\hat{V}$ of the integrable Hamiltonian $\hat{H}_0$ destroy the tori and make the motion ergodic (when each point at the energy shell will be reached sooner or later)**?**

A: **Most of the tori survive weak and smooth enough perturbations**

KAM theorem

Kolmogorov – Arnold – Moser (KAM) theory

A.N. Kolmogorov,
Dokl. Akad. Nauk SSSR, 1954.
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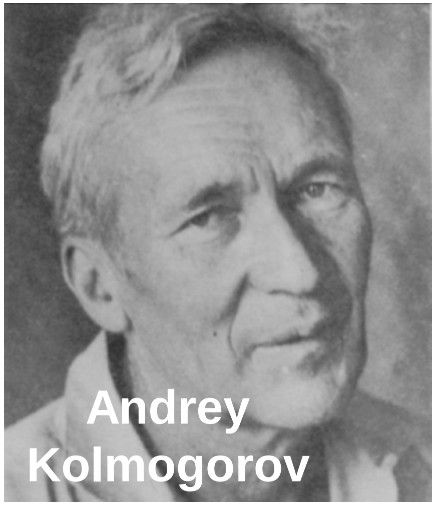
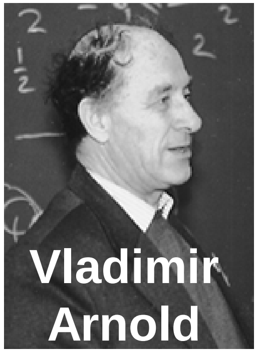
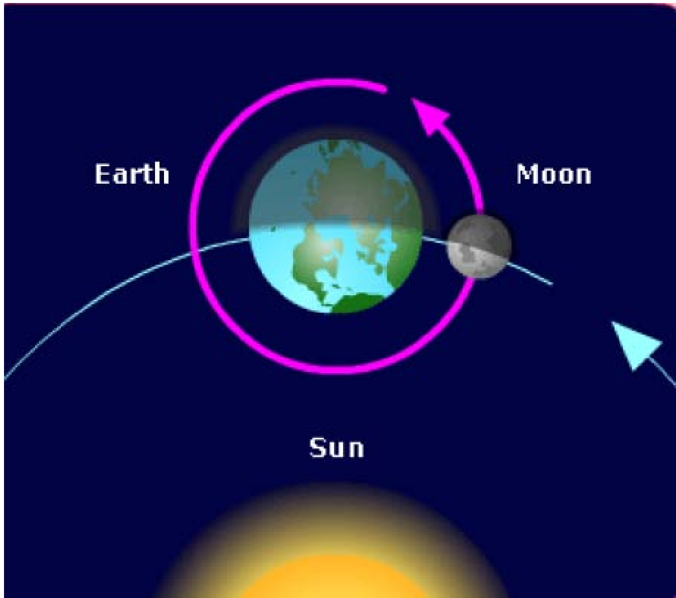
Q: Will an arbitrary weak perturbation $\hat{V}$ of the integrable Hamiltonian $\hat{H}_0$ destroy the tori and make the motion ergodic (i.e. each point at the energy shell would be reached sooner or later) **?**

A: Most of the tori survive weak and smooth enough perturbations

KAM theorem

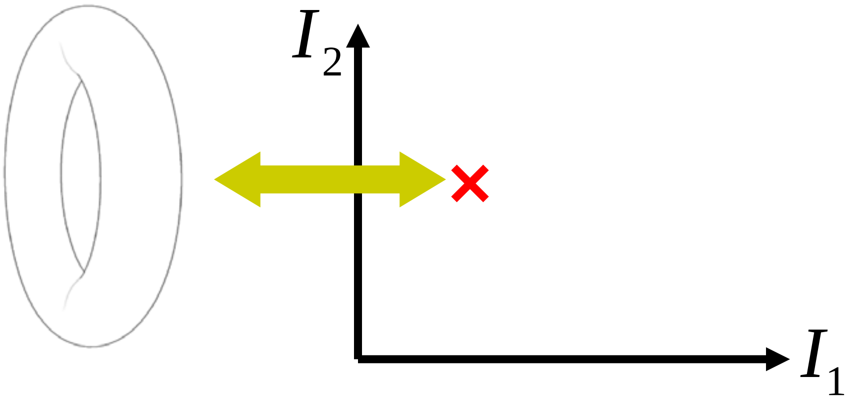




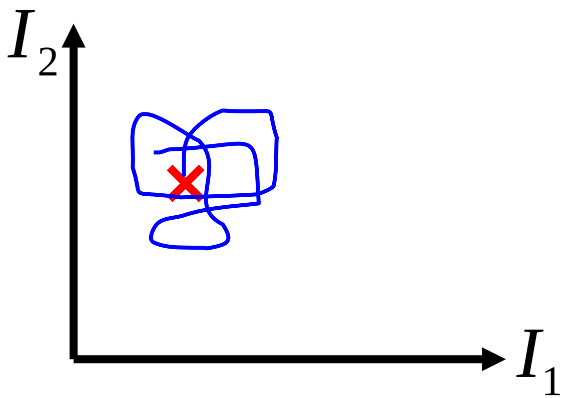


KAM theorem:

Most of the tori survive weak and smooth enough perturbations



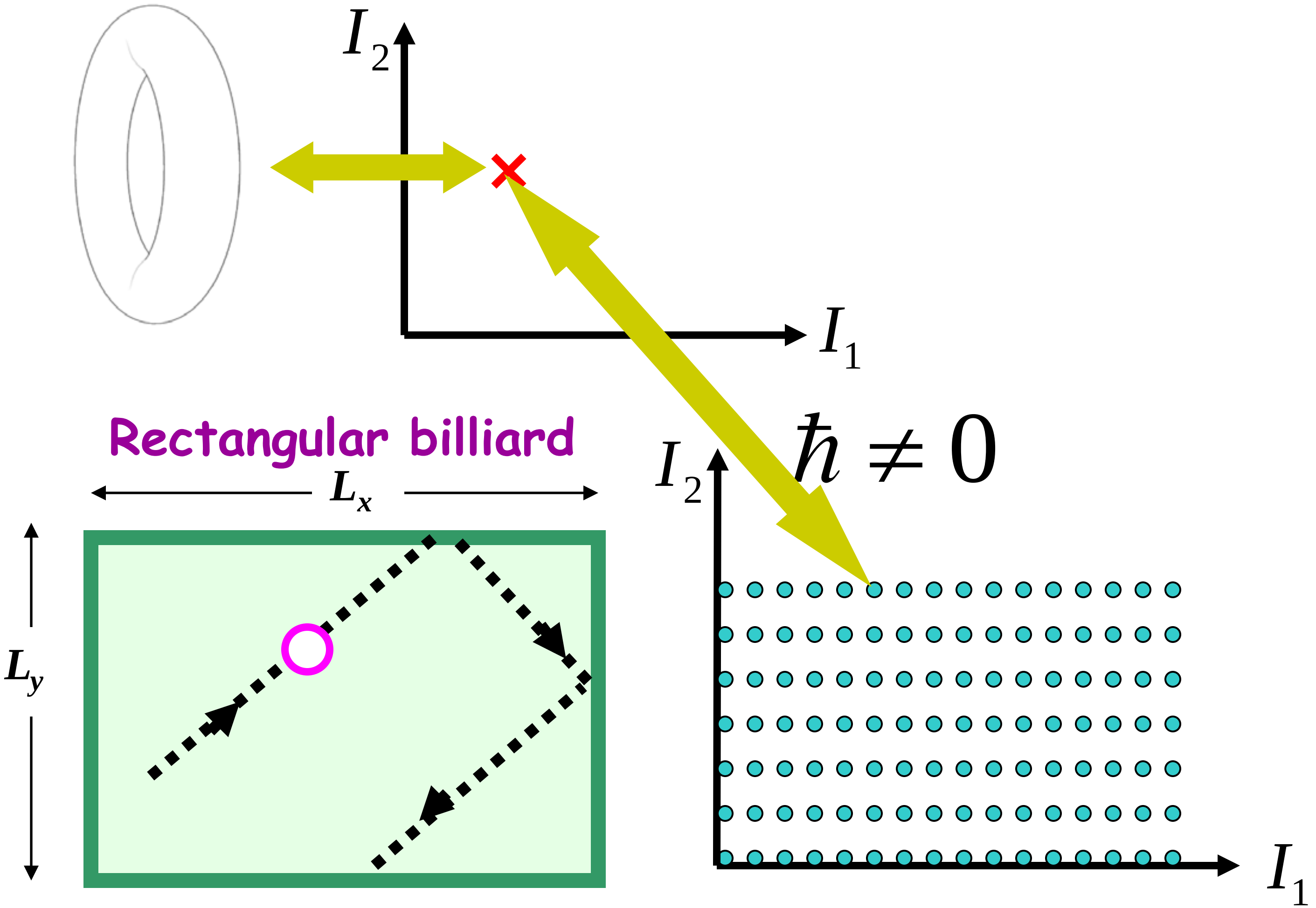
$\hat{V} \neq 0$



Each point in the space of the integrals of motion corresponds to a torus and vice versa

Finite motion.
Localization in the space of the integrals of motion ?

KAM theorem: Most of the tori survive weak and smooth enough perturbations

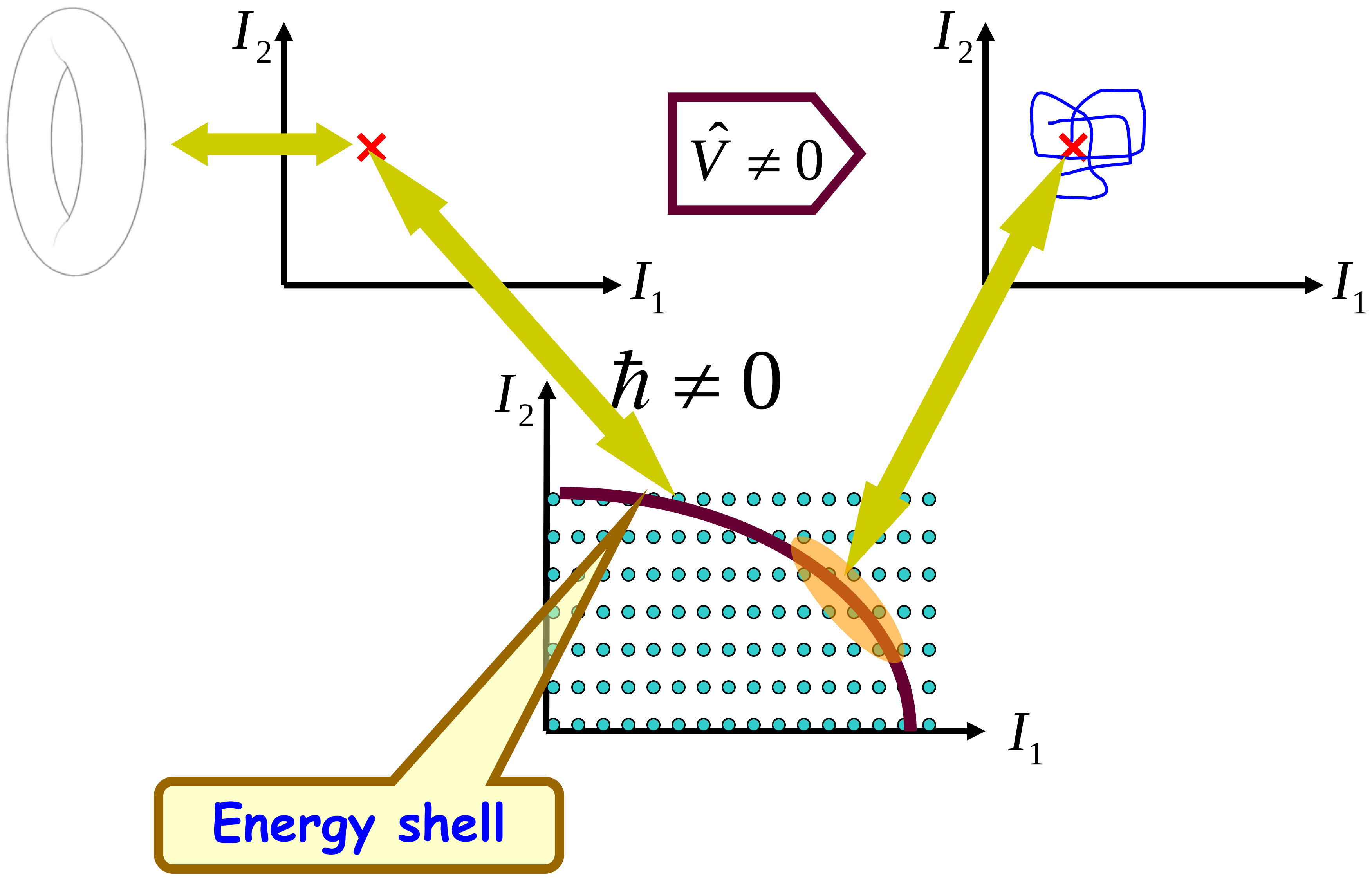


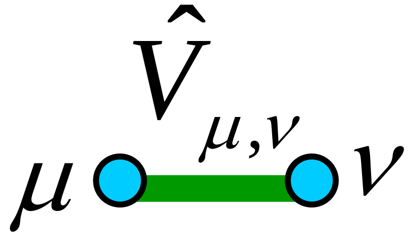
Two integrals of motion

$$I_1 = p_x; \quad I_1 = p_y$$

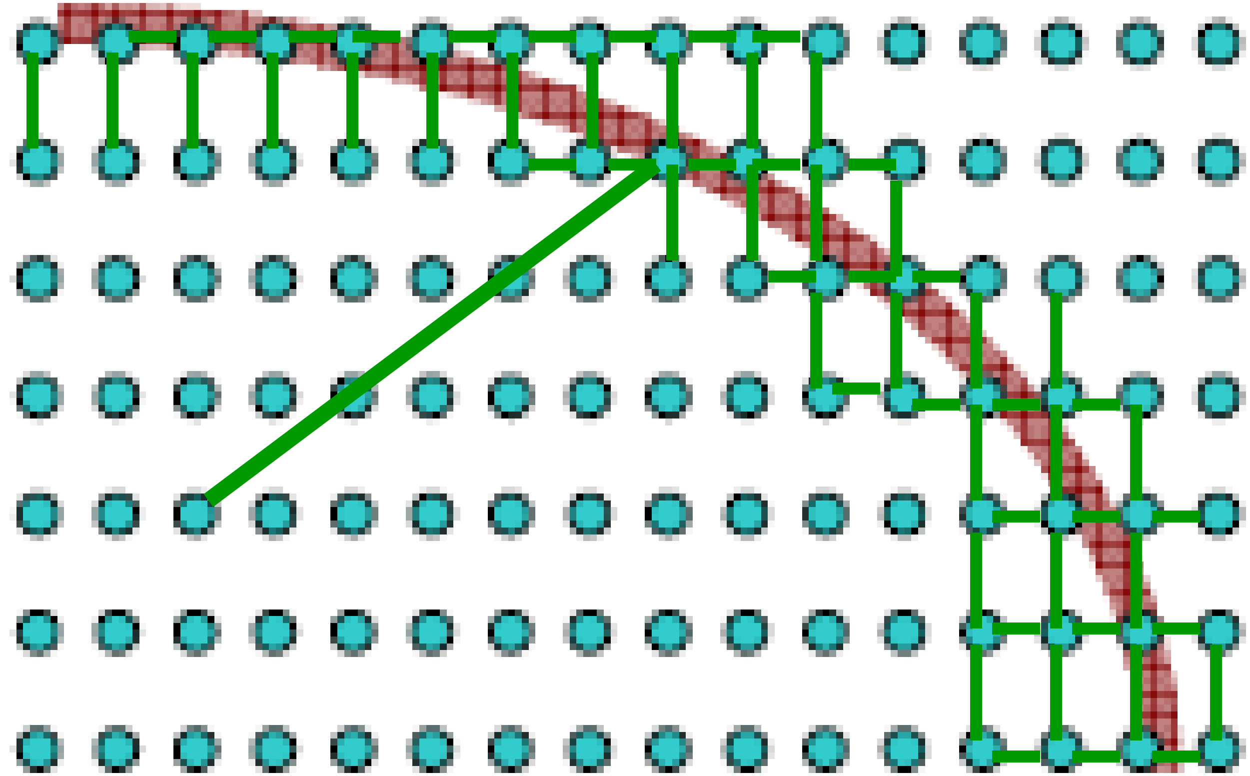
$$p_x = \frac{\pi n}{L_x}; \quad p_y = \frac{\pi m}{L_x}$$

KAM theorem: Most of the tori survive weak and smooth enough perturbations





Matrix element of the perturbation



$$\left|\mu\right\rangle = \left|\vec{I}^{(\mu)}\right\rangle$$

$$\vec{I}^{(\mu)} = \left\{I_1^{(\mu)}, \ldots, I_d^{(\mu)}\right\}$$

One can speak about localization provided that the perturbation is somewhat local in the space of quantum numbers of the original Hamiltonian

AL hops are local – one can distinguish "near" and "far"

KAM perturbation is smooth enough

Glossary

Classical	Quantum
Integrable $H_0 = H_0\left(\vec{I}\right)$	Integrable $\hat{H}_0 = \sum_{\mu} E_{\mu} \left\| \mu \right\rangle \left\langle \mu \right\|, \quad \left\| \mu \right\rangle = \left\| \vec{I} \right\rangle$
KAM	Localized
Ergodic - distributed all over the energy shell Chaotic	Extended ?

~~Strong disorder~~	~~localized~~
~~Weak disorder~~	~~extended~~

Strong disorder	localized
Moderate disorder	extended
No disorder chaotic	extended
No disorder integrable	localized
Too weak disorder int.	localized

Consider an integrable system.
Each state is characterized by a set of quantum numbers.

It can be viewed as a point in the space of quantum numbers. The whole set of the states forms a lattice in this space.

A perturbation that violates the integrability provides matrix elements of the hopping between different sites (Anderson model !?)

Q: Is it possible to tell if the states are localized (in some unknown basis) or extended. **?**

Density of States is not singular at the Anderson transition

This applies only to the average Density of States !

Fluctuations ?

