One-dimensional Quantum Liquids

A REVIEW

by

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Outline

- Quasiparticle description of interacting fermions: D>1 vs. D=1
- Tomonaga-Luttinger: full solution for interacting fermions with linear spectrum, basis for the Luttinger liquid phenomenology
- Fermions with nonlinear spectrum: interaction as perturbation
- New phenomenology: nonlinear Luttinger liquid
- Universality of dynamic responses in nonlinear Luttinger liquid
- Fermions with spin, holons and spinons
- Kinetics of a 1D quantum liquid
- Dynamic Responses of 1D bosonic and spin liquids

Interacting fermions

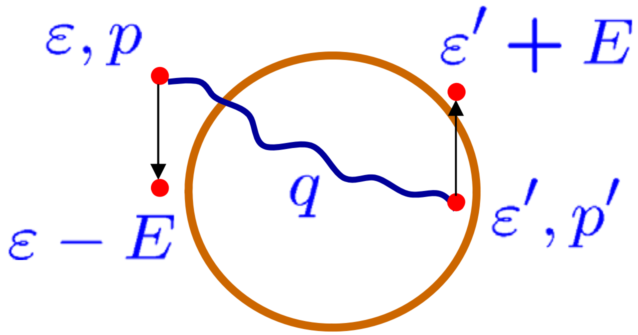
Landau Fermi liquid theory (1956-1958):

excitations of interacting system of fermions ⟷ excitations of free Fermi gas

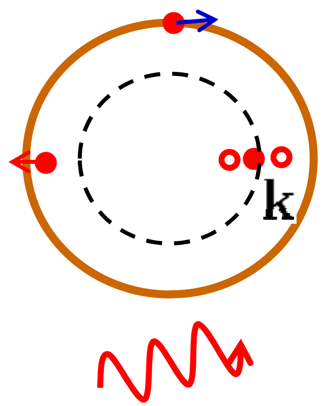
↘ a liquid of weakly interacting **quasiparticles**

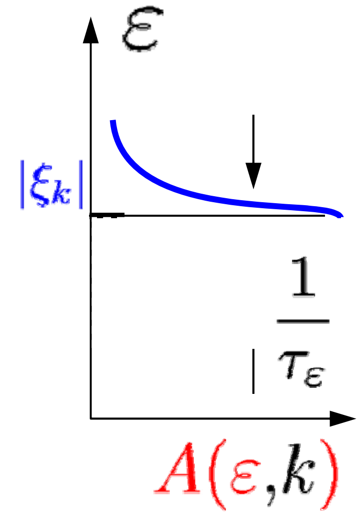
how well the quasiparticles are defined?

Quasiparticles in a Fermi-liquid (D>1)



Visualizing the energy spectrum uncertainty (**ARPES**)





$$\frac{1}{\tau_\varepsilon} \sim \int_0^\varepsilon dE \int_{-E}^0 d\varepsilon'$$

phase space constraint

$$\frac{1}{\tau_\varepsilon} \propto r_s \frac{\varepsilon^2}{\epsilon_F}; \quad r_s \sim \frac{e^2}{\hbar v_F}; \quad \boxed{\frac{1}{\tau_\varepsilon} \ll \varepsilon}$$

