Part II: Functional RG for Fermi systems

A natural way of dealing with many energy scales in interacting electron systems and a powerful source of new approximations.

- applicable to microscopic models (not only field theory)
- no adjustable parameters
- RG treatment of infrared singularities built in

1. Generating functionals
2. Exact flow equations
3. Truncations

1. Generating functionals

Interacting Fermi system with bare action

$$\mathcal{S}[\psi, \bar{\psi}] = -(\bar{\psi}, G_0^{-1}\psi) + V[\psi, \bar{\psi}]$$

$\psi_K, \bar{\psi}_K$ Grassmann variables, K = quantum numbers + Matsubara frequency

G_0 bare propagator, $V[\psi, \bar{\psi}]$ interaction

$(\bar{\psi}, G_0^{-1}\psi) = \sum_K \bar{\psi}_K \, (G_0^{-1}\psi)_K$ with $(G_0^{-1}\psi)_K = \sum_{K'} (G_0^{-1})_{KK'} \, \psi_{K'}$

Spin-$\frac{1}{2}$ fermions with momentum $\mathbf{k}$ and spin orientation σ: $K = (k_0, \mathbf{k}, \sigma)$

Bare propagator in case of translation and spin-rotation invariance:

$$G_0(K) = \frac{1}{ik_0 - \xi_{\mathbf{k}}}$$ (diagonal), where $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu$

Two-particle interaction:

$$V[\psi, \bar{\psi}] = \frac{1}{4} \sum_{K_1, K_2} \sum_{K_1', K_2'} V(K_1', K_2'; K_1, K_2) \, \bar{\psi}_{K_1'} \psi_{K_1} \bar{\psi}_{K_2'} \psi_{K_2}$$

Generating functional for connected Green functions

$$\mathcal{G}[\eta,\bar{\eta}] = -\log\left\{\int\prod_K d\psi_K d\bar{\psi}_K\, e^{-S[\psi,\bar{\psi}]}\, e^{(\bar{\eta},\psi)+(\bar{\psi},\eta)}\right\}$$

Connected m-particle Green function

$$\begin{aligned} G^{(m)}(K_1,\ldots,K_m;K'_1,\ldots,K'_m) &= \\ -\underbrace{\langle\psi_{K_1}\ldots\psi_{K_m}\bar{\psi}_{K'_m}\ldots\bar{\psi}_{K'_1}\rangle_c}_{\text{connected average}} &= \frac{\partial^m}{\partial\eta_{K'_1}\ldots\partial\eta_{K'_m}}\frac{\partial^m}{\partial\bar{\eta}_{K_m}\ldots\partial\bar{\eta}_{K_1}}\mathcal{G}[\eta,\bar{\eta}]\bigg|_{\eta=\bar{\eta}=0} \end{aligned}$$

Legendre transform of $\mathcal{G}[\eta,\bar\eta]$: effective action

$$\Gamma[\psi,\bar\psi] = \mathcal{G}[\eta,\bar\eta] + (\bar\psi,\eta) + (\bar\eta,\psi) \quad \text{with} \quad \psi = -\frac{\partial\mathcal{G}}{\partial\bar\eta} \quad \text{and} \quad \bar\psi = \frac{\partial\mathcal{G}}{\partial\eta}$$

generates one-particle irreducible (1PI) vertex functions $\Gamma^{(m)}$

$$\Gamma^{(1)} = G^{-1} = G_0^{-1} - \Sigma$$

Reciprocity relations at finite source fields:

$$\frac{\partial\Gamma}{\partial\psi} = -\bar\eta \quad , \qquad \frac{\partial\Gamma}{\partial\bar\psi} = \eta$$

$$\mathbf{\Gamma}^{(1)}[\psi,\bar\psi] = \left(\mathbf{G}^{(1)}[\eta,\bar\eta]\right)^{-1}$$

where $\mathbf{\Gamma}^{(1)}$ and $\mathbf{G}^{(1)}$ are matrices of second derivatives ...