4. Spectral statistics and Localization

RANDOM MATRIX THEORY

Spectral statistics

$N \times N$ — ensemble of Hermitian matrices with random matrix element — $N \to \infty$

E_α - spectrum (set of eigenvalues)

$\delta_1 \equiv \left\langle E_{\alpha+1} - E_\alpha \right\rangle$ - mean level spacing, determines the density of states

$\left\langle \ldots\ldots \right\rangle$ - ensemble averaging

$s \equiv \dfrac{E_{\alpha+1} - E_\alpha}{\delta_1}$ - spacing between nearest neighbors

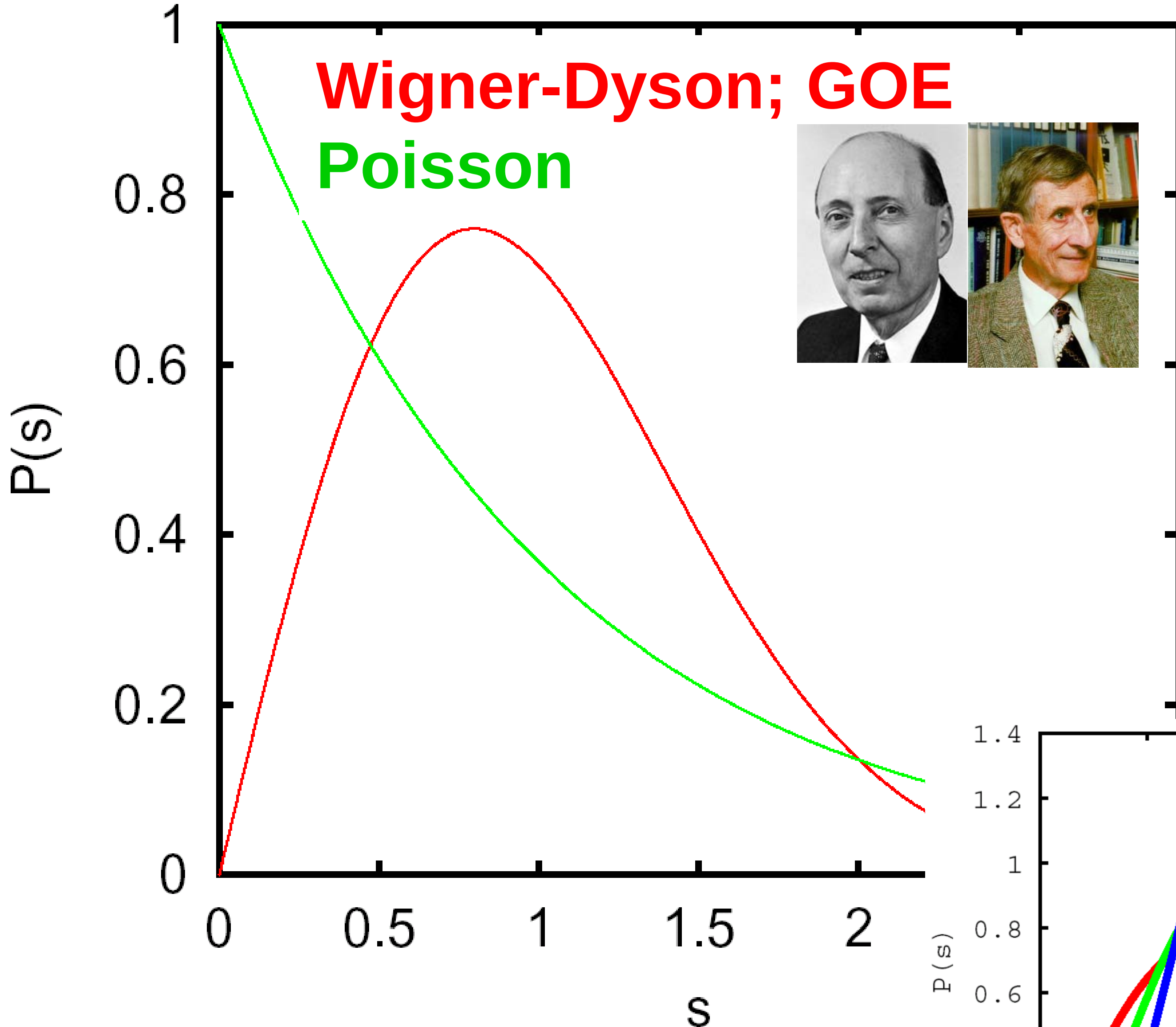
$P(s)$ - distribution function of nearest neighbors spacing between

Spectral Rigidity

Level repulsion

$$P(s=0)=0$$

$$P(s \ll 1) \propto s^\beta \qquad \beta=1,2,4$$

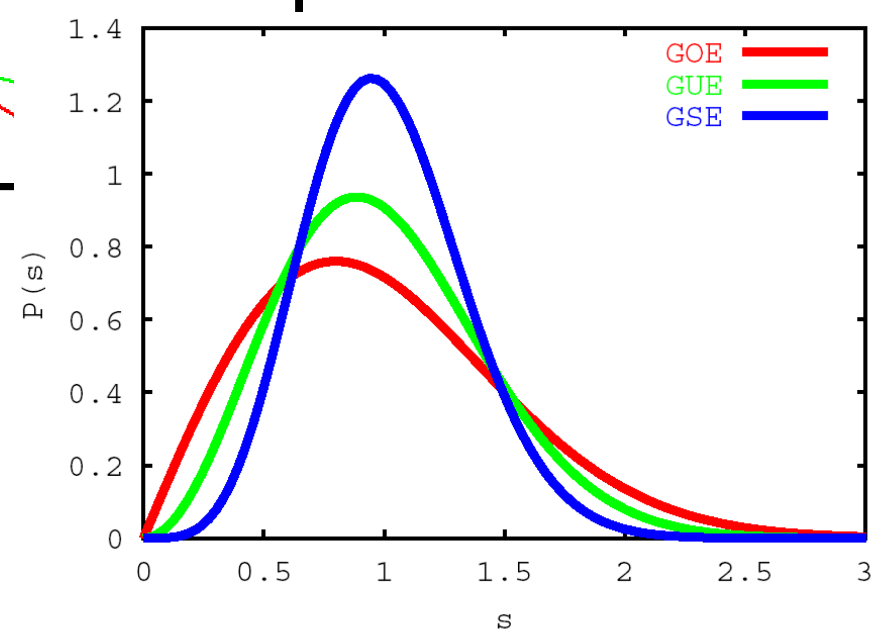


Gaussian
Orthogonal
Ensemble

Orthogonal
$\beta=1$

Unitary
$\beta=2$

Simplectic
$\beta=4$



Poisson – completely uncorrelated levels

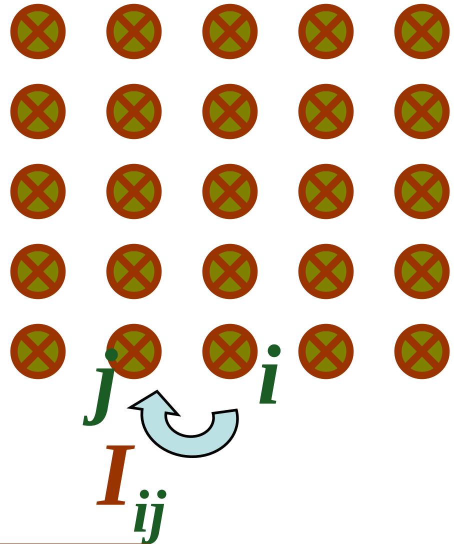
RANDOM MATRICES

$N \times N$ matrices with random matrix elements. $N \to \infty$

Dyson Ensembles

Matrix elements	Ensemble	β	realization
real	orthogonal	1	T-inv potential
complex	unitary	2	broken T-invariance (e.g., by magnetic field)
2×2 matrices	simplectic	4	T-inv, but with spin-orbital coupling

Anderson Model



- *Lattice - tight binding model*
- *Onsite energies* ε_i - *random*
- *Hopping matrix elements* I_{ij}

$-W < \varepsilon_i < W$

uniformly distributed

Is there much in common between Random Matrices and Hamiltonians with random potential ?

Q: What are the spectral statistics of a finite size Anderson model ?

Anderson Transition

Strong disorder

$$I < I_c$$

Insulator

All eigenstates are localized

Localization length ξ

The eigenstates, which are localized at different places will not repel each other

Poisson spectral statistics

Weak disorder

$$I > I_c$$

Metal

There appear states extended all over the whole system

Any two extended eigenstates repel each other

Wigner – Dyson spectral statistics

Anderson Localization and Spectral Statistics

Localized states
Insulator

Poisson spectral statistics

Extended states
Metal

Wigner-Dyson spectral statistics

Consider an integrable system.
Each state is characterized by a set of quantum numbers.

It can be viewed as a point in the space of quantum numbers. The whole set of the states forms a lattice in this space.

A perturbation that violates the integrability provides matrix elements of the hopping between different sites (Anderson model !?)

Weak enough hopping:
Localization - Poisson
Strong hopping:
transition to Wigner-Dyson

Extended states: Level repulsion, anticrossings, Wigner-Dyson spectral statistics

Localized states: Poisson spectral statistics

Invariant (basis independent) definition

Many-Body Localization

BA, Gefen, Kamenev & Levitov, 1997
Basko, Aleiner & BA, 2005. . .

Example: Random Ising model in the perpendicular field

Will not discuss today in detail

$$\hat{H} = \underbrace{\sum_{i=1}^{N} B_i \hat{\sigma}_i^z + \sum_{i \neq j} J_{ij} \hat{\sigma}_i^z \hat{\sigma}_j^z}_{\text{Random Ising model in a parallel field}} + I \sum_{i=1}^{N} \hat{\sigma}_i^x \equiv \hat{H}_0 + I \sum_{i=1}^{N} \hat{\sigma}_i^x$$

Perpendicular field

$\vec{\sigma}_i$ - Pauli matrices, $\sigma_i^z = \pm \frac{1}{2}$

$i = 1, 2, \ldots, N; \quad N >> 1$

Without perpendicular field all σ_i^z commute with the Hamiltonian, i.e. they are integrals of motion

$$\hat{H} = \sum_{i=1}^{N} B_i \hat{\sigma}_i^z + \underbrace{\sum_{i \neq j} J_{ij} \hat{\sigma}_i^z \hat{\sigma}_j^z}_{} + I \sum_{i=1}^{N} \hat{\sigma}_i^x \equiv \hat{H}_0 + I \sum_{i=1}^{N} \hat{\sigma}_i^x$$

Random Ising model in a parallel field

Perpendicular field

$\vec{\sigma}_i$ - Pauli matrices

$i = 1, 2, \ldots, N; \quad N >> 1$

Without perpendicular field all σ_i^z commute with the Hamiltonian, i.e. they are integrals of motion

Anderson Model on *N*-dimensional cube

$\left\{ \sigma_i^z \right\}$ determines a site

$$H_0\left(\left\{ \sigma_i \right\}\right)$$

onsite energy

$$\hat{\sigma}^x = \hat{\sigma}^+ + \hat{\sigma}^-$$

hoping between nearest neighbors

$$\hat{H} = \sum_{i=1}^{N} B_i \hat{\sigma}_i^z + \sum_{i \neq j} J_{ij} \hat{\sigma}_i^z \hat{\sigma}_j^z + I \sum_{i=1}^{N} \hat{\sigma}_i^x \equiv \hat{H}_0 + I \sum_{i=1}^{N} \hat{\sigma}_i^x$$

Anderson Model on *N*-dimensional cube

Usually:

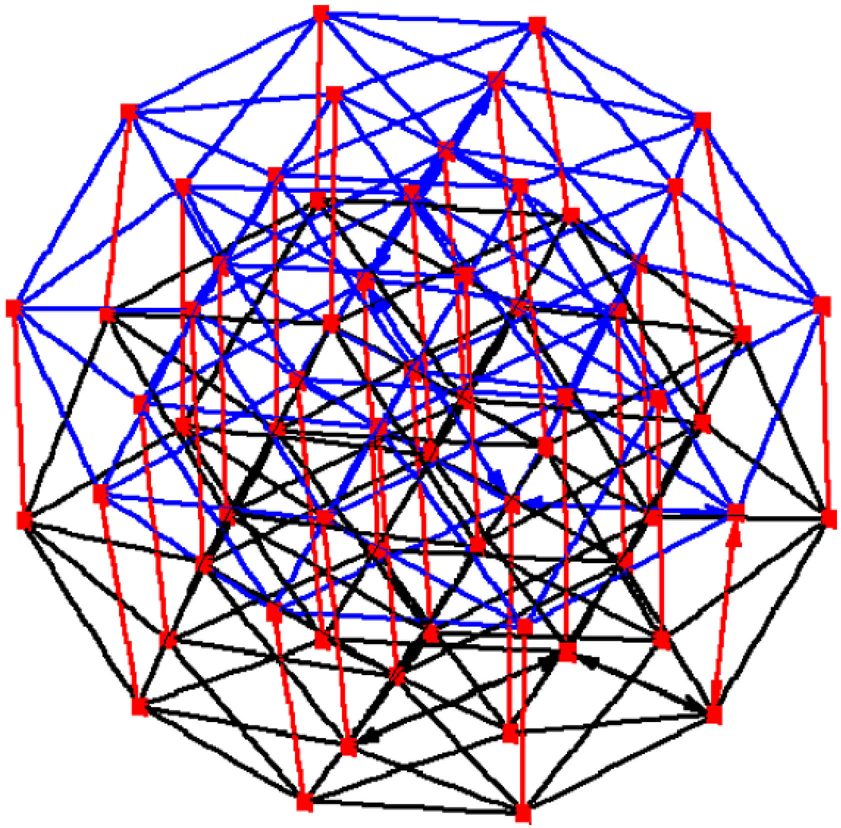
# of dimensions $d \to const$

system linear size $L \to \infty$

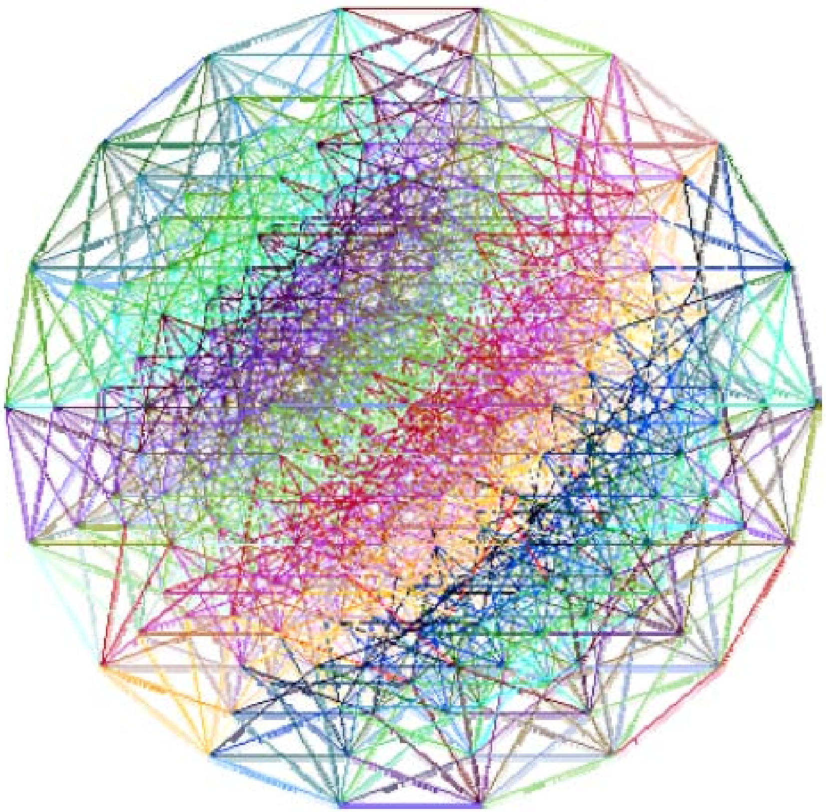
Here:

# of dimensions $d = N \to \infty$

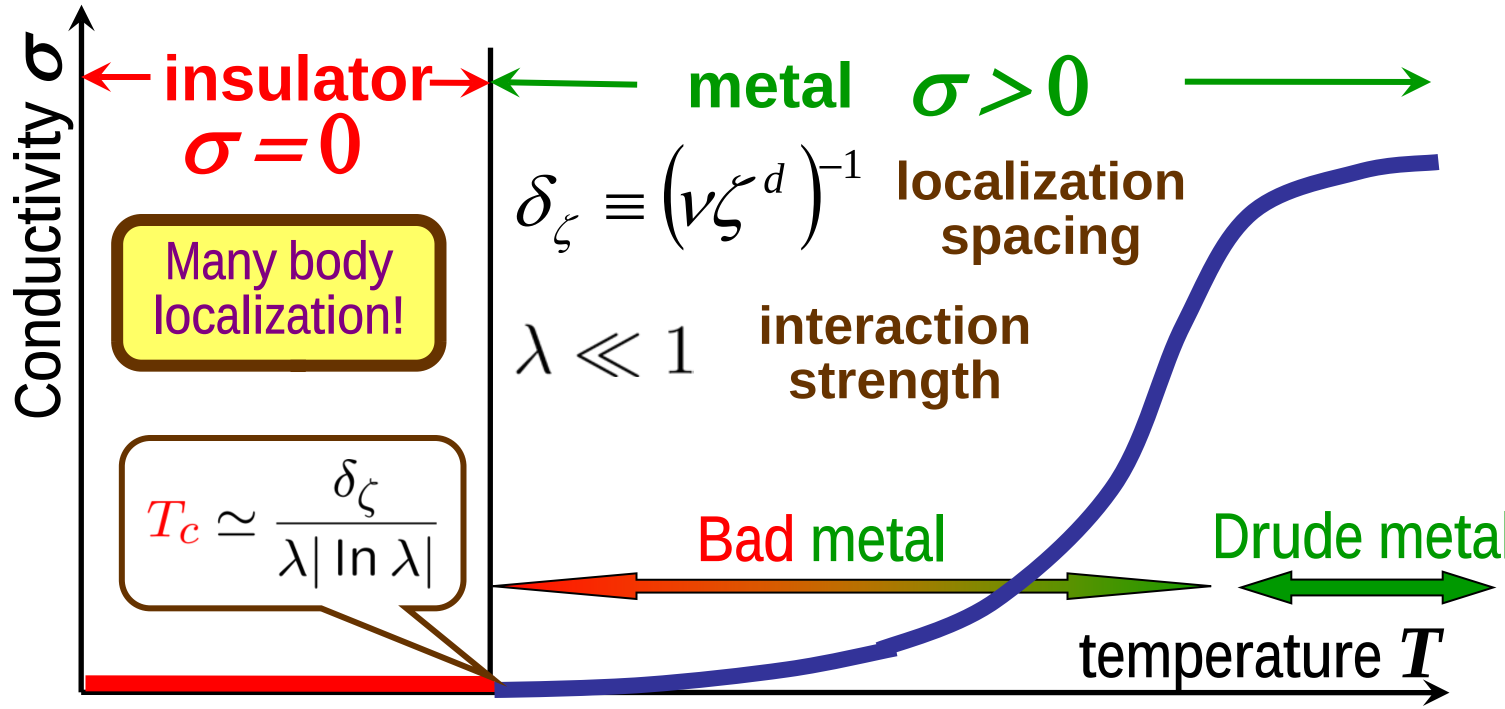
system linear size $L = 1$



6-dimensional cube



9-dimensional cube



Definitions:

Insulator $\sigma = 0$ **Metal** $\sigma \neq 0$

not $d\sigma/dT < 0$ **not** $d\sigma/dT > 0$

Many-Body Localization

1D bosons + disorder

1D Localization

Exactly solved:
all states are localized

Gertsenshtein & Vasil'ev, 1959

Conjectured:

Mott & Twose, 1961

-
-
-

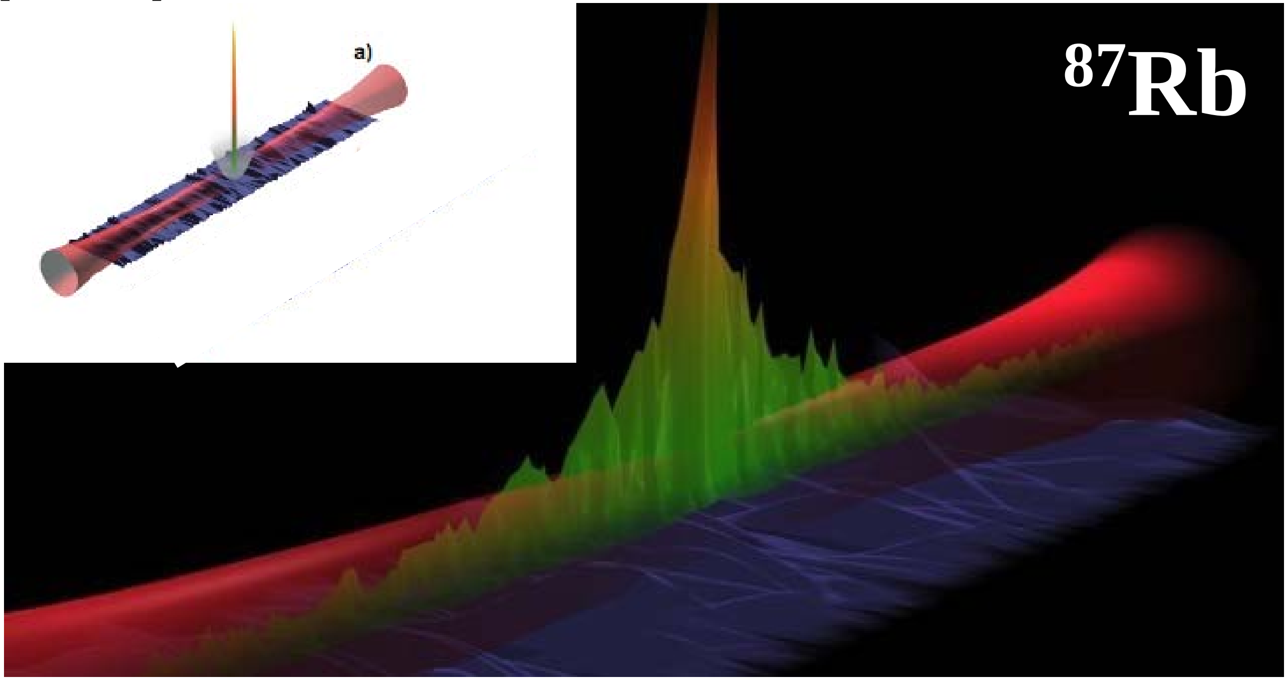
1-particle problem → correct for bosons as well as for fermions

Bosons without disorder

- Bose - Einstein condensation
- Bose-condensate even at weak enough repulsion
- Even in $1d$ case at $T=0$ – "algebraic superfluid"
- Finite temperature – Normal fluid

Localization of cold atoms

Billy et al. “Direct observation of Anderson localization of matter waves in a controlled disorder”. Nature <u>453</u>, 891- 894 (2008).