consequence of phase space constraint

$$\varepsilon = E_i - E_f = |\xi_k|$$

$$\varepsilon = E_i - \tilde{E}_f > |\xi_k|$$

$$\varepsilon = E_i - \tilde{\tilde{E}}_f < |\xi_k|$$

Interacting fermions: D>1

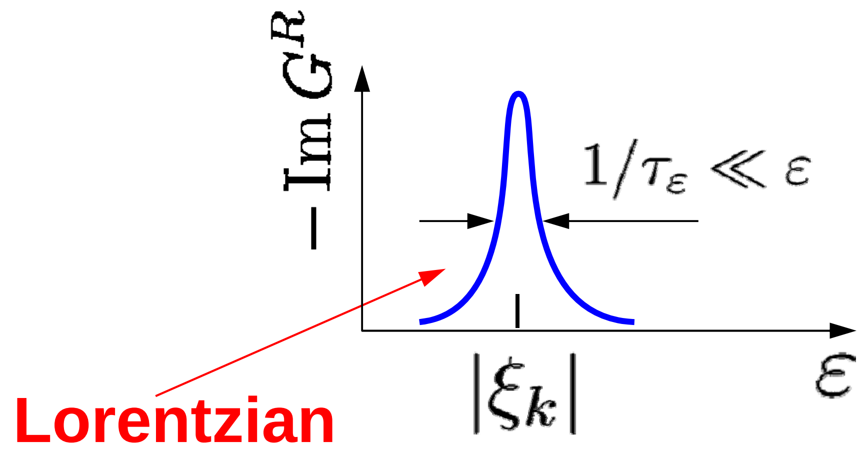
spectral function $A(\varepsilon,\boldsymbol{k}) = -\mathrm{Im}\, G^R$

$$G^R(\varepsilon,k) = \frac{1}{\varepsilon - \xi(k) - \Sigma(\varepsilon,k)} = \frac{1}{\varepsilon - \tilde{\xi}_{\boldsymbol{k}} - i/2\tau_\varepsilon}$$

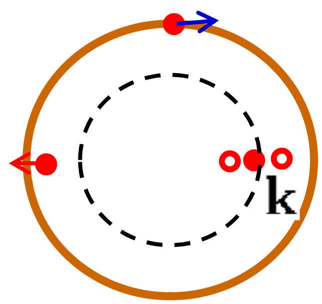
a hole in a 3D Fermi liquid:

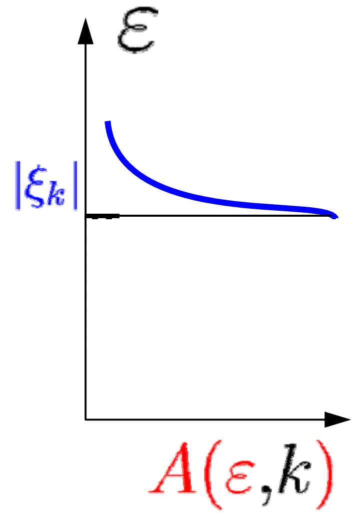
$$-\mathrm{Im}\Sigma(\varepsilon = -|\xi_k|) = \frac{1}{2\tau_\varepsilon} \propto r_s^2 \frac{\varepsilon^2}{\epsilon_F}$$

$$r_s = e^2/\hbar v_F$$



Peculiarity of D=1





$$\varepsilon = E_i - E_f = |\xi_k|$$
$$\varepsilon = E_i - \tilde{E}_f > |\xi_k|$$
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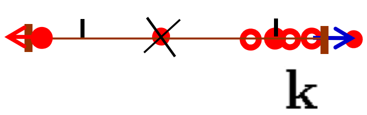
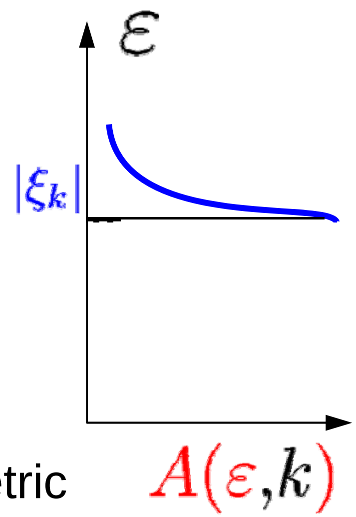
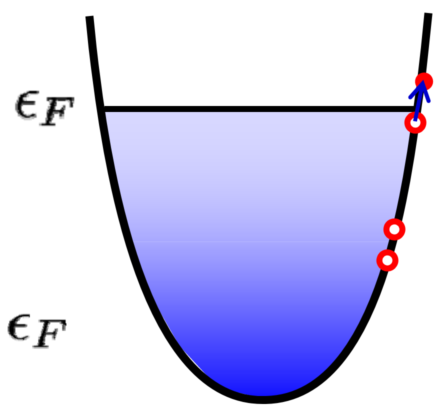