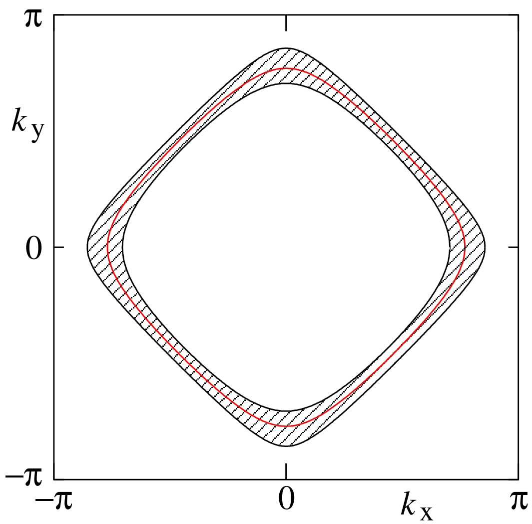
$$\mathbf{G}^{(1)}[\eta, \bar{\eta}] = \begin{pmatrix} -\dfrac{\partial^2 \mathcal{G}}{\partial \bar{\eta}_K \partial \eta_{K'}} & \dfrac{\partial^2 \mathcal{G}}{\partial \bar{\eta}_K \partial \bar{\eta}_{K'}} \\ \dfrac{\partial^2 \mathcal{G}}{\partial \eta_K \partial \eta_{K'}} & -\dfrac{\partial^2 \mathcal{G}}{\partial \eta_K \partial \bar{\eta}_{K'}} \end{pmatrix} = -\begin{pmatrix} \langle \psi_K \bar{\psi}_{K'} \rangle & \langle \psi_K \psi_{K'} \rangle \\ \langle \bar{\psi}_K \bar{\psi}_{K'} \rangle & \langle \bar{\psi}_K \psi_{K'} \rangle \end{pmatrix}$$

$$\mathbf{\Gamma}^{(1)}[\psi, \bar{\psi}] = \begin{pmatrix} \dfrac{\partial^2 \Gamma}{\partial \bar{\psi}_K \partial \psi_{K'}} & \dfrac{\partial^2 \Gamma}{\partial \bar{\psi}_K \partial \bar{\psi}_{K'}} \\ \dfrac{\partial^2 \Gamma}{\partial \psi_K \partial \psi_{K'}} & \dfrac{\partial^2 \Gamma}{\partial \psi_K \partial \bar{\psi}_{K'}} \end{pmatrix}$$

2. Exact flow equations

Impose infrared cutoff at energy scale $\Lambda > 0$, e.g. a momentum cutoff

$$G_0^\Lambda(k_0, \mathbf{k}) = \frac{\Theta^\Lambda(\mathbf{k})}{ik_0 - \xi_\mathbf{k}} \quad \text{with} \quad \Theta^\Lambda(\mathbf{k}) = \Theta(|\xi_\mathbf{k}| - \Lambda)$$



Momentum space region around the Fermi surface excluded by a sharp momentum cutoff in a 2D lattice model

Cutoff regularizes divergence of $G_0(k_0, \mathbf{k})$ in $k_0 = 0$, $\xi_\mathbf{k} = 0$ (Fermi surface)

Other choices: smooth cutoff, frequency cutoff,
mixed momentum-frequency cutoff $\Theta^\Lambda(\sqrt{\xi_\mathbf{k}^2 + k_0^2})$

Cutoff excludes "soft modes" below scale Λ from functional integral.

Λ-dependent functionals $\mathcal{G}^{\Lambda}[\eta, \bar{\eta}]$ and $\Gamma^{\Lambda}[\psi, \bar{\psi}]$.

Functionals $\mathcal{G}$ and Γ recovered for $\Lambda \to 0$.