Roati et al. “Anderson localization of a non-interacting Bose-Einstein condensate“. Nature <u>453</u>, 895-898 (2008).

No interaction !

Thermodynamics of ideal Bose-gas in the presence of disorder is a pathological problem: all particles will occupy the localized state with the lowest energy

$\Rightarrow$ Need repulsion

Q: 1D Bosons + disorder + weak repulsion ?

Weakly interacting bosons

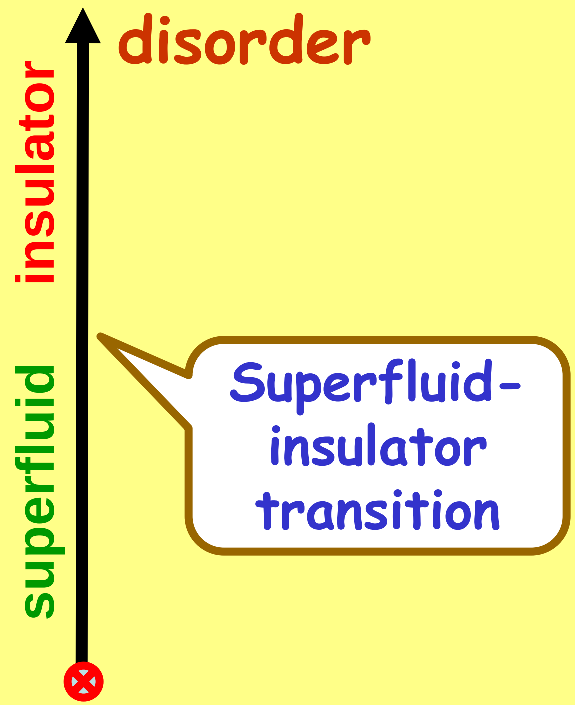
- Bose - Einstein condensation
- Bose-condensate even at weak enough repulsion
- Even in 1D case at T=0 – "algebraic superfluid"

1. No interaction

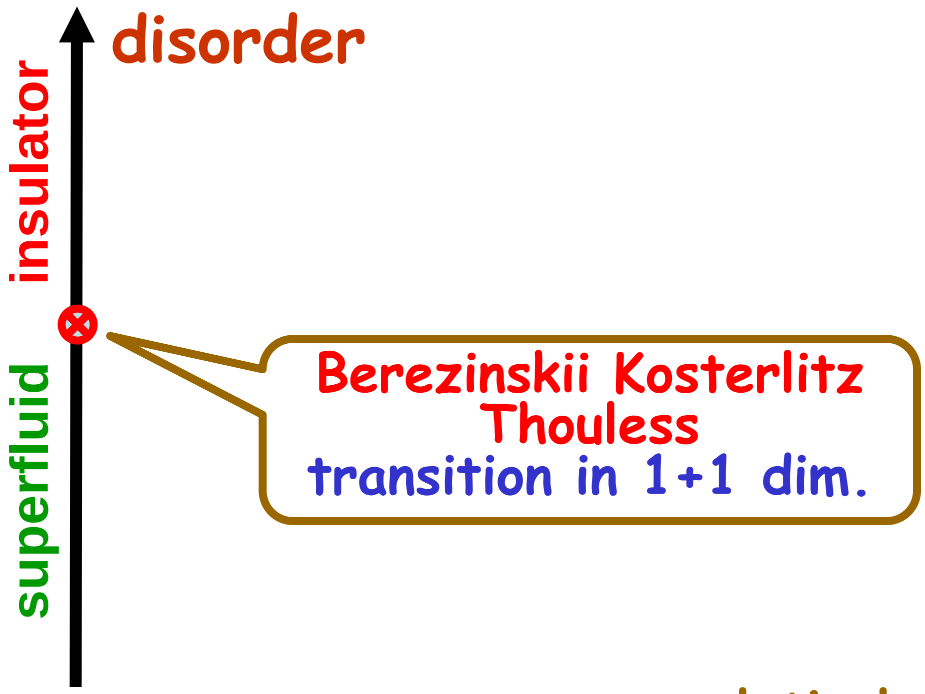
2. No disorder

Superfluid-insulator transition

3. Weak repulsion



T=0 Superfluid - Insulator Quantum Phase Transition



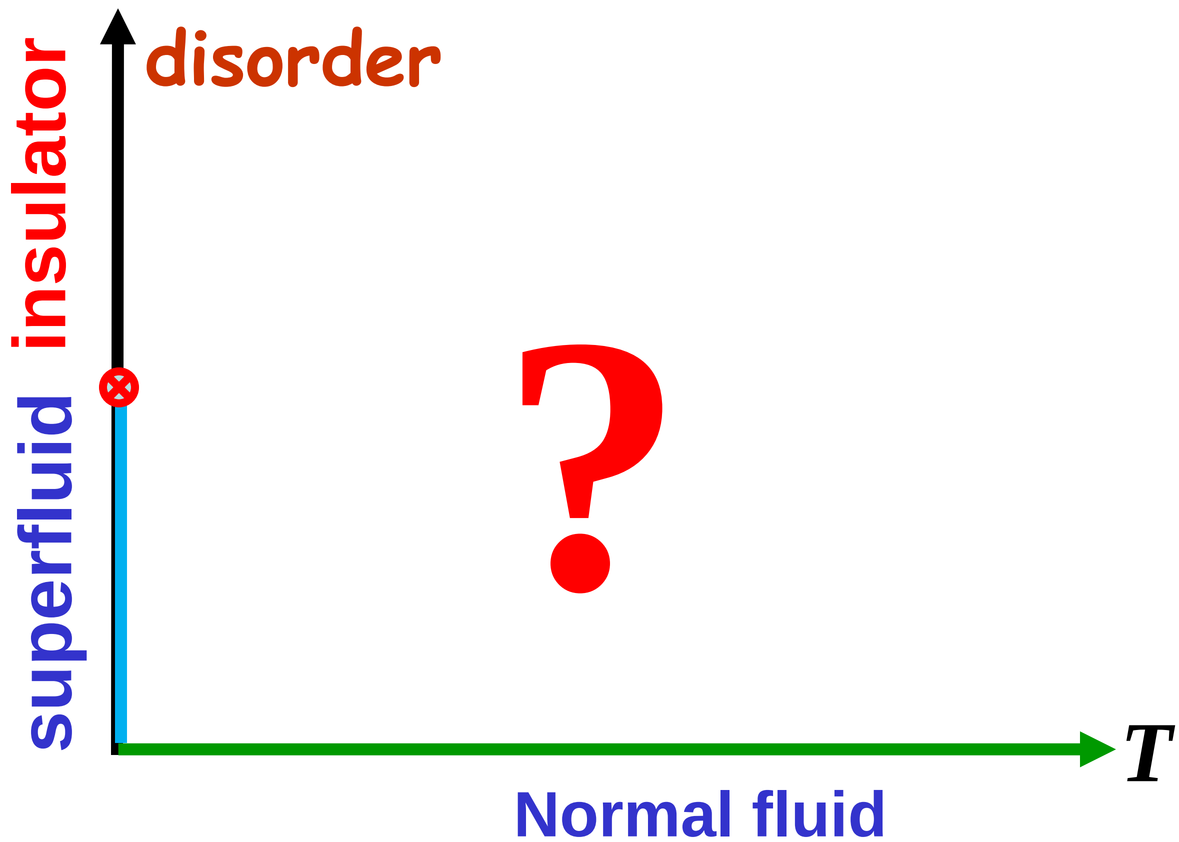
T. Giamarchi and H. J. Schulz, *Phys. Rev*., **B37**, #1(1988).

relatively strong interaction

E. Altman, Y. Kafri, A. Polkovnikov & G. Refael, *Phys. Rev. Lett*., **100**, 170402 (2008).

G.M. Falco, T. Nattermann, & V.L. Pokrovsky, *Phys. Rev*., **B80**, 104515 (2009).

weak interaction



Is it a normal fluid at any temperature?

Dogma

There can be no phase transitions at a finite temperature in 1D

Van Howe, Landau

Reason

Thermal fluctuation destroy any long range correlations in 1D

$T\neq0$ Normal fluid – Insulator Phase Transition:

Neither normal fluids nor glasses (insulators) exhibit long range correlations

still

True phase transition: singularities in transport (rather than thermodynamic) properties

What is insulator?

Perfect Insulator

Zero DC conductivity at finite temperatures

Possible if the system is decoupled from any outside bath

Normal metal (fluid)

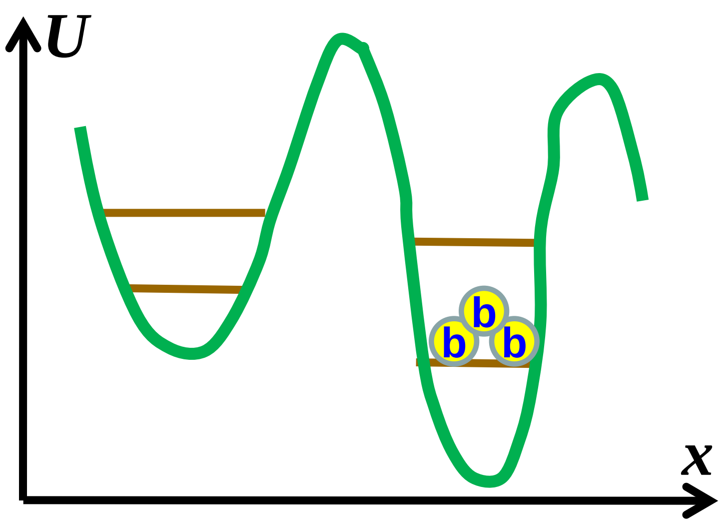
Finite (even if very small) DC conductivity at finite temperatures

1D Luttinger liquid: bosons = fermions ?

Bosons with infinitely strong repulsion $\approx$ Free fermions

Free bosons $\approx$ Fermions with infinitely strong attraction

Weakly interacting bosons $\approx$ Fermions with strong attraction



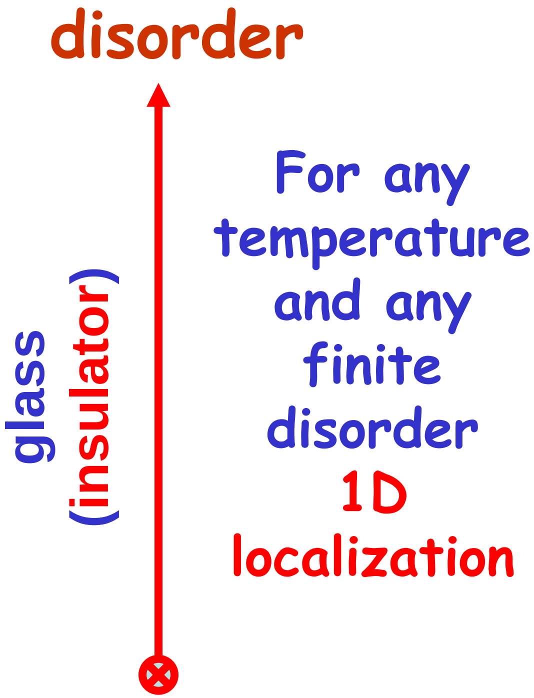
As soon as the occupation numbers become large the analogy with fermions is not too useful

1D Weakly Interacting Bosons + Disorder

Aleiner, BA & Shlyapnikov, 2010, Nature Physics, to be published

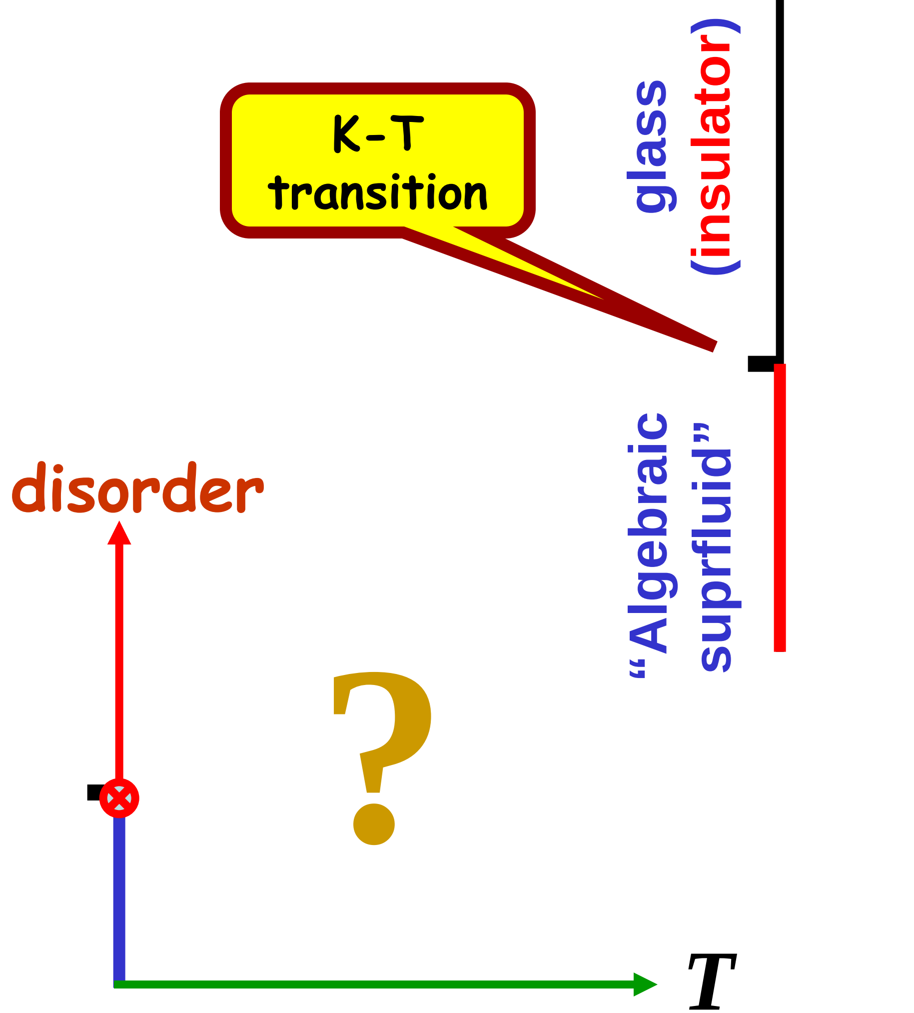
cond-mat 0910.4534

1. No interaction

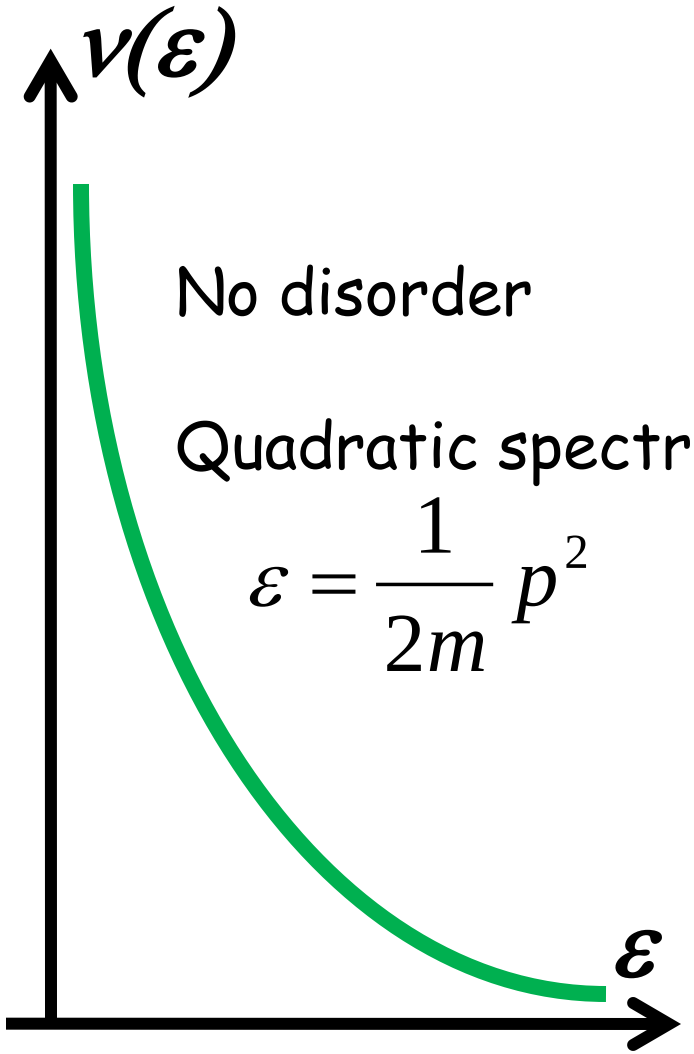


2. No disorder

3. *T*=0



Density of States $\nu(\varepsilon)$ in one dimension



No disorder

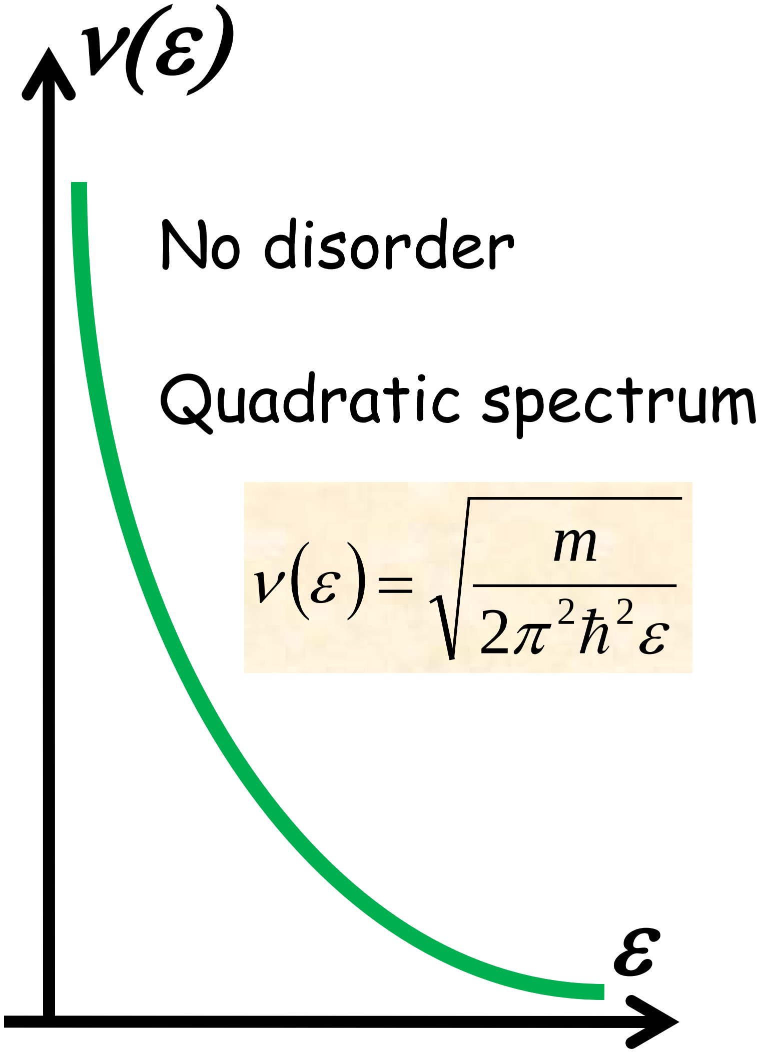
Quadratic spectrum

$$\varepsilon = \frac{1}{2m} p^2$$

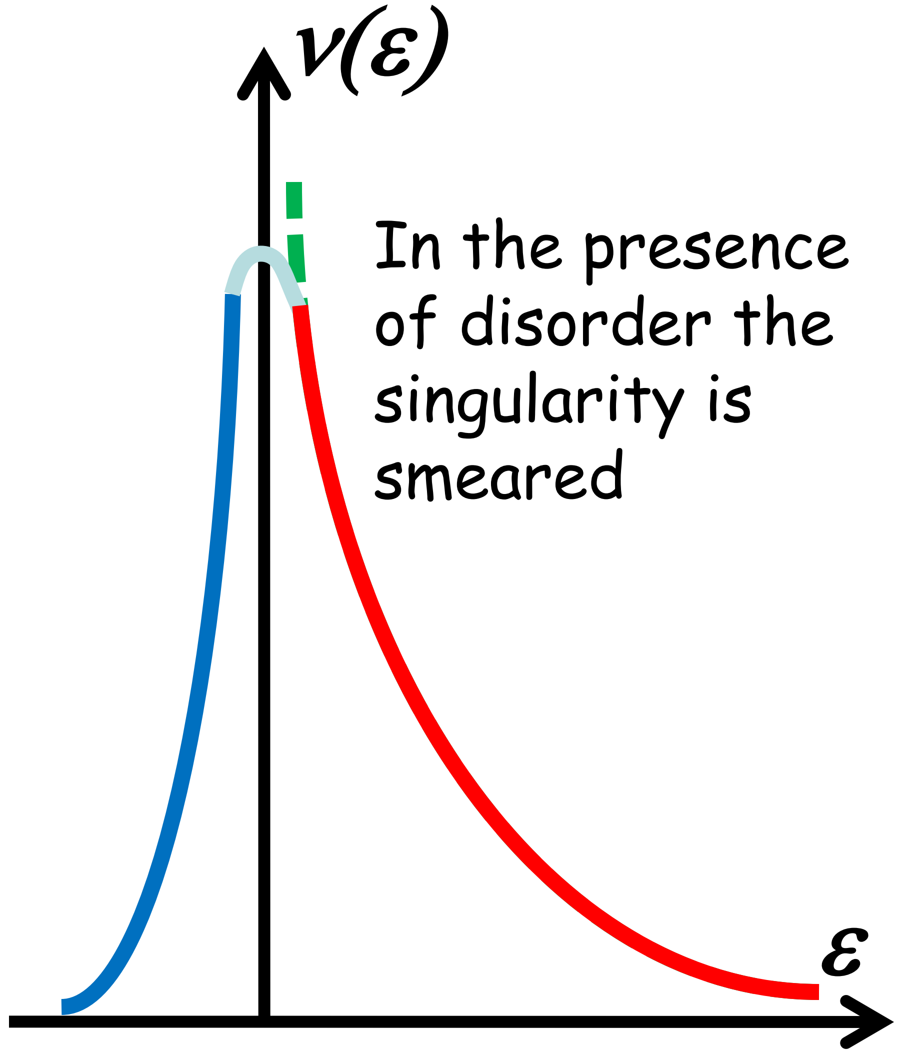
$$\nu(\varepsilon) = \sqrt{\frac{m}{2\pi^2 \hbar^2 \varepsilon}}$$