Asymmetric

$$\varepsilon = E_i - E_f = |\xi_k|$$
$$\varepsilon = E_i - \tilde{E}_f > |\xi_k|$$
$$\varepsilon = E_i - \tilde{\tilde{E}}_f \geq |\xi_k|$$



$$\xi_k = \frac{k^2}{2m} - \epsilon_F$$

Peculiarity of D=1

$$G^R(\varepsilon, k) = \frac{1}{\varepsilon - \xi(k) - \Sigma(\varepsilon, k)}$$

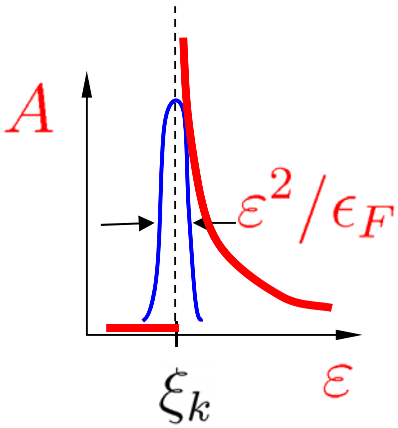
$$-\mathrm{Im}\Sigma(\varepsilon\,, \xi_k) \propto V_{ee}^2 \cdot (\varepsilon - \xi_k)\theta(\varepsilon - |\xi_k|)$$

$$\xi_k < 0$$

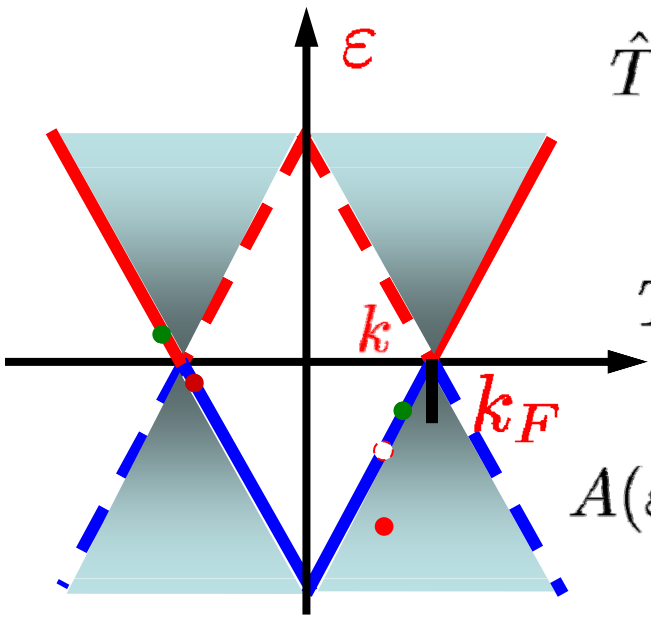
$$A(\varepsilon, k) = -\mathrm{Im}\, G_R(\varepsilon, k) \propto V_{ee}^2 \frac{\theta(\varepsilon - |\xi_k|)}{\varepsilon - |\xi_k|}$$

$$\Sigma(\varepsilon\,, \xi_k) \propto V_{ee}^2 \cdot (\varepsilon - \xi_k) \ln(\varepsilon - \xi_k)$$

non-analytic at mass shell



Spectral function: Perturbation theory



$$\hat{T} \propto \hat{V}_{\rm int} \frac{1}{E - \hat{H}_0} \hat{V}_{\rm tunn} + \ldots$$

$$T \propto V_{LR} \frac{1}{\varepsilon - \xi_k} V_{\rm tunn}$$

$$A(\varepsilon, k) \propto \int dk_L^r dk_L^g dk_R^g \left| \frac{V_{LR}}{\varepsilon - \xi_k} \right|^2 \delta(\Sigma k)\delta(\Sigma \varepsilon)$$