Exact flow equation for Γ^Λ:

$$\frac{d}{d\Lambda}\Gamma^\Lambda[\psi,\bar\psi] = -(\bar\psi, \dot Q_0^\Lambda \psi) - \frac{1}{2}\mathrm{tr}\left[\dot{\mathbf{Q}}_0^\Lambda \left(\mathbf{\Gamma}^{(1)\Lambda}[\psi,\bar\psi]\right)^{-1}\right]$$

$$Q_0^\Lambda = (G_0^\Lambda)^{-1} \qquad \dot Q_0^\Lambda = \partial_\Lambda Q_0^\Lambda$$

$$\mathbf{Q}_0^\Lambda = \begin{pmatrix} Q_{0,KK'}^\Lambda & 0 \\ 0 & -Q_{0,K'K}^\Lambda \end{pmatrix} \qquad \mathbf{\Gamma}^{(1)\Lambda}[\psi,\bar\psi] = \begin{pmatrix} \dfrac{\partial^2\Gamma^\Lambda}{\partial\bar\psi_K \partial\psi_{K'}} & \dfrac{\partial^2\Gamma^\Lambda}{\partial\bar\psi_K \partial\bar\psi_{K'}} \\ \dfrac{\partial^2\Gamma^\Lambda}{\partial\psi_K \partial\psi_{K'}} & \dfrac{\partial^2\Gamma^\Lambda}{\partial\psi_K \partial\bar\psi_{K'}} \end{pmatrix}$$

Wetterich '93, Morris '94, Salmhofer + Honerkamp '01

(derivation later)

Expansion in fields:

$$\mathbf{\Gamma}^{(1)\Lambda}[\psi,\bar{\psi}] = (\mathbf{G}^\Lambda)^{-1} - \tilde{\mathbf{\Sigma}}^\Lambda[\psi,\bar{\psi}]$$

where $\mathbf{G}^\Lambda = \left(\mathbf{\Gamma}^{(1)\Lambda}[\psi,\bar{\psi}]\big|_{\psi=\bar{\psi}=0} \right)^{-1} = \begin{pmatrix} G^\Lambda_{KK'} & 0 \\ 0 & -G^\Lambda_{K'K} \end{pmatrix}$

$\tilde{\mathbf{\Sigma}}^\Lambda[\psi,\bar{\psi}]$ contains all contributions to $\mathbf{\Gamma}^{(1)\Lambda}[\psi,\bar{\psi}]$ which are at least quadratic in the fields.