$\sqrt{}$ - singularity

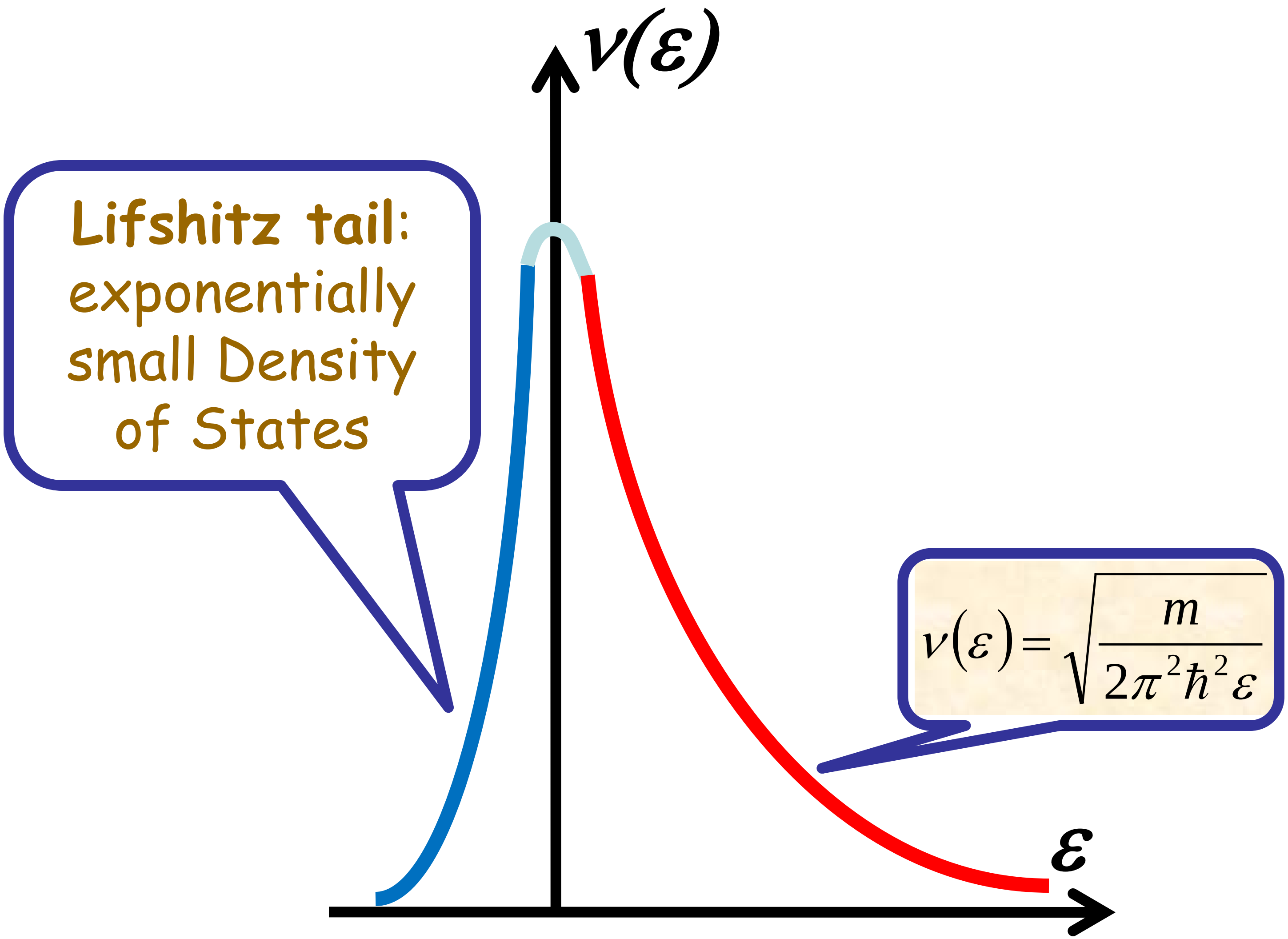
Density of States $\nu(\varepsilon)$ in one dimension



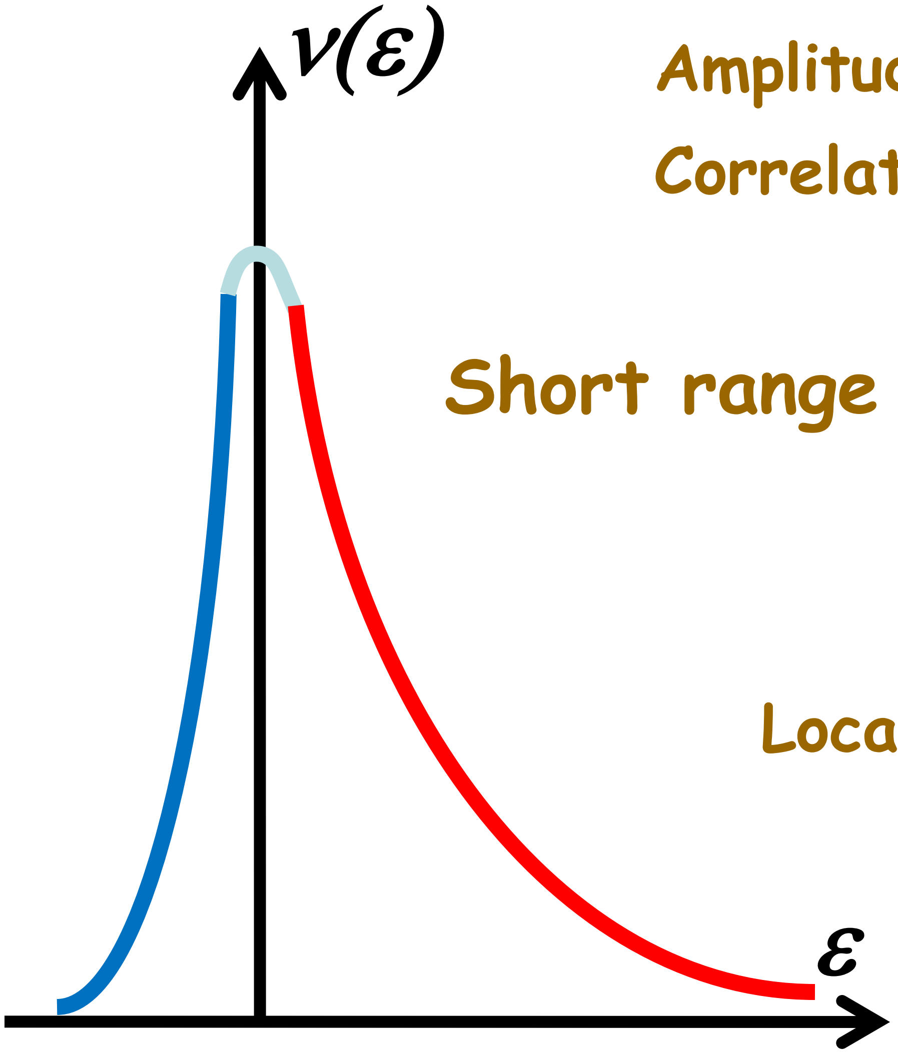
$$\nu(\varepsilon) = \sqrt{\frac{m}{2\pi^2\hbar^2\varepsilon}}$$



Density of States $\nu(\varepsilon)$ in one dimension



Weak disorder – random potential $U(x)$



Random potential $U(x)$:

Amplitude U_0

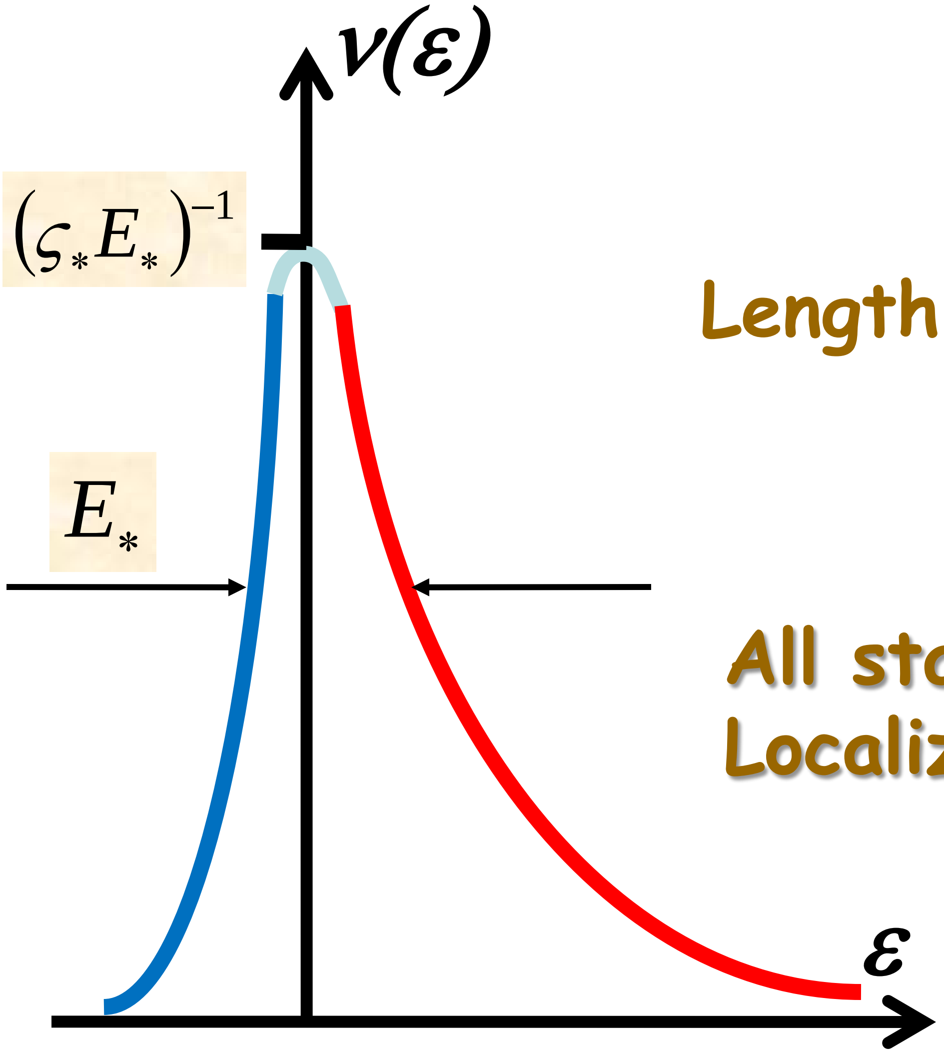
Correlation length σ

Short range disorder: $U_0 << \frac{\hbar^2}{m\sigma^2}$

$\Downarrow$

Localization length $\zeta >> \sigma$

Characteristic scales:



Energy

$$E_* \equiv \left(\frac{U_0^4 \sigma^2 m}{\hbar^2} \right)^{1/3}$$

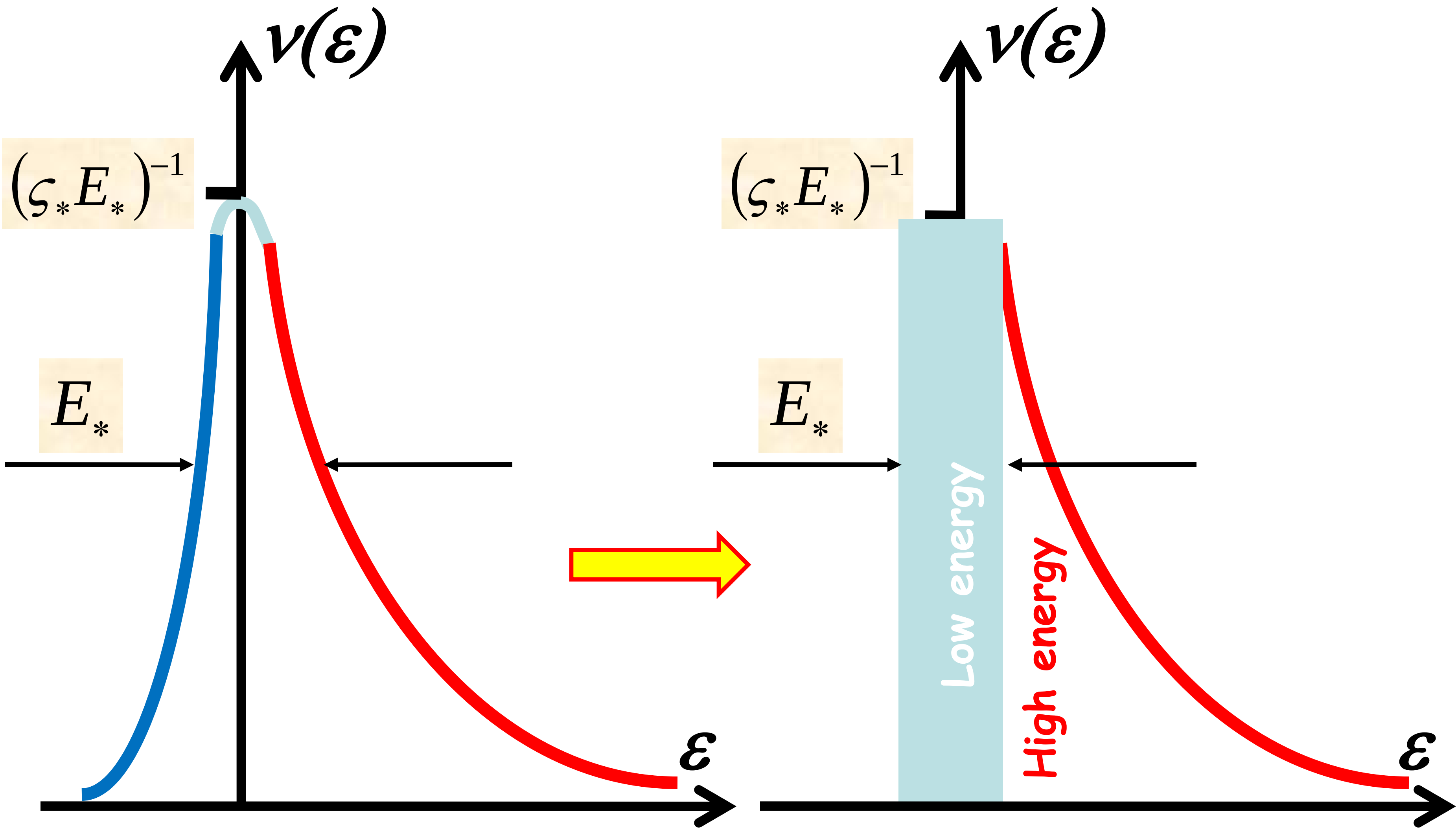
Length

$$\varsigma_* \equiv \left(\frac{\hbar^4}{U_0^2 \sigma m} \right)^{1/3} >> \sigma$$

All states are localized

Localization length:

$$\varsigma(\varepsilon) \sim \begin{cases} \varsigma_* & \varepsilon \sim E_* \\ \varsigma_* \dfrac{\varepsilon}{E_*} & \varepsilon >> E_* \end{cases}$$



$\nu(\varepsilon)$
$(\varsigma_* E_*)^{-1}$
E_*
ε
$\nu(\varepsilon)$
$(\varsigma_* E_*)^{-1}$
E_*
Low energy
High energy
ε