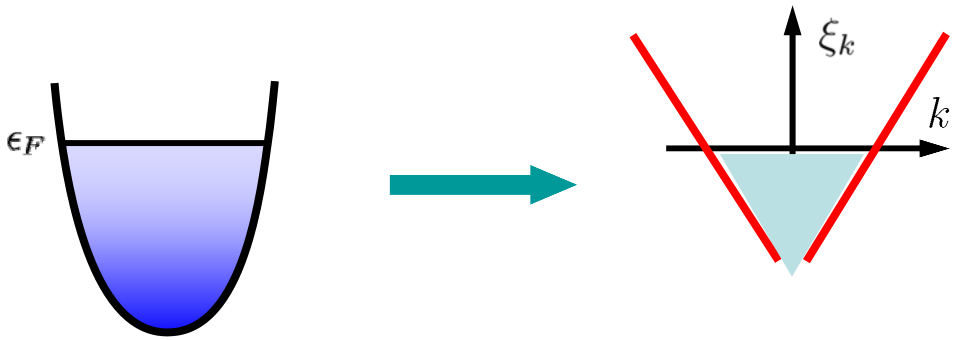
$$k_L^g - k_L^r = \frac{\varepsilon - \xi_k}{v}$$

3 integrations, 2 conservation laws: one integration left

$$A(\varepsilon, k) = -\operatorname{Im} G_R(\varepsilon, k) \propto V_{LR}^2 \frac{\theta(\varepsilon - \xi_k)}{\varepsilon - \xi_k}$$

Tomonaga-Luttinger Model

Simplification: Interacting fermions with linear energy spectrum



Progress of Theoretical Physics Vol. 5, No. 4, July~August, 1950

Remarks on Bloch's Method of Sound Waves applied to Many-Fermion Problems

Sin-itiro TOMONAGA

... an assembly of Fermi particles can be described by a quantized field of sound waves in the Fermi gas, where the sound field obeys Bose statistics, is proved in the one-dimensional case...

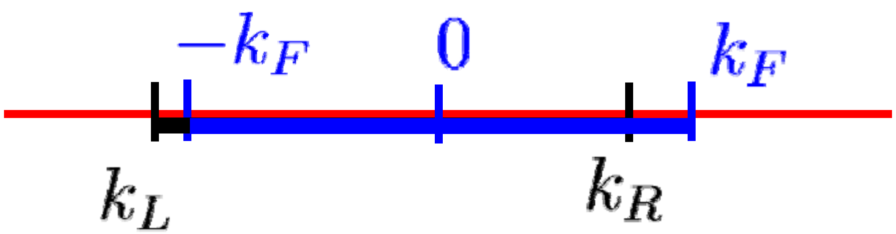
... The field equation for the sound wave is found to be linear irrespective of the absence or presence of mutual interaction between particles, so that this method is a very useful means of dealing with many-Fermion problems.

Bosonization (and Spectrum Curvature)

Haldane, 1983

$$k_{L,R}(x) \pm k_F \rightarrow \partial_x \varphi_{L,R}$$

excess number of left (*L*), right (*R*) movers



$$\xi_k = \pm v_F k + \frac{k^2}{2m}$$

$$H_{\mathrm{K}}(x) = \int_{k_L(x)}^{k_R(x)} \xi_k \, dk - \bar{E}_{\mathrm{K}} \rightarrow \frac{v_F}{2}(\partial_x \varphi_R)^2 + \frac{1}{m}(\partial_x \varphi_R)^3 + (R \leftrightarrow L)$$

$$H_{int}(x) = V_{LL}(\partial_x \varphi_L)^2 + V_{RR}(\partial_x \varphi_R)^2 + 2V_{LR}(\partial_x \varphi_L)(\partial_x \varphi_R)$$

$\varphi_{L,R} \leftrightarrow \varphi \pm \vartheta$ excess density ($n(x) = \partial_x \varphi$), momentum ($\propto \partial_x \vartheta$)