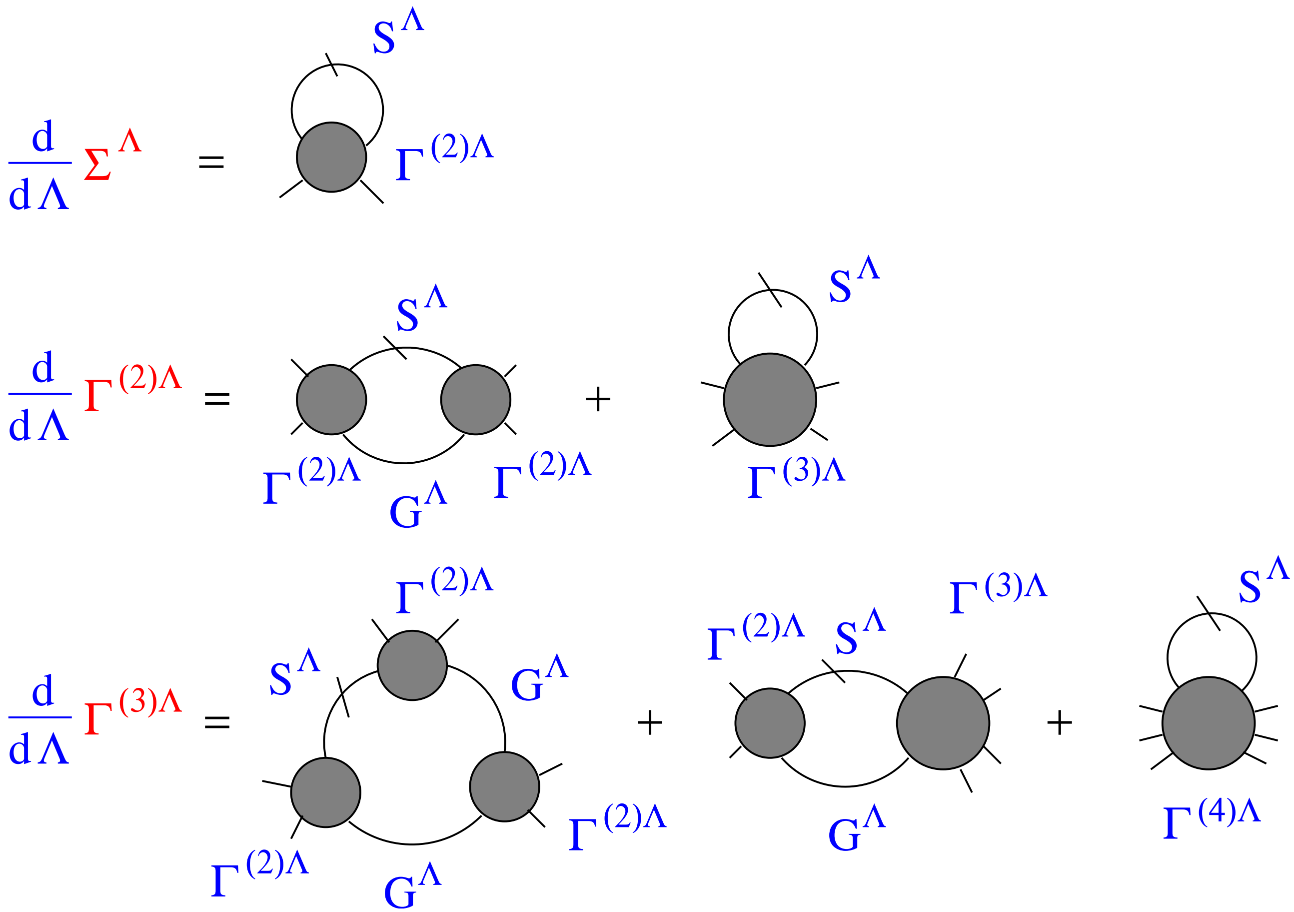
$$\left(\mathbf{\Gamma}^{(1)\Lambda}[\psi,\bar{\psi}] \right)^{-1} = \left(1 - \mathbf{G}^\Lambda \tilde{\mathbf{\Sigma}}^\Lambda\right)^{-1} \mathbf{G}^\Lambda = \left[1 + \mathbf{G}^\Lambda \tilde{\mathbf{\Sigma}}^\Lambda + (\mathbf{G}^\Lambda \tilde{\mathbf{\Sigma}}^\Lambda)^2 + \dots \right] \mathbf{G}^\Lambda \;\Rightarrow$$

$$\frac{d}{d\Lambda}\Gamma^\Lambda = -\mathrm{tr}\left[\dot{Q}^\Lambda_0 G^\Lambda\right] - \left(\bar{\psi}, \dot{Q}^\Lambda_0 \psi\right) + \frac{1}{2}\mathrm{tr}\left[\mathbf{S}^\Lambda(\tilde{\mathbf{\Sigma}}^\Lambda + \tilde{\mathbf{\Sigma}}^\Lambda \mathbf{G}^\Lambda \tilde{\mathbf{\Sigma}}^\Lambda + \dots)\right]$$

where $\mathbf{S}^\Lambda = -\mathbf{G}^\Lambda \dot{\mathbf{Q}}^\Lambda_0 \mathbf{G}^\Lambda = \frac{d}{d\Lambda}\,\mathbf{G}^\Lambda\big|_{\mathbf{\Sigma}^\Lambda \text{ fixed}}$ *"single scale propagator"*

Expand $\Gamma^\Lambda[\psi,\bar{\psi}]$ in powers of ψ and $\bar{\psi}$, compare coefficients $\Rightarrow$

Flow equations for self-energy $\Sigma^\Lambda = Q_0^\Lambda - \Gamma^{(1)\Lambda}$, two-particle vertex $\Gamma^{(2)\Lambda}$, and many-particle vertices $\Gamma^{(3)\Lambda}$, $\Gamma^{(4)\Lambda}$, etc.