Finite density Bose-gas with repulsion

Density n

Two more energy scales

Temperature of quantum degeneracy $T_d \equiv \frac{\hbar^2 n^2}{m}$

Interaction energy per particle ng

Two dimensionless parameters

$\kappa \equiv E_* / ng$ Characterizes the strength of **disorder**

$\gamma \equiv ng / T_d$ Characterizes the **interaction** strength

Strong disorder $\kappa >> 1$

Weak interaction $\gamma << 1$

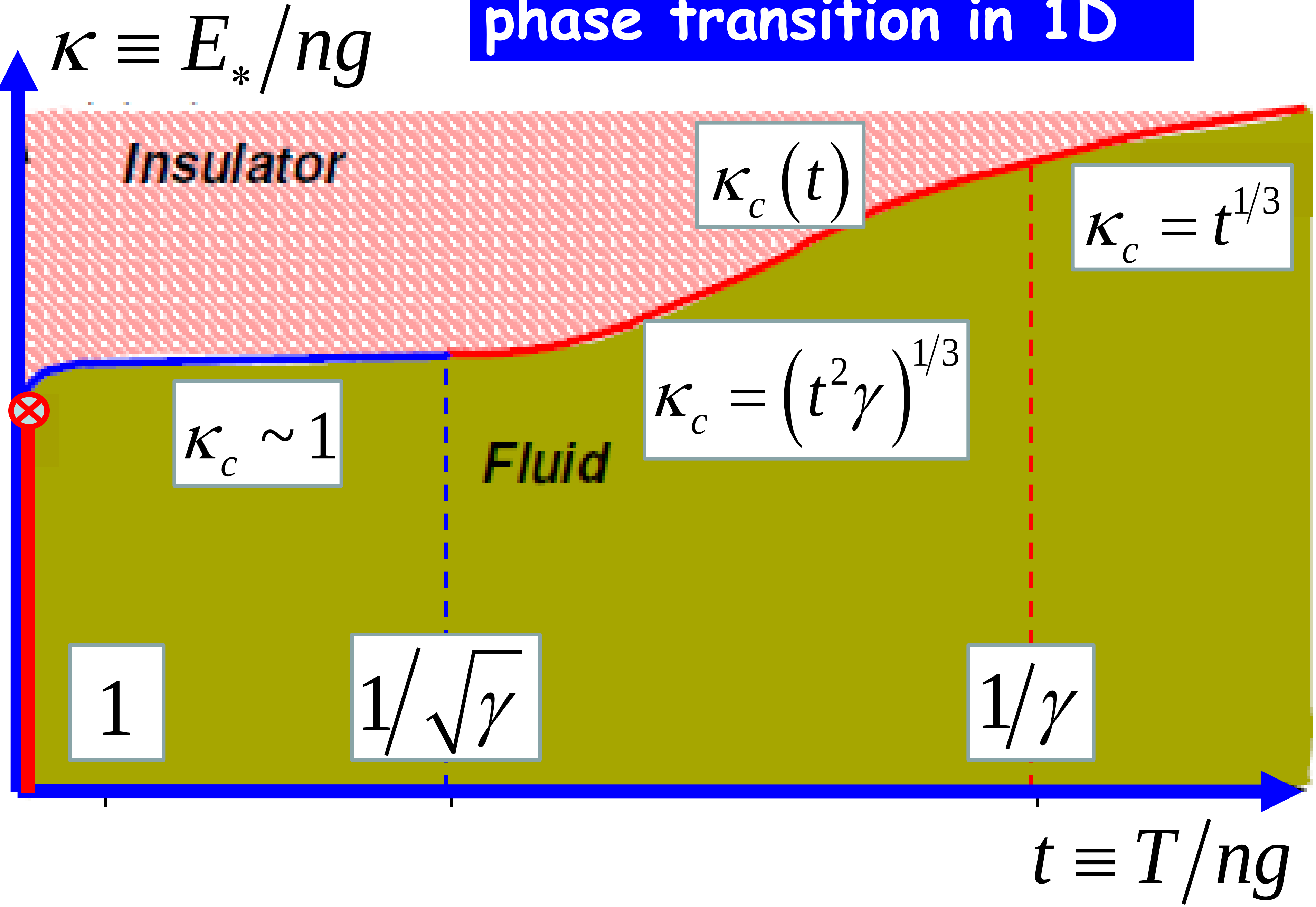
Dimensionless temperature $t = T/ng$

Critical temperature T_c $t_c = t_c(\kappa, \gamma)$

Critical disorder $\kappa_c = \kappa_c(t, \gamma)$

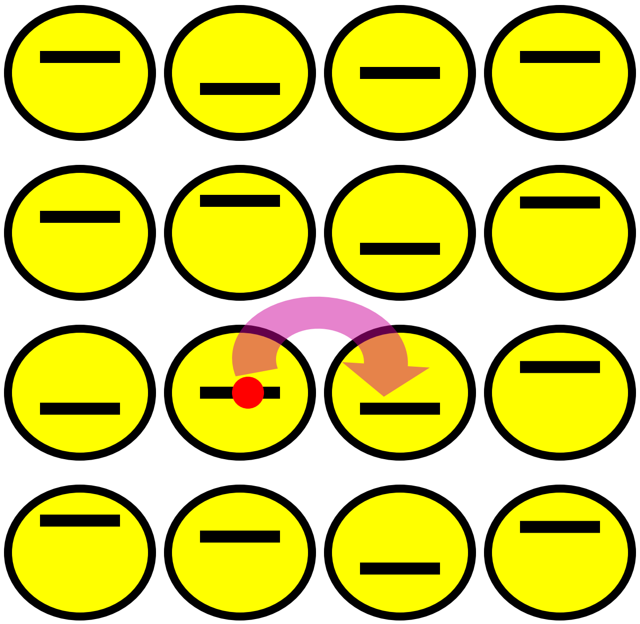
Phase transition line on the t, κ - plane

Finite temperature phase transition in 1D



Conventional Anderson Model

- one particle,
- one level per site,
- onsite disorder
- nearest neighbor hoping



Basis: $\left|i\right\rangle,\quad i$ **labels sites**

Hamiltonian: $\hat{H} = \hat{H}_0 + \hat{V}$

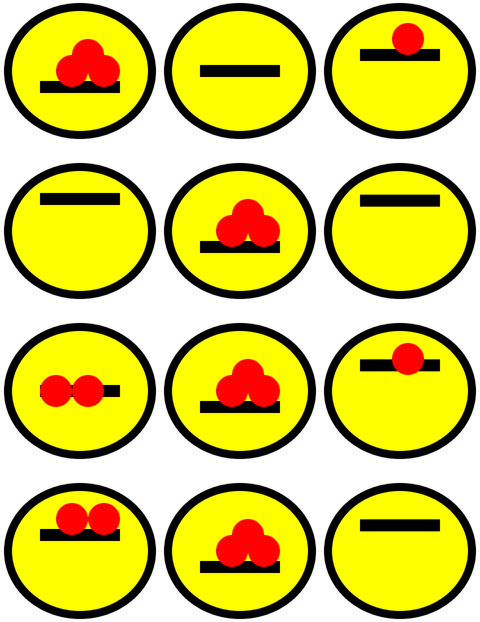
$$\hat{H}_0 = \sum_i \varepsilon_i \left|i\right\rangle\left\langle i\right| \qquad \hat{V} = \sum_{i,j=n.n.} I\left|i\right\rangle\left\langle j\right|$$

Transition: happens when the hoping matrix element exceeds the energy mismatch

The same for many-body localization

Many body Anderson-like Model

- many particles,
- several particles per site.
- interaction



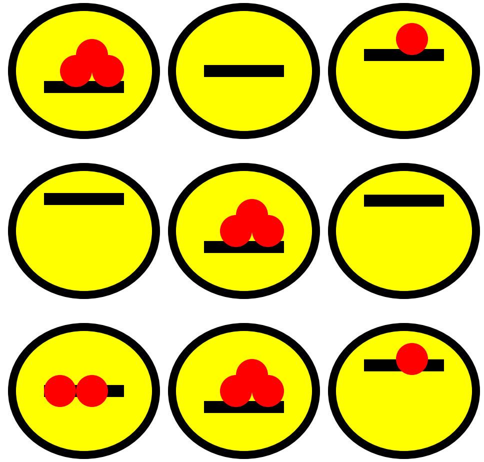
Basis: $\left|\mu\right\rangle \equiv \left|\left\{n_i\right\}\right\rangle$

i label sites

$n_i = 0, 1, 2, 3, \ldots$
occupation numbers

Many body Anderson-like Model

- many particles,
- several particles per site.
- interaction



Basis: $|\mu\rangle$

$$\mu = \{ n_i \}$$

i **labels sites**

$n_i = 0, 1, 2, \ldots$ **occupation numbers**

Hamiltonian:

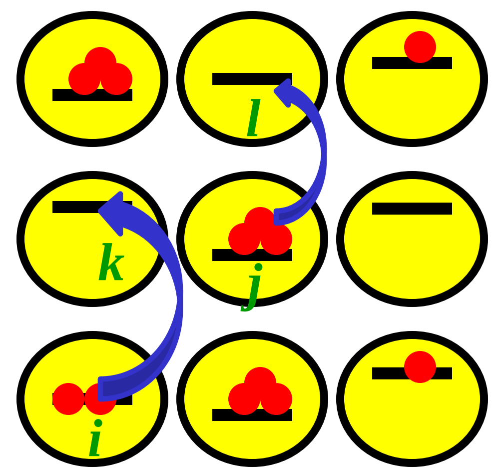
$$\hat{H} = \hat{H}_0 + \hat{V}$$

$$\hat{H}_0 = \sum_{\mu} E_\mu |\mu\rangle\langle\mu|$$

$$\hat{V} = \sum_{\mu,\eta(\mu)} I |\mu\rangle\langle\eta(\mu)|$$

$$|\eta(\mu)\rangle = |.., n_i - 1, .., n_j - 1, .., n_k + 1, .., n_l^\delta + 1, ..\rangle$$

$$i, j, k, l = n.n.$$



Conventional Anderson Model

Basis: $|i\rangle$

i labels sites

$$\hat{H} = \sum_i \varepsilon_i |i\rangle\langle i| + \sum_{i,j=n.n.} I |i\rangle\langle j|$$

Many body Anderson-like Model

Basis: $|\mu\rangle, \ \mu = \{ n_i^\alpha \}$

i labels sites

$n_i = 0, 1, 2, \ldots$ occupation numbers

$$\hat{H} = \sum_\mu E_\mu |\mu\rangle\langle\mu| + \sum_{\mu,\nu(\mu)} I |\mu\rangle\langle\nu(\mu)|$$

"nearest neighbors":

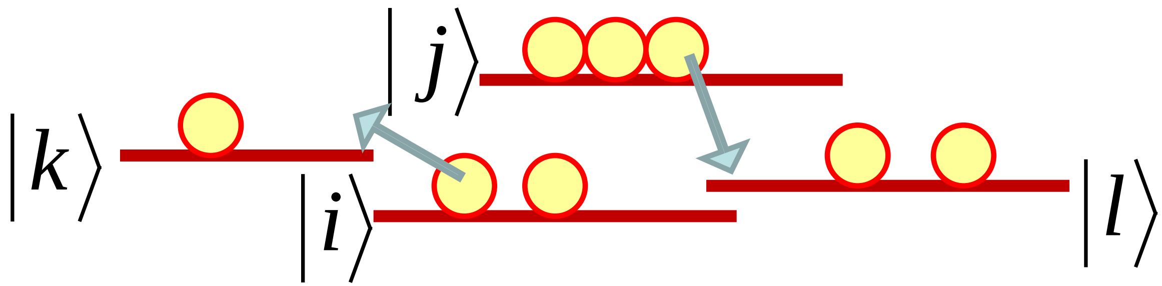
$$|\nu(\mu)\rangle = |.., n_i - 1, .., n_j - 1, .., n_k + 1, .., n_l^\delta + 1, ..\rangle$$

$$i, j, k, l = n.n.$$

Transition temperature: $T_c \equiv t_c\left(ng\right)$

$$\left|i\right\rangle,\left|j\right\rangle \Rightarrow \left|k\right\rangle,\left|l\right\rangle$$

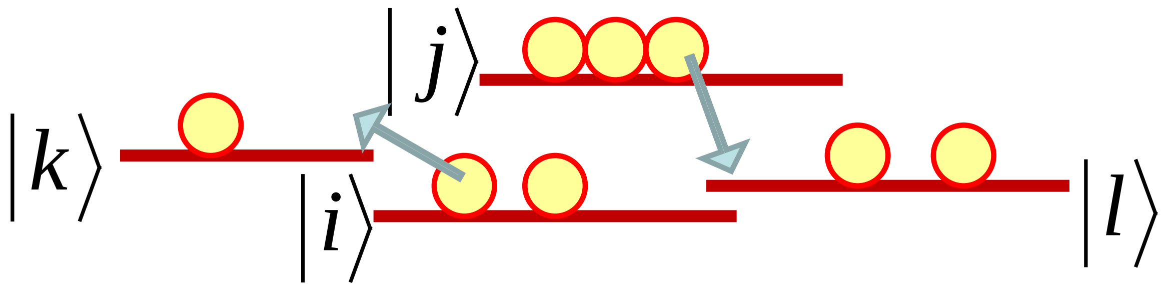
transition



Transition temperature: $T_c \equiv t_c(ng)$

$|i\rangle, |j\rangle \Rightarrow |k\rangle, |l\rangle$

transition



$\Delta_{ij,kl} \equiv \varepsilon_i + \varepsilon_j - \varepsilon_k - \varepsilon_l$ energy mismatch

$I_{ij,kl}$ matrix element

Decay of a state $|i\rangle$

Δ typical mismatch

N_1 typical # of channels

I typical matrix element

Anderson condition:

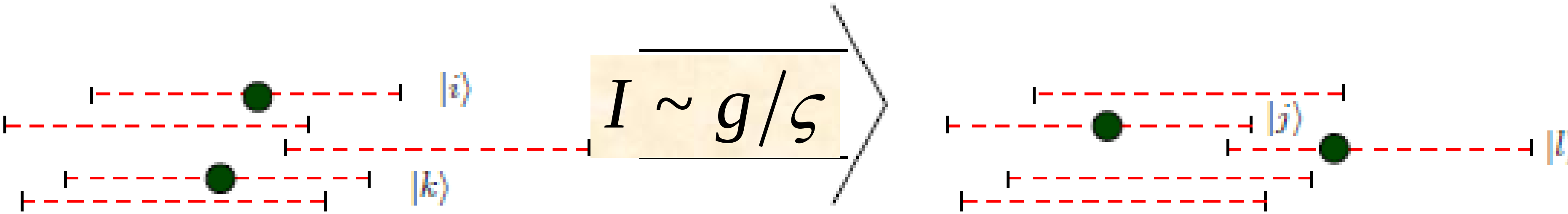
$$I(T) \begin{matrix} \gg \Delta(T)/N_1(T) \\ \ll \Delta(T)/N_1(T) \end{matrix}$$

extended

localized

High temperatures: $T >> T_d$ ⟺ $t >> \gamma^{-1}$

Bose-gas is not degenerated;
occupation numbers either **0** or **1**



Matrix element of the transition

$$I \sim g/\varsigma(\varepsilon = T) \sim (gE_*)/(\varsigma_* T)$$

should be compared with the minimal energy mismatch

$$(\nu\varsigma)^{-1}/(n\varsigma) \sim \left(\nu n \varsigma_*{}^2 T^2\right)^{-1} E_*^2$$

Localization spacing δ_ς

Number of channels

$$\boxed{\kappa_c(t) \propto t^{1/3} \qquad t\gamma >> 1}$$

Intermediate temperatures: $\gamma^{-1/2} << t << \gamma^{-1}$

1. $T << T_d$ $\Longleftrightarrow$ $t\gamma << 1$

2. Bose-gas is degenerated; occupation numbers either $>>1$.

3. Typical energies $|\mu| = T^2/T_d$, μ is the chemical potential. Correct as long as

$$|\mu| >> ng, E_* \Longleftrightarrow t\sqrt{\gamma} >> 1$$

multiple occupation $N(\varepsilon) \sim \frac{T}{\varepsilon}$

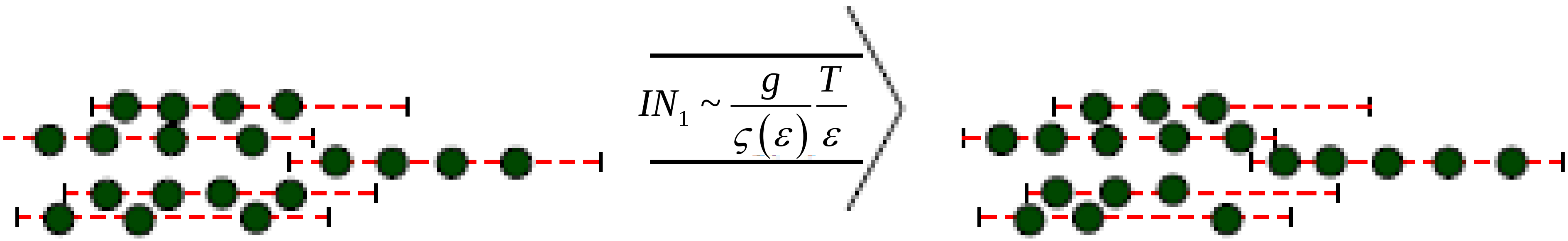
4. Characteristic energies $\varepsilon \sim |\mu|$ $\begin{matrix} << T \\ >> ng, E_* \end{matrix}$

We are still dealing with the high energy states

Intermediate temperatures: $\gamma^{-1/2} << t << \gamma^{-1}$

$$|\mu| = T^2 / T_d >> ng, E_* \qquad T << T_d$$

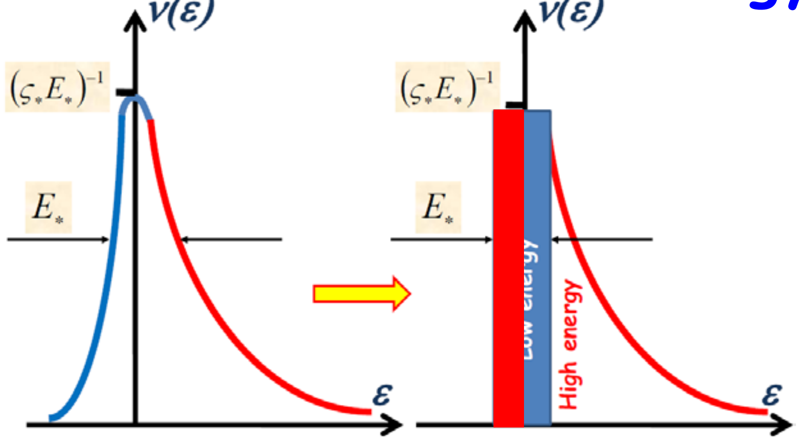
Bose-gas is degenerated; typical energies ~
$|\mu| >> T \Longrightarrow$ **occupation numbers** $>> 1 \Longrightarrow$ **matrix elements are enhanced**



$$\boxed{\kappa_c(t) \propto t^{2/3} \gamma^{1/3} \qquad \sqrt{\gamma} << t\gamma << 1}$$

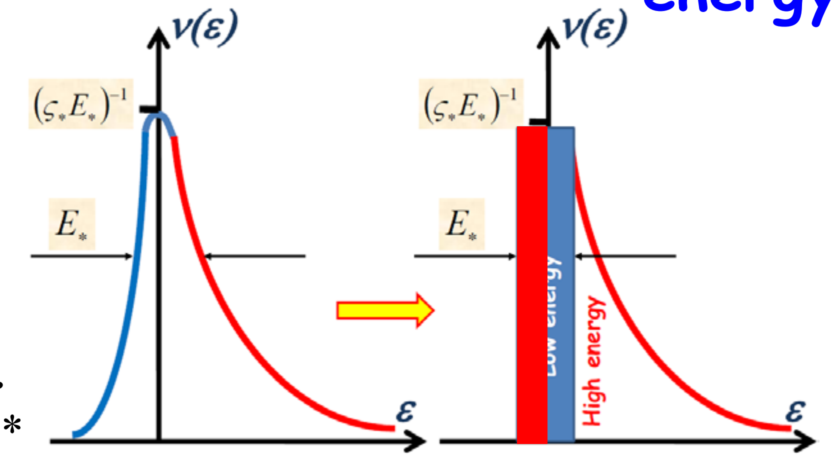
Low temperatures: $t << \gamma^{-1/2}$ Start with T=0

Suppose $\kappa \equiv E_*/ng >> 1$ ⟹ $|\mu| << E_*$ ⟹ Bosons occupy only small fraction of low energy states $\varepsilon_i < \mu$



Low temperatures: $t << \gamma^{-1/2}$ Start with T=0

Suppose $\kappa \equiv E_*/ng >> 1 \Longrightarrow |\mu| << E_* \Longrightarrow$ Bosons occupy only small fraction of low energy states $\varepsilon_i < \mu$

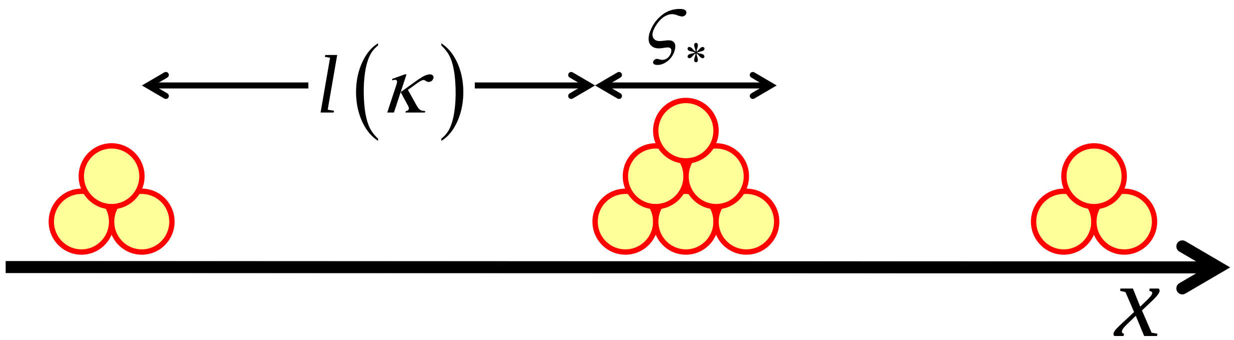


Localization length ς_*

Occupation #: $(\mu - \varepsilon_i)\varsigma_*/g$

DoS: $\nu(\varepsilon) = (E_*\varsigma_*)^{-1}$

$$\Longrightarrow n = \frac{\mu^2}{2gE_*} \Longrightarrow \mu = E_*/\sqrt{\kappa}$$