Quantized Displacement Fields (Bosonization)

$$\mathcal{H} = \frac{v}{2\pi} \int dx \left[\frac{1}{K} (\partial_x \varphi)^2 + K (\partial_x \vartheta)^2 \right]$$

conjugate variable (momentum)

$[\varphi(x), \vartheta(y)] \propto \mathrm{sign}(x - y)$

field of displacements, $n(x) = \partial_x \varphi$

$$v = \sqrt{\frac{\pi n_0}{m} \frac{\partial \mu}{\partial n}}$$

Galilean invariance

$(V_{LL} = V_{LR} = V_{RR})$

$$K = \frac{\pi n_0}{m v}$$

Dynamics is controlled by two parameters of the liquid:

v and K

$K < 1$	$K = 1$	$K > 1$
repulsion	free fermions	attraction

Dynamic Structure Factor

Perturbation: $$\mathcal{H}_{\text{ext}} = \int \hat{n}(x) U(x,t) dx$$

Linear response to *U(q,ω)*: density-density correlation function

$$\chi(q,\omega) = \left\langle -i\theta(t)\left[\hat{n}(x,t), \hat{n}^{\dagger}(0,0)\right]\right\rangle_{q,\omega}$$

dynamic structure factor:

$$S(q,\omega) = \int dx\, dt\, e^{i(\omega t - qx)} \langle \hat{n}(x,t)\hat{n}(0,0)\rangle = 2\,\mathrm{Im}\chi(q,\omega)$$

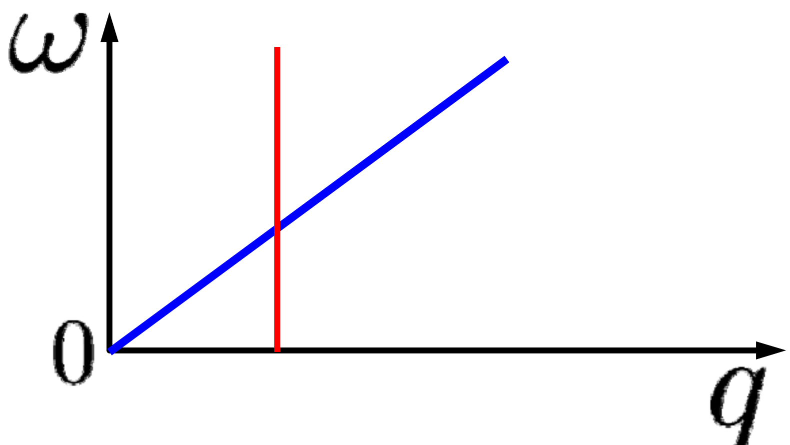
at $T = 0$ (FDT)

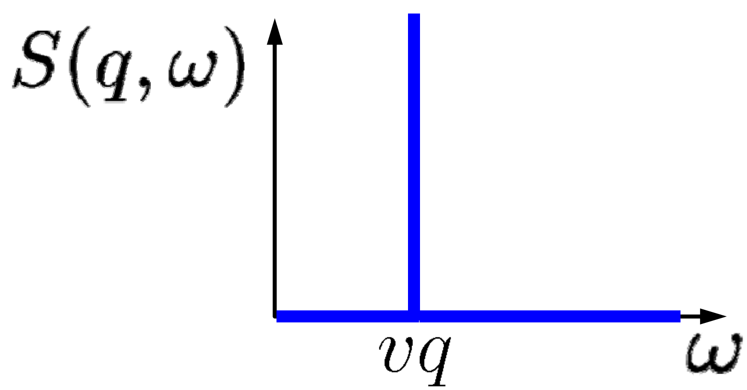
Structure factor of a Luttinger liquid

$$n(x) = \partial_x \varphi$$ "acoustic phonons"