Hierarchy of 1-loop diagrams; all one-particle irreducible

Initial conditions:

$\Sigma^{\Lambda_0} =$ bare single-particle potential (if any)

$\Gamma^{(2)\Lambda_0} =$ antisymmetrized bare two-particle interaction

$\Gamma^{(m)\Lambda_0} = 0$ for $m \geq 3$

Derivation of flow equation:

$$e^{-\mathcal{G}^\Lambda[\eta,\bar\eta]} = \int \prod_K d\psi_K d\bar\psi_K \, e^{(\bar\psi, Q_0^\Lambda \psi)} \, e^{-V[\psi,\bar\psi]} \, e^{(\bar\eta,\psi)+(\bar\psi,\eta)}$$

Take Λ-derivative on both sides $\Rightarrow$

$$\begin{aligned} -(\partial_\Lambda \mathcal{G}^\Lambda) \, e^{-\mathcal{G}^\Lambda} &= \int \prod_K d\psi_K d\bar\psi_K \, (\bar\psi, \dot Q_0^\Lambda \psi) \, e^{(\bar\psi, Q_0^\Lambda \psi)} \, e^{-V[\psi,\bar\psi]} \, e^{(\bar\eta,\psi)+(\bar\psi,\eta)} \\ &= -(\partial_\eta, \dot Q_0^\Lambda \partial_{\bar\eta}) \, e^{-\mathcal{G}^\Lambda[\eta,\bar\eta]} \end{aligned}$$

$\Rightarrow$ Flow equation for $\mathcal{G}^\Lambda$

$$\frac{d}{d\Lambda} \mathcal{G}^\Lambda[\eta,\bar\eta] = \left(\frac{\partial \mathcal{G}^\Lambda}{\partial \eta}, \dot Q_0^\Lambda \frac{\partial \mathcal{G}^\Lambda}{\partial \bar\eta} \right) + \mathrm{tr} \left(\dot Q_0^\Lambda \, \frac{\partial^2 \mathcal{G}^\Lambda}{\partial \bar\eta \partial \eta} \right)$$

Legendre transform

$$\Gamma^\Lambda[\psi,\bar\psi] = \mathcal{G}^\Lambda[\eta^\Lambda,\bar\eta^\Lambda] + (\bar\psi,\eta^\Lambda) + (\bar\eta^\Lambda,\psi)$$

Note that η^Λ and $\bar\eta^\Lambda$ are Λ-dependent functions of ψ and $\bar\psi$.

$$\frac{d}{d\Lambda}\Gamma^\Lambda[\psi,\bar\psi] = \frac{d}{d\Lambda}\mathcal{G}^\Lambda[\eta^\Lambda,\bar\eta^\Lambda] + (\bar\psi,\partial_\Lambda\eta^\Lambda) + (\partial_\Lambda\bar\eta^\Lambda,\psi)$$

The total derivative acts also on the Λ-dependence of η^Λ and $\bar\eta^\Lambda$.

$$\frac{\partial\mathcal{G}^\Lambda}{\partial\bar\eta} = -\psi\,,\ \frac{\partial\mathcal{G}^\Lambda}{\partial\eta} = \bar\psi \quad \Rightarrow \quad \frac{d}{d\Lambda}\Gamma^\Lambda[\psi,\bar\psi] = \frac{d}{d\Lambda}\mathcal{G}^\Lambda[\eta^\Lambda,\bar\eta^\Lambda]\big|_{\eta^\Lambda,\bar\eta^\Lambda\ \mathrm{fixed}}$$

Insert flow equation for $\mathcal{G}^\Lambda$ and use reciprocity relations between derivatives of $\mathcal{G}^\Lambda$ and Γ^Λ

$\Rightarrow$ Flow equation for Γ^Λ

$$\frac{d}{d\Lambda}\Gamma^\Lambda[\psi,\bar\psi] = -(\bar\psi,\dot{Q}_0^\Lambda\psi) - \frac{1}{2}\mathrm{tr}\left[\dot{\mathbf{Q}}_0^\Lambda\left(\mathbf{\Gamma}^{(1)\Lambda}[\psi,\bar\psi]\right)^{-1}\right]$$

Alternative functional RG versions:

- Polchinski flow equations
- Wick ordered flow equations

3. Truncations

Infinite hierarchy of flow equations usually unsolvable.