$l(\kappa) = \varsigma_*\sqrt{\kappa} >> \varsigma_*$

Occupation

$nl(\kappa)/\varsigma_* = \gamma^{-1/2} >> 1$

Low temperatures: $t \ll \gamma^{-1/2}$

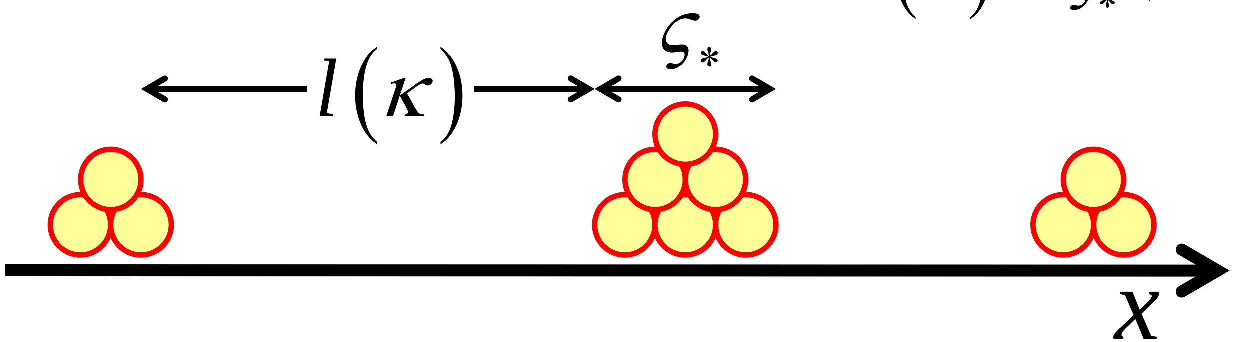
$\kappa \equiv E_* / ng \gg 1$ ⟹ "lakes"

Occupation

$nl(\kappa)/\varsigma_* = \gamma^{-1/2} \gg 1$

Distance

$l(\kappa) = \varsigma_* \sqrt{\kappa} \gg \varsigma_*$



$l(\kappa) \gg \varsigma_*$ ⟹ Strong insulator

$\kappa \to \kappa_c$

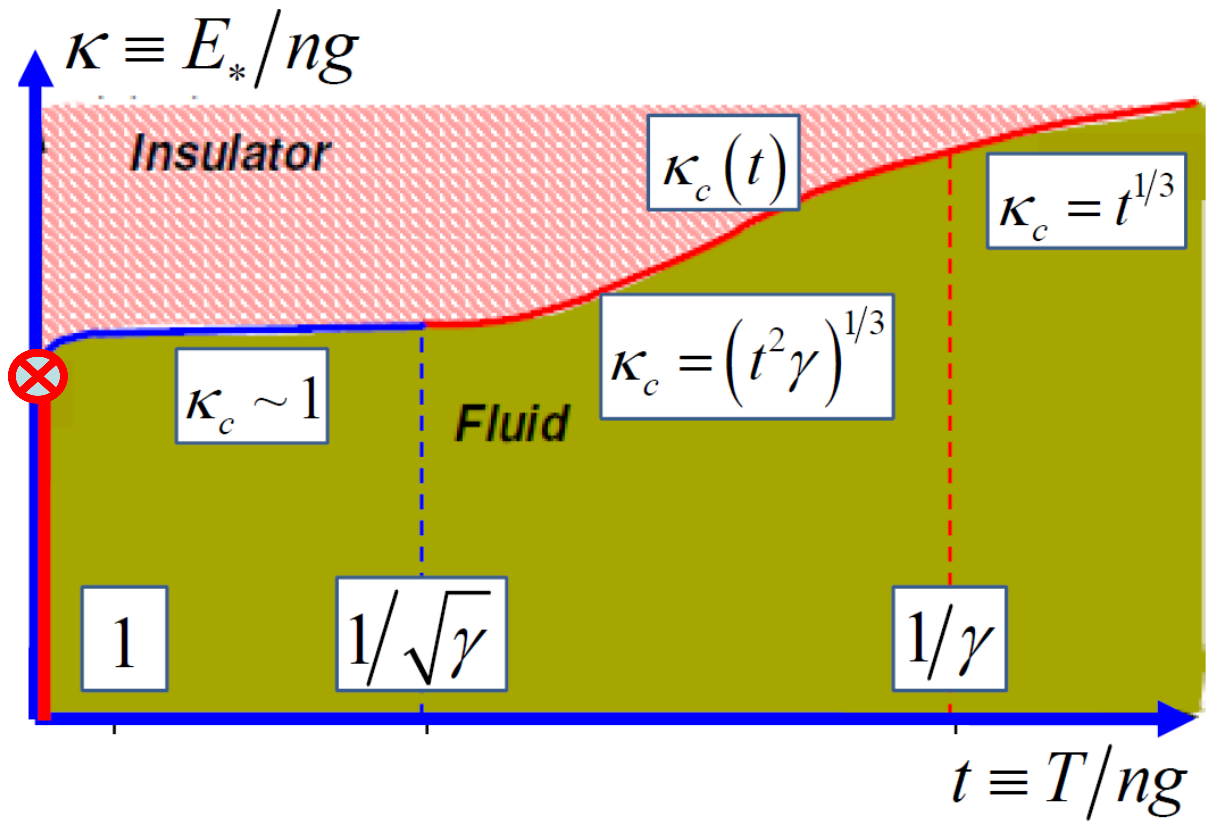
$l(\kappa) \ll \varsigma_*$ ⟹ Insulator – Superfluid transition in a chain of "Josephson junctions"

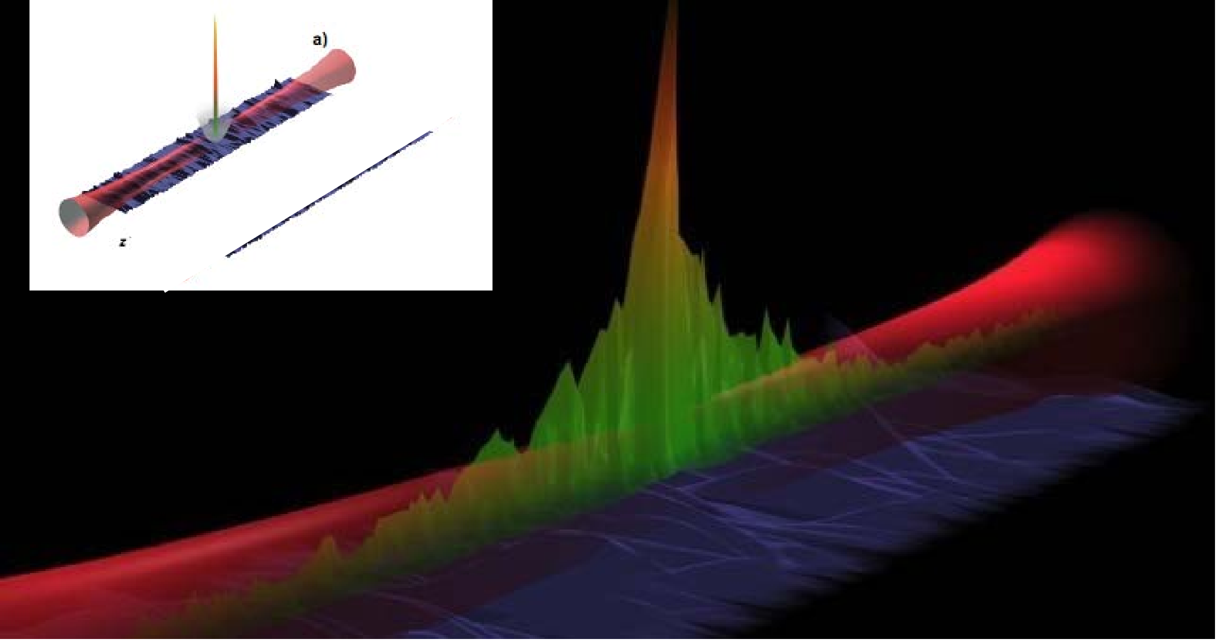
Low temperatures: $t << \gamma^{-1/2}$

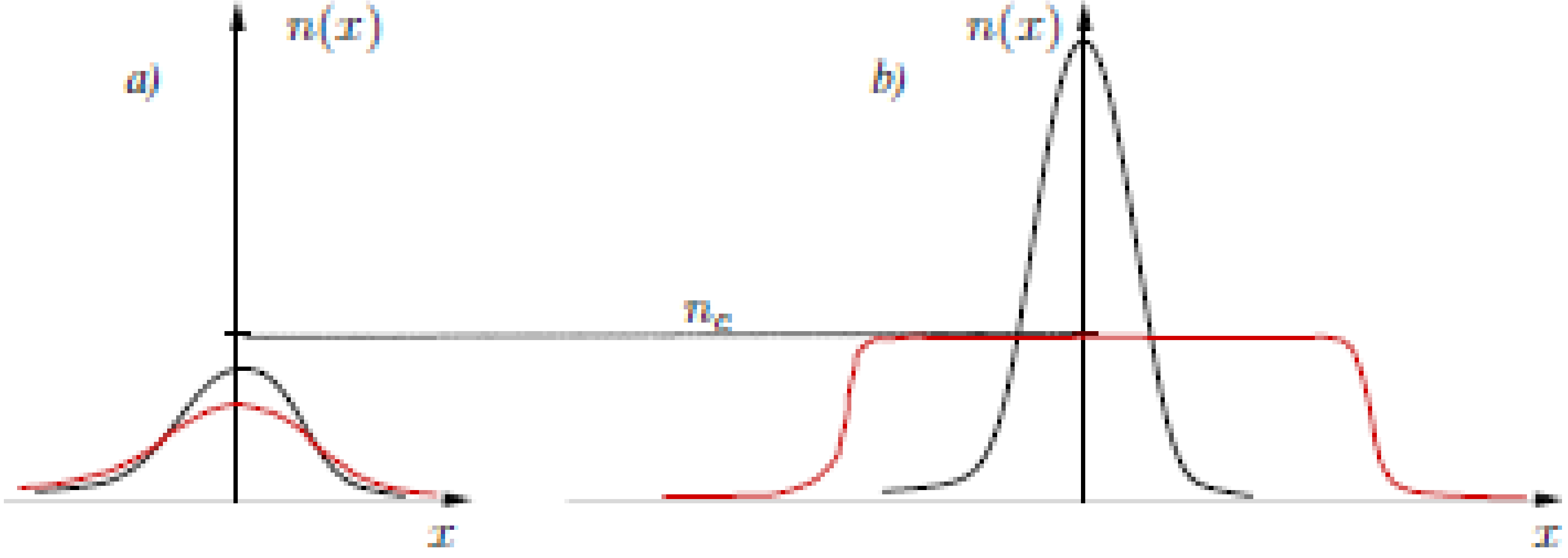
$\kappa \equiv E_* / ng >> 1 \Longrightarrow$ Strong insulator

$T = 0$ transition $\kappa_c \sim 1$

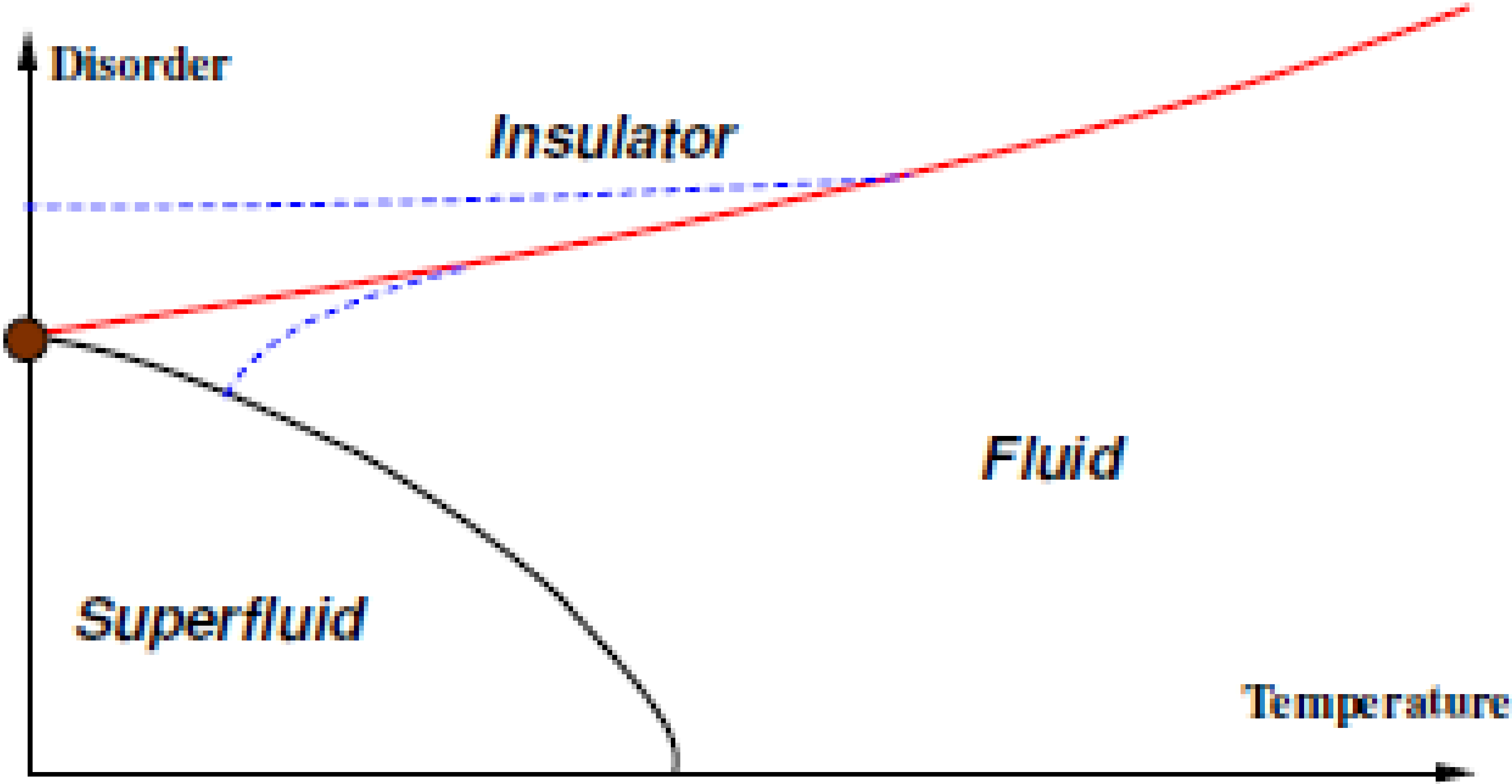
$\Rightarrow \kappa_c \sim 1$ for $t << \gamma^{-1/2}$



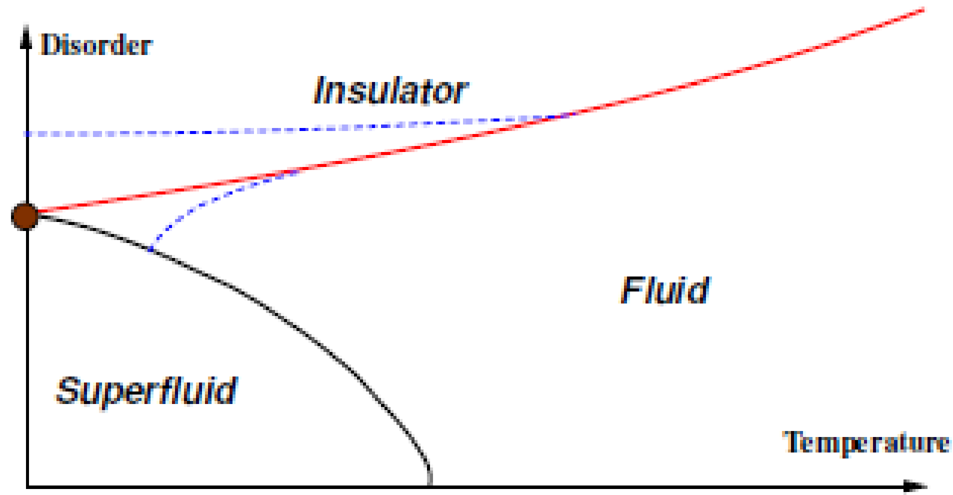

a)
z


$n(x)$
a)
x
$n(x)$
b)
n_c
x

Disordered interacting bosons in two dimensions



Disordered interacting bosons in two dimensions



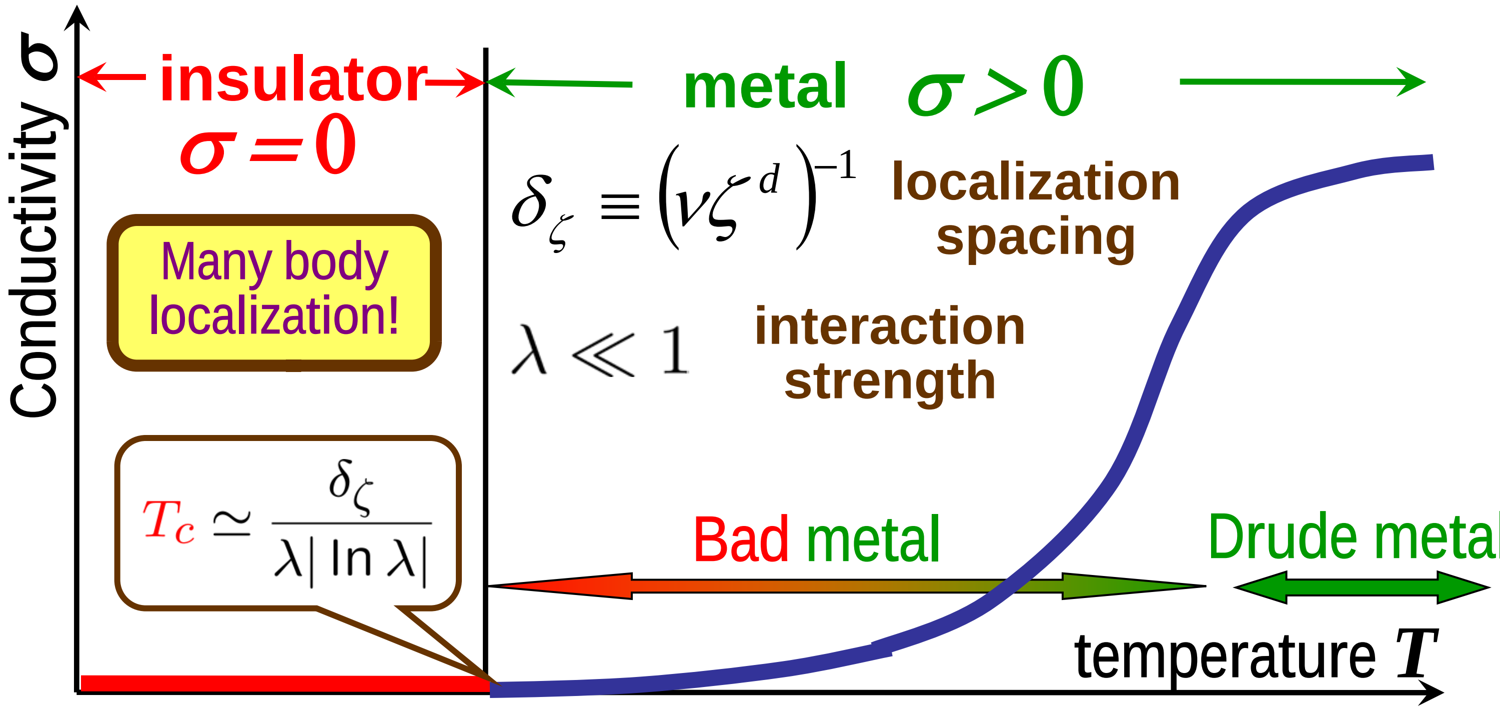
Justification:

1. At $T=0$ normal state is unstable with respect to either insulator or superfluid.

2. At finite temperature in the vicinity of the critical disorder the insulator can be thought of as a collection of "lakes", which are disconnected from each other. The typical size of such a "lake" diverges. This means that the excitations in the insulator state are localized but the localization length can be arbitrary large. Accordingly the many –body delocalization is unavoidable at an arbitrary low but finite T.