$$S(q,\omega) = \left\langle n(q,\omega) n(-q,-\omega) \right\rangle \propto \left\langle \varphi(q,\omega)\varphi(-q,-\omega) \right\rangle$$

$$\sim q\, \delta(\omega - vq)$$



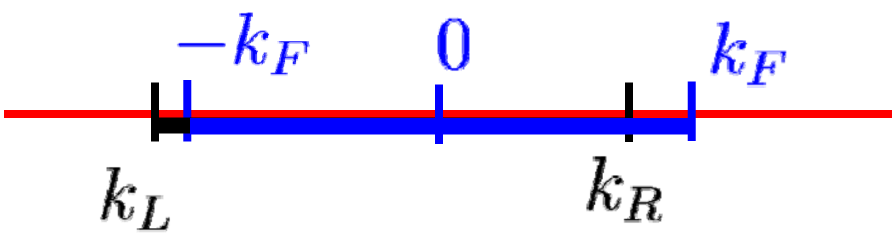


How does the dispersion curvature affect the structure factor ?

$$\xi_k = \pm vk + \frac{k^2}{2m}$$

Spectrum Curvature in Bosonization

Haldane, 1983



$$k_{L,R}(x) \pm k_F \rightarrow \partial_x \varphi_{L,R}$$

$$\varphi \pm \vartheta \leftrightarrow \varphi_{L,R}$$

$$\xi_k = \pm v k + \frac{k^2}{2m}$$

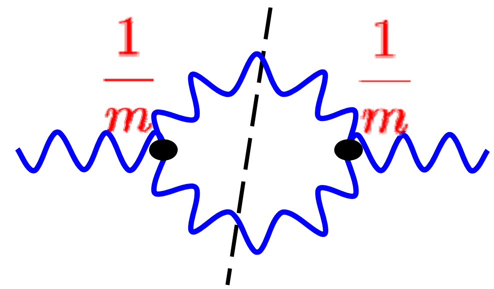
$$H_{\mathrm{K}}(x) = \int_{k_L(x)}^{k_R(x)} \xi_k \, dk - \bar{E}_{\mathrm{K}} \rightarrow \frac{v}{2}(\partial_x \varphi_R)^2 + \frac{1}{m}(\partial_x \varphi_R)^3 + (R \leftrightarrow L)$$

$$H_{int}(x) = V_{LL}(\partial_x \varphi_L)^2 + V_{RR}(\partial_x \varphi_R)^2 + 2V_{LR}(\partial_x \varphi_L)(\partial_x \varphi_R)$$

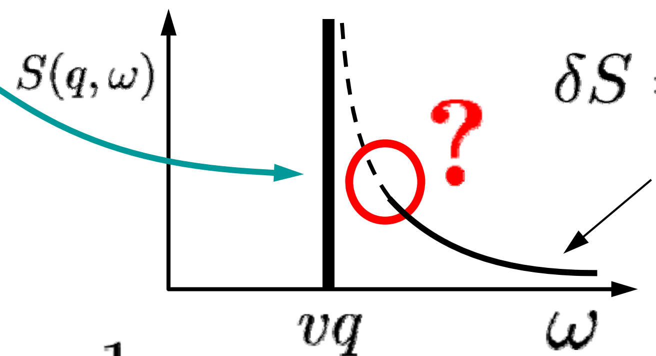
Curvature as a perturbation

$$H(x) = \left[\frac{v}{2} + V_{RR}\right](\partial_x\varphi_R)^2 + V_{LR}(\partial_x\varphi_L)(\partial_x\varphi_R)$$

$$+\frac{1}{m}(\partial_x\varphi_R)^3 + (R \leftrightarrow L)$$



$$S(q,\omega) \propto q\delta(\omega - vq)$$



$$\delta S = \frac{f(V_{LR})}{v}\frac{(q^2/m)^2}{\omega^2 - (vq)^2}$$

Pustilnik et al 03, Pereira et al 06

$$\Sigma^{(2)}_{Boson}(\omega = vq, q) = \frac{1}{m^2}\cdot\infty$$

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 4, NUMBER 9 SEPTEMBER 1963

An Exactly Soluble Model of a Many-Fermion System*

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Department of Physics, Columbia University, New York, New York
(Received 2 April 1963)