Two types of approximation:

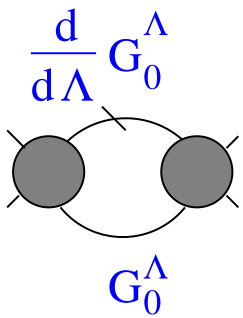
- Truncation of hierarchy at finite order
- Simplified parametrization of effective interactions

Truncations can be justified for weak coupling or small phase space.

Simple truncations in one-particle irreducible fRG:

- Set $\Gamma^{(3)\Lambda} = 0$, neglect self-energy feedback in flow of $\Gamma^{(2)\Lambda}$:

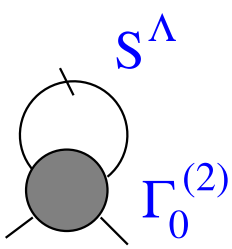
$$\frac{d}{d\Lambda}\Gamma^{(2)\Lambda} =$$



Unbiased stability analysis
at weak coupling;
d-wave superconductivity
in 2D Hubbard model

- Compute flow of self-energy with bare interaction (neglecting flow of $\Gamma^{(2)\Lambda}$):

$$\frac{d}{d\Lambda}\Sigma^{\Lambda} =$$



Captures properties
of isolated impurities
in 1D Luttinger liquid

Power counting:

Which interaction terms are important at low energy?

Conventional power counting procedure:
rescale momenta, frequencies and fields after mode elimination
such that quadratic part of action remains invariant;
see how interaction terms scale.

Consider 1D chiral Fermi system with
linear dispersion $\xi_k = v\, k$ at $T = 0$

Effective action

$$\mathcal{S}^\Lambda = \int dk_0 \int_{-\Lambda}^{\Lambda} dk\, (ik_0 - vk)\, \bar\psi_{k_0,k}\psi_{k_0,k} - V^\Lambda[\psi, \bar\psi]$$

Mode elimination reduces Λ: $\Lambda' = \Lambda/s$, $s > 1$

Rescale momentum and frequency: $k = k'/s$, $k_0 = k'_0/s \quad \Rightarrow \quad |k'| \le \Lambda$

$$dk_0\, dk\, (ik_0 - vk) = [dk'_0\, dk'\, (ik'_0 - vk')]/s^3$$

Compensate by rescaling fields $\psi = s^{3/2}\psi'$, $\bar{\psi} = s^{3/2}\bar{\psi}'$

Now see scaling of interaction terms:

2-particle interaction: $g \int \prod_{j=1}^{3} \underbrace{dk_{j0}\, dk_j}_{s^{-2}} \underbrace{\bar{\psi}\, \bar{\psi}\, \psi\, \psi}_{s^6}$ invariant, "marginal"

k-dependence of g: $g(k) = g(0) + \underbrace{\sum_j \gamma_j k_j}_{\text{extra } s^{-1}} + \dots$ "irrelevant"

3-particle interaction: $(s^{-2})^5\, (s^{3/2})^6 = s^{-1}$ irrelevant **if** $g_3(0)$ finite

Usually $g_3(0)$ of order Λ^{-1} ! Not irrelevant !

Power counting in $d > 1$ cannot be done (easily) by scaling, since quadratic term *cannot be restored by homogeneous scaling of momenta!*

Better look directly at behavior of Feynman diagrams.

Interactions generally "less relevant" in $d > 1$
due to stronger phase space restrictions.