Phononless conductance

Many-body Localization of fermions



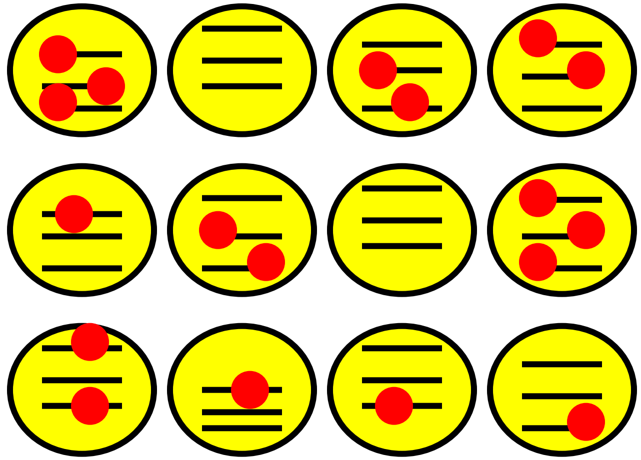
Definitions:

Insulator $\sigma = 0$ **Metal** $\sigma \neq 0$

not $d\sigma/dT < 0$ **not** $d\sigma/dT > 0$

Many body Anderson-like Model

- many particles,
- several levels per site,
- onsite disorder
- local interaction



Basis: $\left|\mu\right\rangle$

$$\mu = \left\{ n_i^\alpha \right\}$$

i labels sites α labels levels

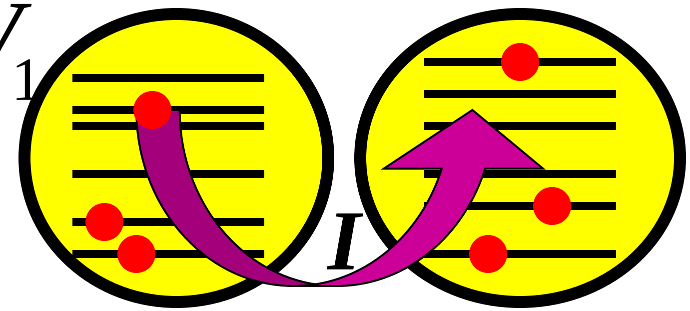
$n_i^\alpha = 0,1$ occupation numbers

Hamiltonian:

$$\hat{H} = \hat{H}_0 + \hat{V}_1 + \hat{V}_2$$

$$\hat{H}_0 = \sum_\mu E_\mu \left|\mu\right\rangle\left\langle\mu\right|$$

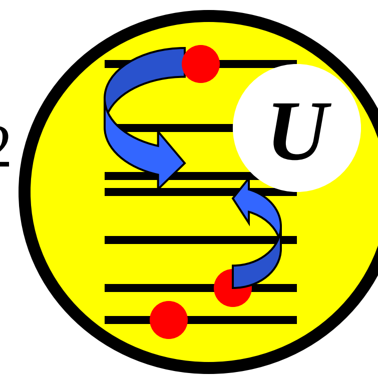
$\hat{V}_1$



$$\hat{V}_1 = \sum_{\mu,\nu(\mu)} I \left|\mu\right\rangle\left\langle\nu(\mu)\right|$$

$$\left|\nu(\mu)\right\rangle = \left|.., n_i^\alpha - 1, .., n_j^\beta + 1, ..\right\rangle, \quad i, j = n.n.$$

$\hat{V}_2$



$$\hat{V}_2 = \sum_{\mu,\eta(\mu)} U \left|\mu\right\rangle\left\langle\eta(\mu)\right|$$

$$\left|\nu(\mu)\right\rangle = \left|.., n_i^\alpha - 1, .., n_i^\beta - 1, .., n_i^\gamma + 1, .., n_i^\delta + 1, ..\right\rangle$$

Conventional Anderson Model

Basis: $|i\rangle$

i labels sites

$$\hat{H} = \sum_i \varepsilon_i |i\rangle\langle i| + \sum_{i,j=n.n.} I |i\rangle\langle j|$$

Many body Anderson-like Model

Basis: $|\mu\rangle, \quad \mu = \{n_i^\alpha\}$

i labels sites

α labels levels

$n_i^\alpha = 0,1$ occupation numbers

$$\hat{H} = \sum_\mu E_\mu |\mu\rangle\langle\mu| + \sum_{\mu,\nu(\mu)} I |\mu\rangle\langle\nu(\mu)| + \sum_{\mu,\eta(\mu)} U |\mu\rangle\langle\eta(\mu)|$$

Two types of "nearest neighbors":

$$|\nu(\mu)\rangle = |..,n_i^\alpha - 1,..,n_j^\beta + 1,..\rangle, \quad i, j = n.n.$$

$$|\eta(\mu)\rangle = |..,n_i^\alpha - 1,..,n_i^\beta - 1,..,n_i^\gamma + 1,..,n_i^\delta + 1,..\rangle$$

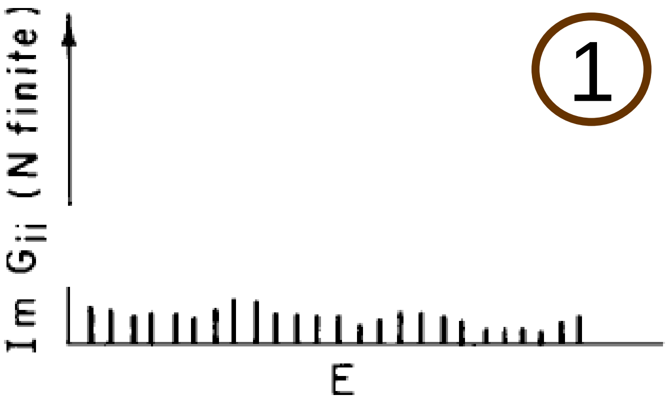
Anderson's recipe:

1. **take descrete spectrum E_μ of H_0**

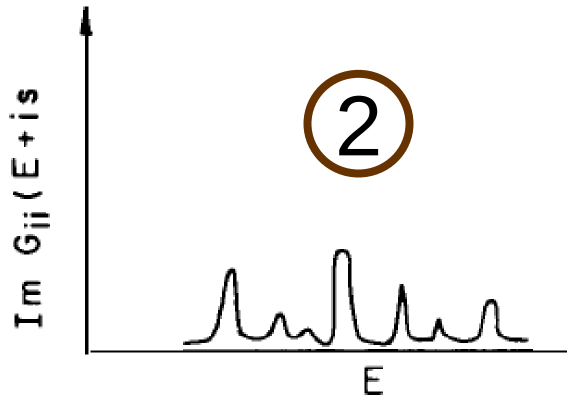
2. **Add an infinitesimal *Im* part $i\eta$ to E_μ**

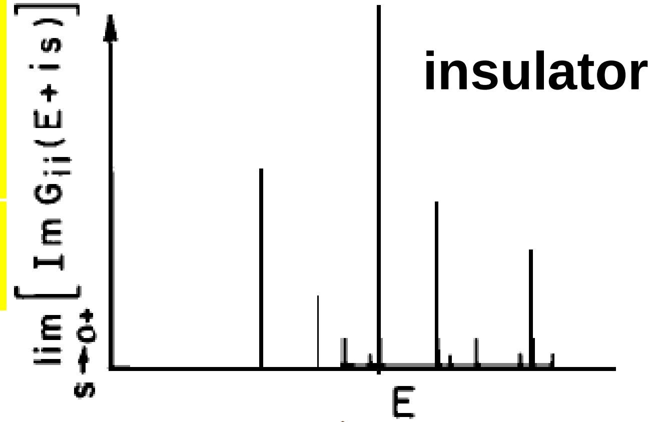
3. **Evaluate $Im\Sigma_\mu$**

1



2

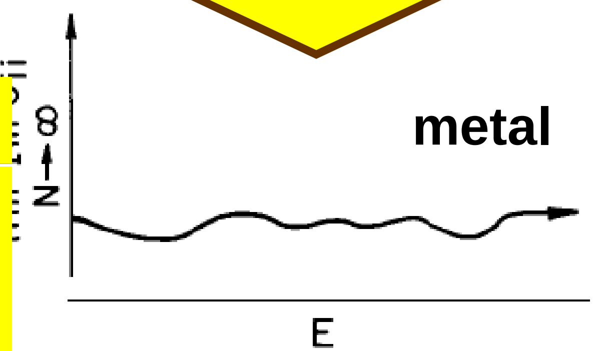




4 ***limits***

1) $N \to \infty$

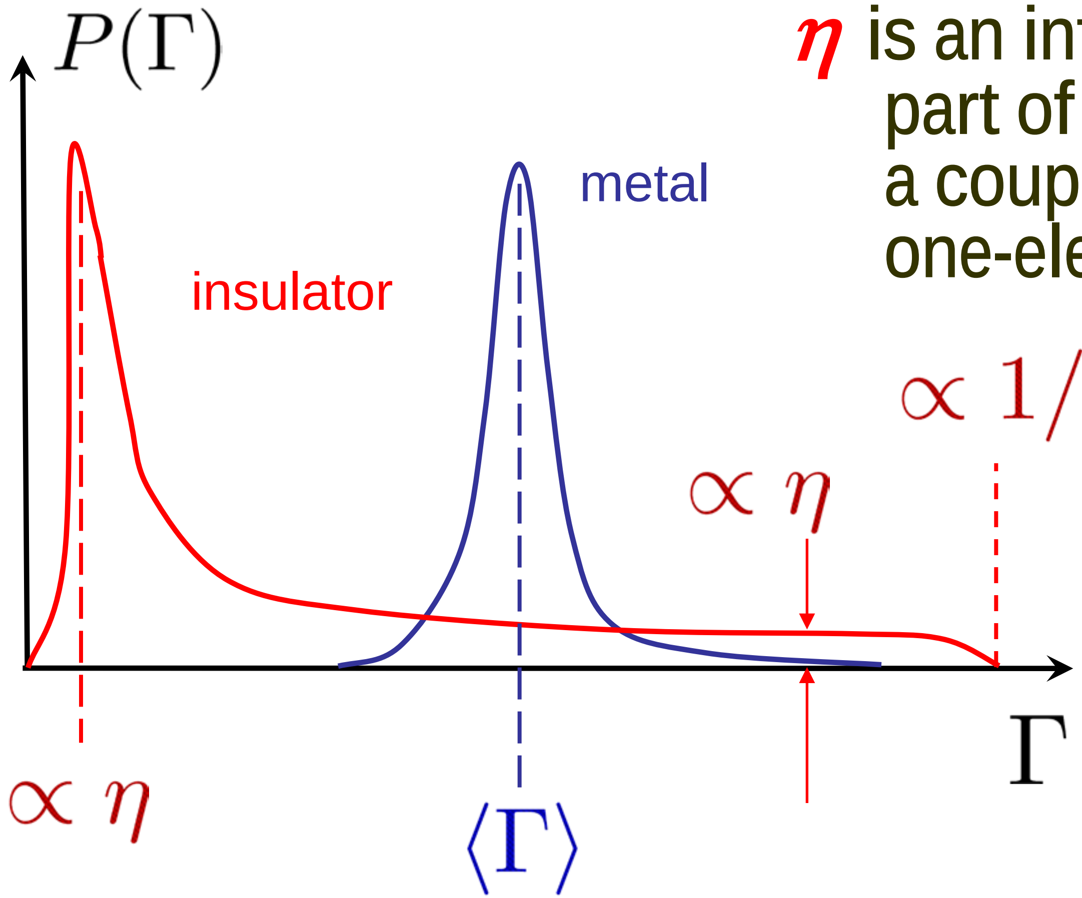
2) $\eta \to 0$



4. **take limit $\eta \to 0$ but only after $N \to \infty$**

5. **"What we really need to know is the *probability distribution* of $Im\Sigma$, *not* its average…"** **!**

Probability Distribution of Γ=*Im* Σ



η is an infinitesimal width (***Im*** part of the self-energy due to a coupling with a bath) of one-electron eigenstates

Look for:

$$\lim_{\eta\to+0}\lim_{V\to\infty} P(\Gamma>0) = \begin{cases} >0; & \textit{metal} \\ 0; & \textit{insulator} \end{cases}$$

Stability of the insulating phase: NO spontaneous generation of broadening

$$\Gamma_\alpha(\varepsilon) = 0$$

is always a solution

$$\varepsilon \to \varepsilon + i\eta$$

linear stability analysis

$$\frac{\Gamma}{(\varepsilon-\xi_\alpha)^2+\Gamma^2} \to \pi\delta(\varepsilon-\xi_\alpha) + \frac{\Gamma}{(\varepsilon-\xi_\alpha)^2}$$

After ***n*** iterations of the equations of the Self Consistent Born Approximation

$$P_n(\Gamma) \propto \frac{\eta}{\Gamma^{3/2}}\left(const\,\frac{\lambda T}{\delta_\zeta}\ln\frac{1}{\lambda}\right)^n$$

first $n \to \infty$

then $\eta \to 0$

$(\ldots) < 1$ – insulator is stable !

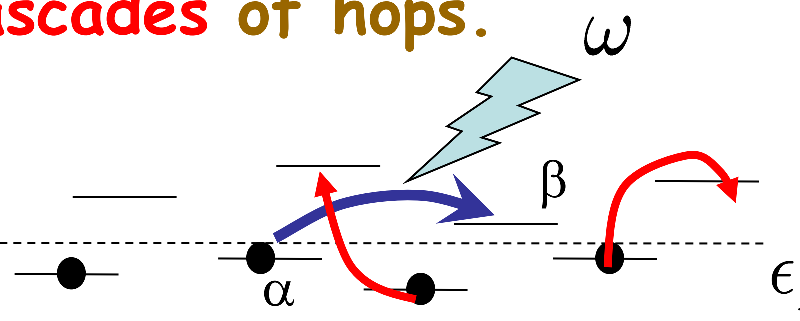
Physics of the transition: cascades

Conventional wisdom:

For phonon assisted hopping one phonon – one electron hop

It is maybe correct at low temperatures, but the higher the temperature the easier it becomes to create e-h pairs.

Therefore with increasing the temperature the typical number of pairs created n_c (i.e. the number of hops) increases. Thus phonons create cascades of hops.



Typical size of the cascade $\approx$ Localization length

Physics of the transition: cascades

Conventional wisdom:

For phonon assisted hopping one phonon – one electron hop

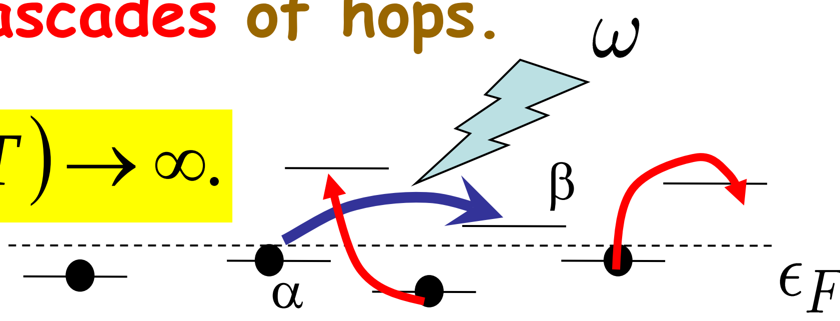
It is maybe correct at low temperatures, but the higher the temperature the easier it becomes to create e-h pairs.

Therefore with increasing the temperature the typical number of pairs created $\boldsymbol{n_c}$ (i.e. the number of hops) increases. Thus phonons create cascades of hops.

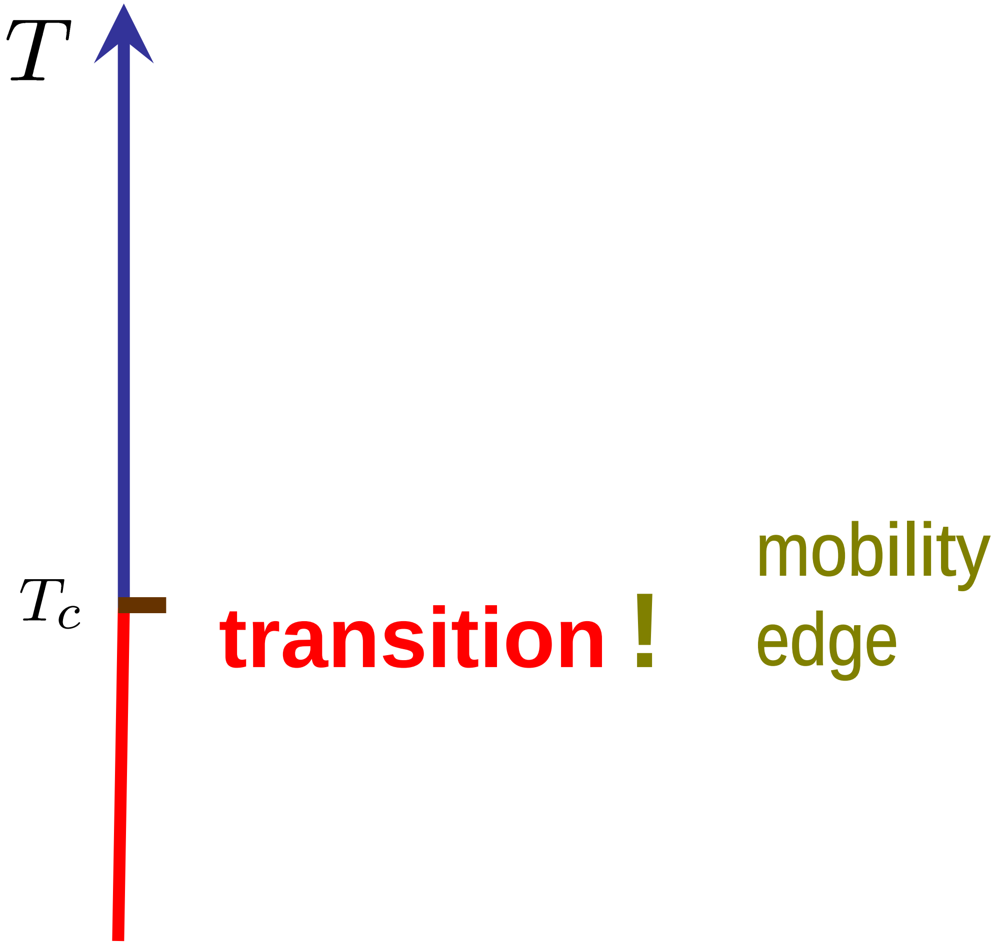
At some temperature $\boxed{T = T_c \quad n_c(T) \to \infty.}$

This is the critical temperature.

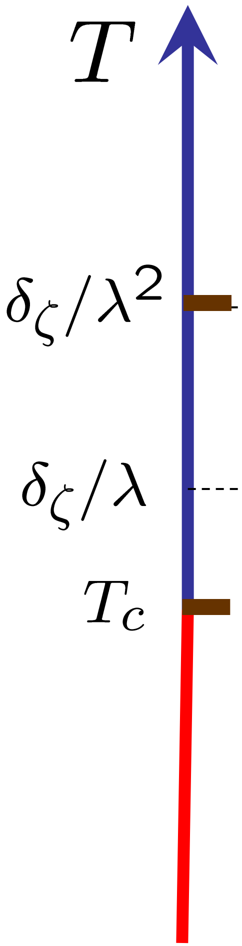
Above T_c one phonon creates infinitely many pairs, i.e., phonons are not needed for charge transport.