An exactly soluble model of a one-dimensional many-fermion system is discussed. The model has a fairly realistic interaction between pairs of fermions. An exact calculation of the momentum distribution in the ground state is given. It is shown that there is no discontinuity in the momentum distribution in this model at the Fermi surface, but that the momentum distribution has infinite slope there. Comparison with the results of perturbation theory for the same model is also presented, and it is shown that, for this case at least, the perturbation and exact answers behave qualitatively alike. Finally, the response of the system to external fields is also discussed.

I. INTRODUCTION

WE shall be concerned in this paper with a model of a many-fermion system which is exactly soluble. The model is quite unrealistic for two reasons: it is one-dimensional and the fermions are massless. On the other hand, it has the realistic feature that there is a true pair interaction between the particles. It is very closely related to the well-known Thirring Model[1] in field theory, though slightly more general. Our main interest in the model is in connection with the question of whether or not a sharp Fermi Surface (F.S.) exists in the exact ground state.

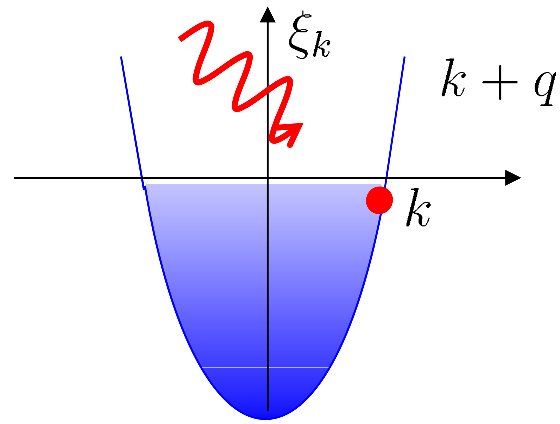
(1) Higher dimensions – Fermi-liquid theory (1956 - …)

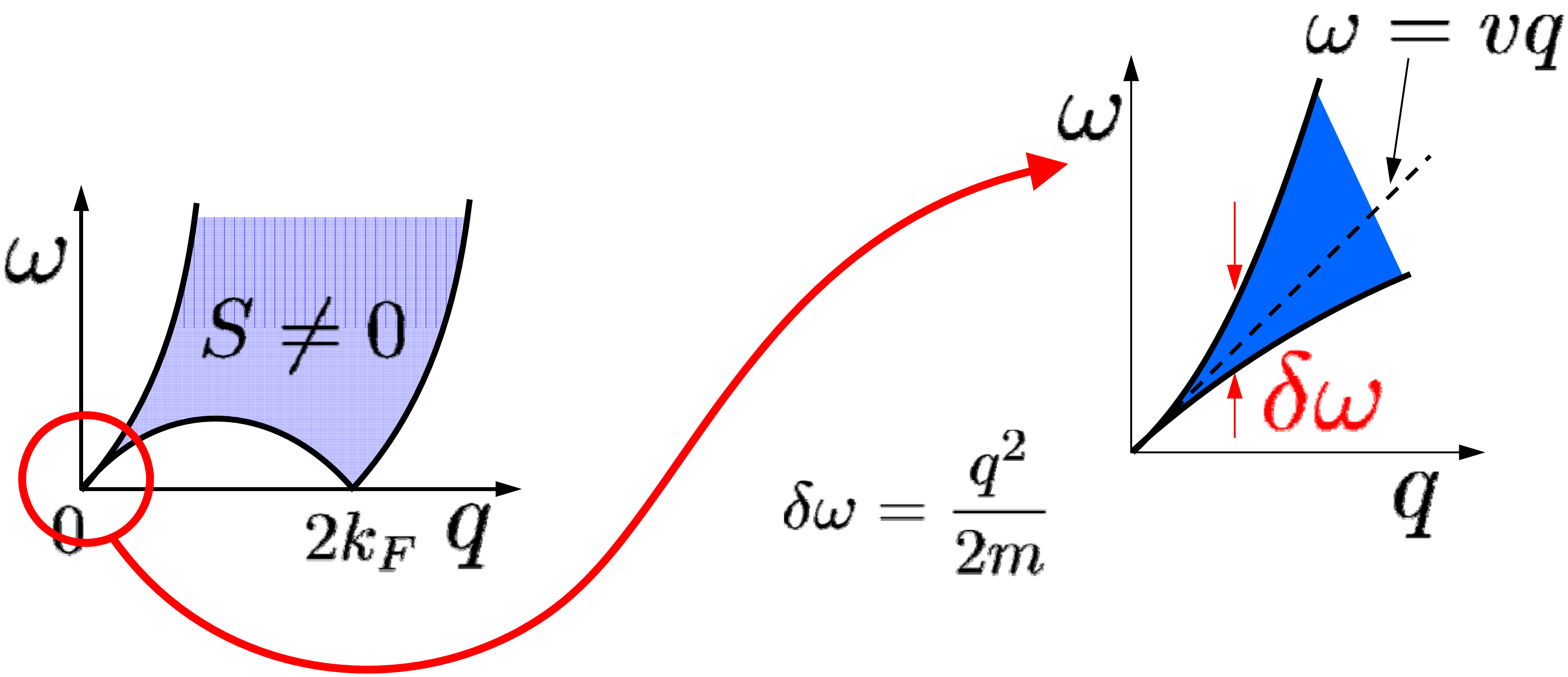
(2) One-dimensional fermions with mass (a part of these lectures)