Many-body mobility edge



Many-body mobility edge



Large E (high T): extended states

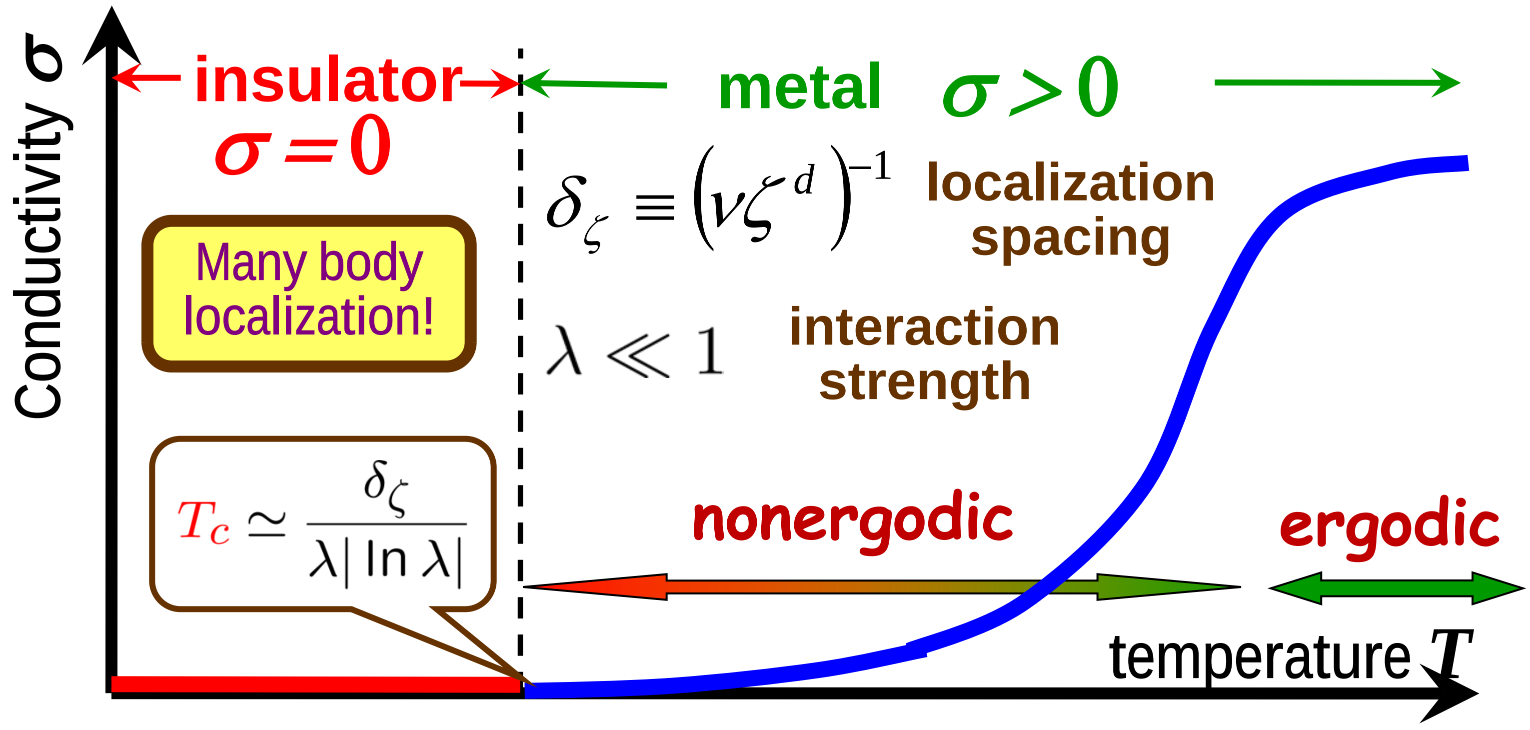
(good metal)

Fermi Golden Rule
hopping (bad metal)

mobility
edge

transition !

Finite T normal metal – insulator transition is another example of the many-body localization



Definition: We will call a quantum state $|\mu\rangle$ **ergodic** if it occupies the number N_μ of sites N_μ on the Anderson lattice, which is proportional to the total number of sites N:

$$\frac{N_\mu}{N} \xrightarrow[N\to\infty]{} 0$$

nonergodic

$$\frac{N_\mu}{N} \xrightarrow[N\to\infty]{} const > 0$$

ergodic

Localized states are obviously not ergodic: $N_\mu \xrightarrow[N\to\infty]{} const$

Q: Is each of the extended state ergodic **?**

A: In **3D** probably **YES**, for ***d>4*** - probably **NO**

Nonergodic states

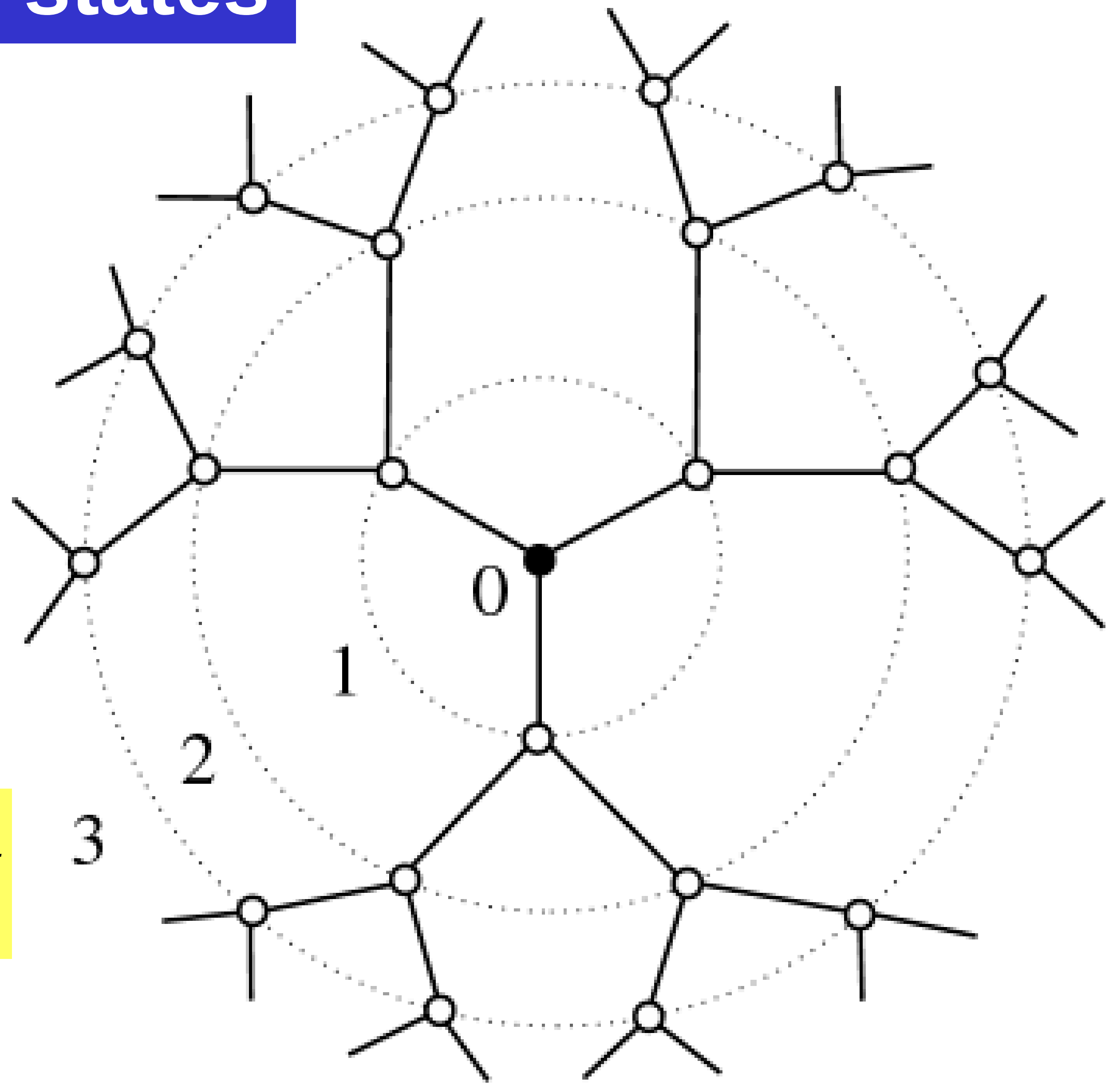
Cayley tree (Bethe lattice)

$$I_c = \frac{W}{K \ln K}$$

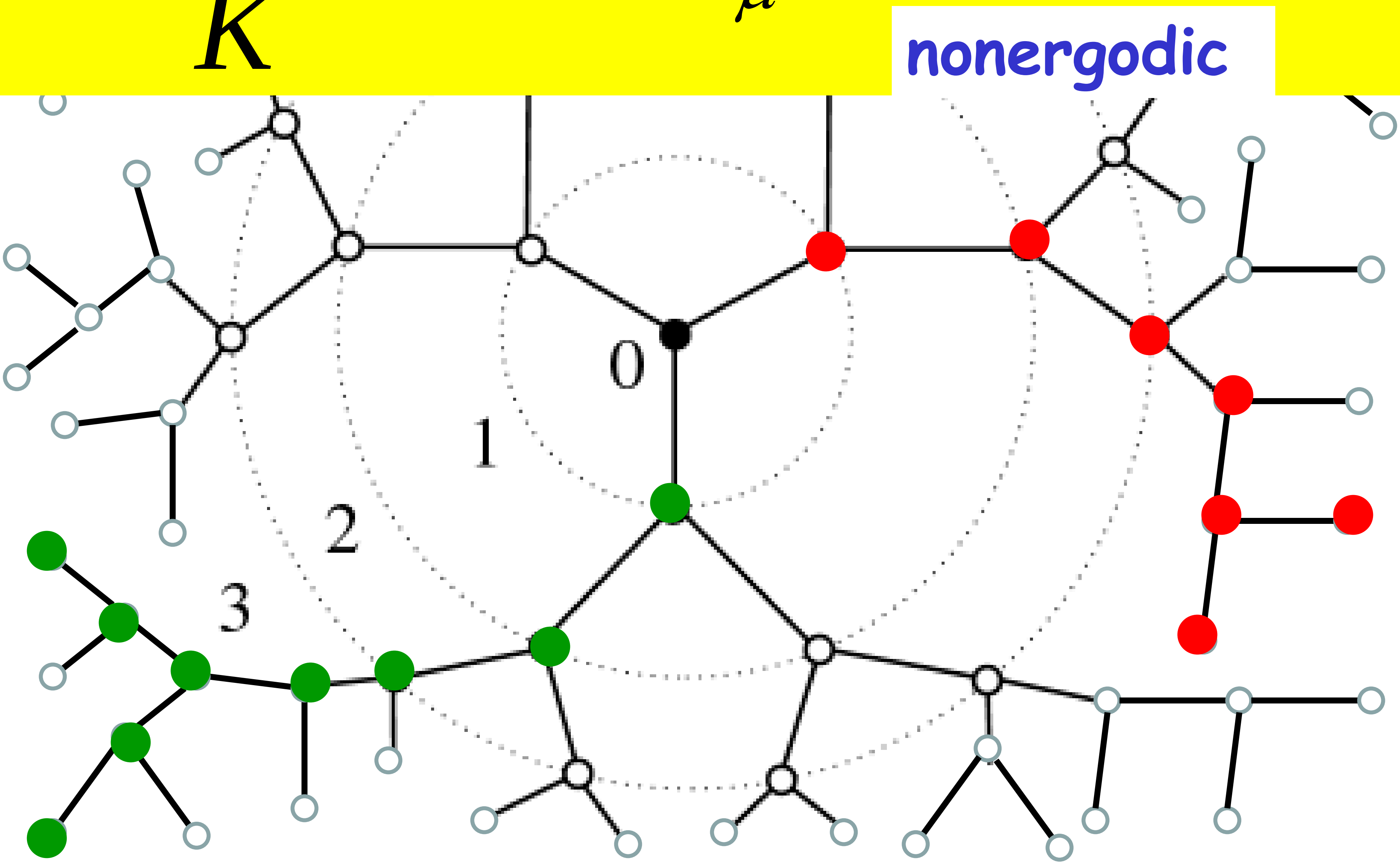
K is the branching number

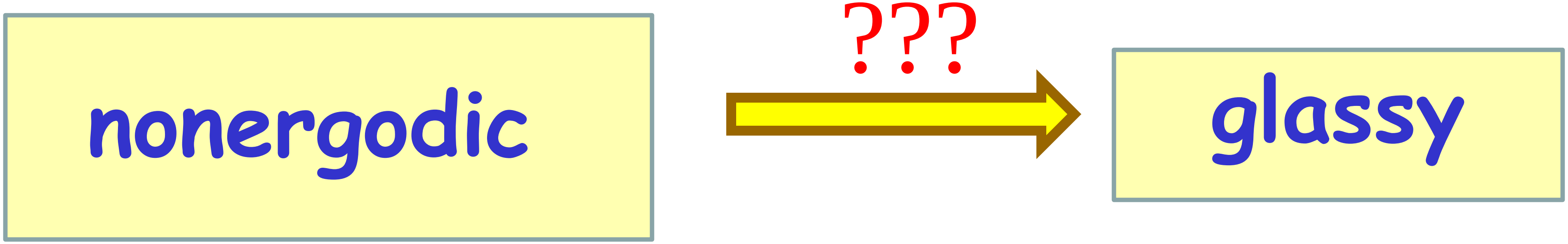
$$I_c < I < W$$

Extended but not ergodic



$$I \approx \frac{W}{K} \quad \Rightarrow \quad N_\mu \approx \ln N << N$$




nonergodic
???
glassy

Main postulate of the Gibbs StatMech- equipartition (microcanonical distribution):

In the equilibrium all states with the same energy are realized with the same probability.

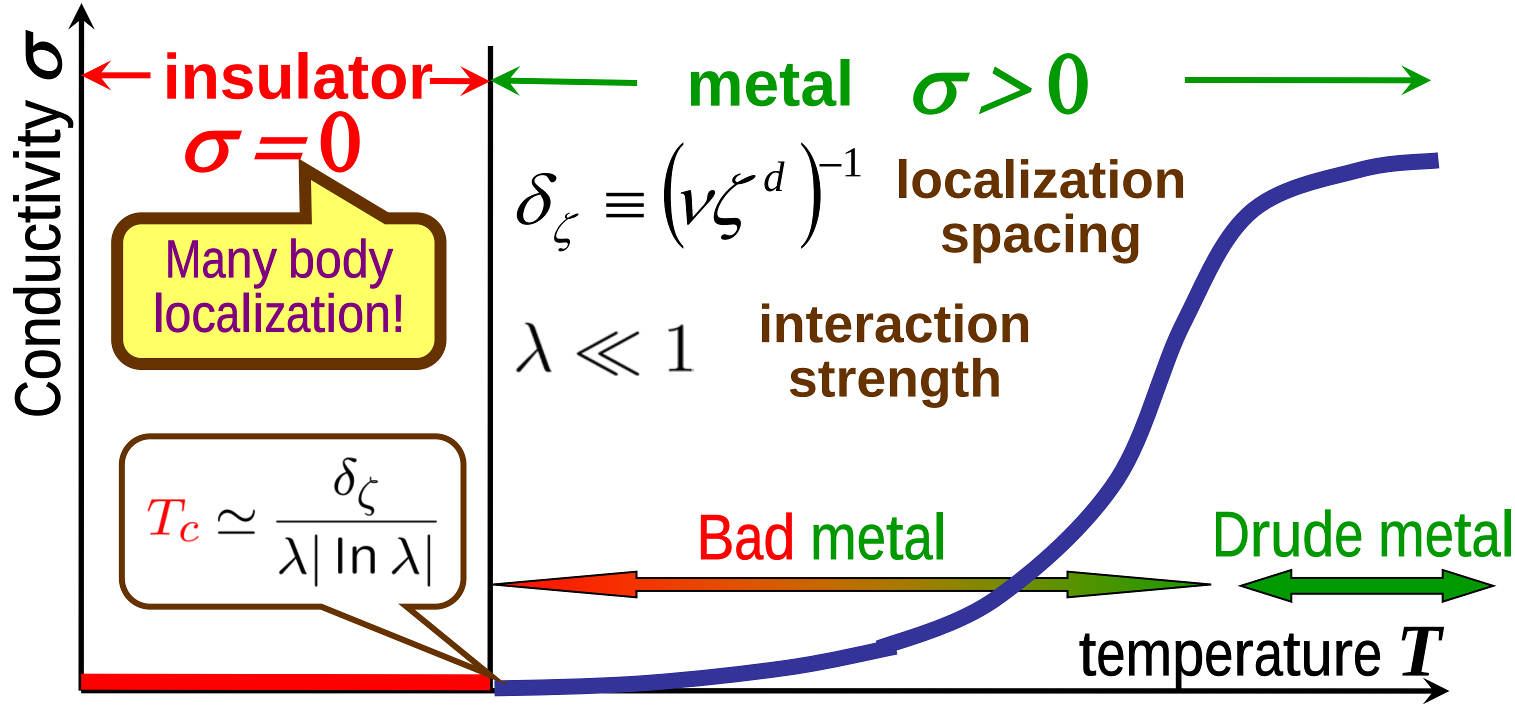
Without interaction between particles the equilibrium would never be reached – each one-particle energy is conserved.

Common believe: Even weak interaction should drive the system to the equilibrium.

Is it always true?

Lecture 3.

4. Speculations



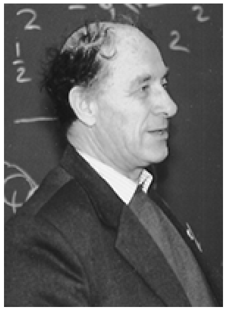
Q: What happens in the classical limit $\hbar \to 0$?

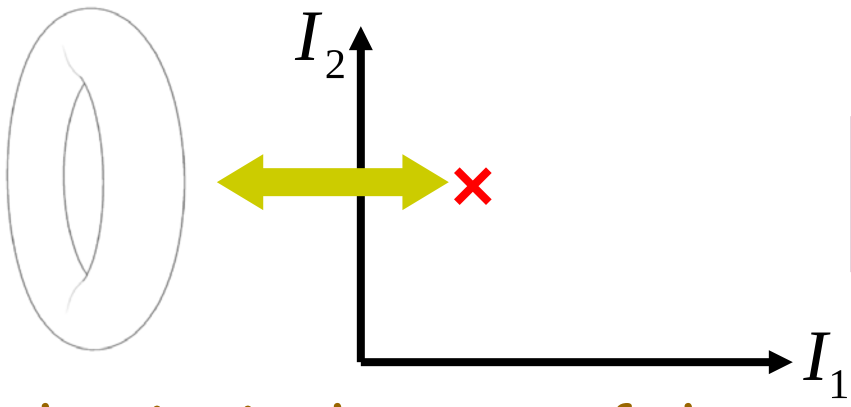
Speculations: 1. No transition $T_c \to 0$

2. Bad metal still exists

Reason: Arnold diffusion

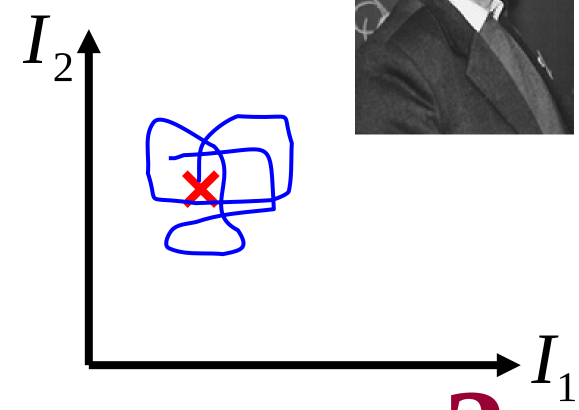
Arnold diffusion





Each point in the space of the **integrals of motion** corresponds to a torus and vice versa

$$\hat{V} \neq 0$$



Finite motion?

$$d = 2$$

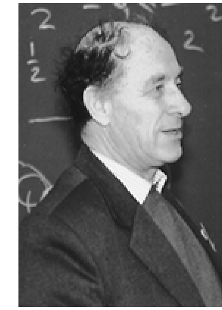
All classical trajectories correspond to a finite motion

$$d > 2$$

Most of the trajectories correspond to a finite motion

However small fraction of the trajectories goes infinitely far

Arnold diffusion



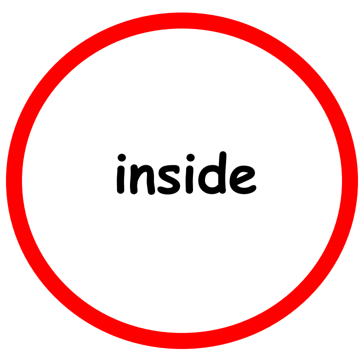
1. Most of the tori survive – KAM

2. Classical trajectories do not cross each other

space	# of dimensions
real space	d
phase space	$2d$
energy shell	$2d-1$
tori	d

$$d = 2 \quad \Rightarrow \quad d_{en.shell} - d_{tori} = 1$$

Each torus has “inside” and “outside”



$$d = 2 \quad \Rightarrow \quad d_{en.shell} - d_{tori} = 1$$

A torus does not have “inside” and “outside” as a ring in >2 dimensions

Speculations:

1. Arnold diffusion $\Longleftrightarrow$ Nonergodic (bad) metal

2. Appearance of the transition (finite T_c) – quantum localization of the Arnold diffusion

Conclusions

Anderson Localization provides a relevant language for description of a wide class of physical phenomena – far beyond conventional Metal to Insulator transitions.

Transition between integrability and chaos in quantum systems

Interacting quantum particles + strong disorder. Three types of behavior:

ordinary ergodic metal
"bad" nonergodic metal
"true" insulator

A closed system without a bath can relaxation to a microcanonical distribution only if it is an ergodic metal