Back to free fermions

Lehmann (Golden rule – like) representation

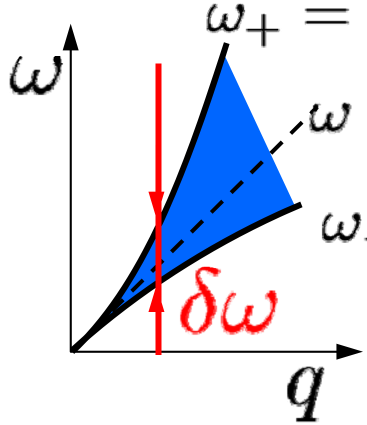
$$S(q,\omega) = 2\pi \sum_{k=k_F-q}^{k_F} \delta\left[\omega - (\xi_{k+q} - \xi_k)\right]$$

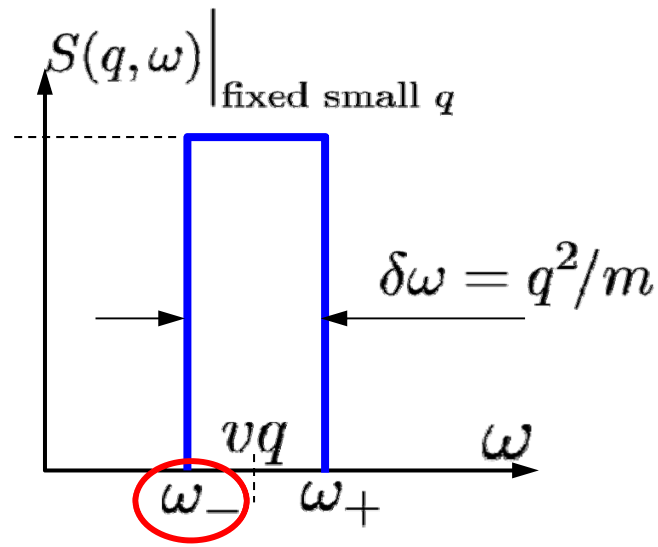




$$\delta\omega = \frac{q^2}{2m}$$

Curvature: free fermions perspective





$$\delta\omega = q^2/m \sim \omega^2/\epsilon_F$$

$\Longrightarrow$

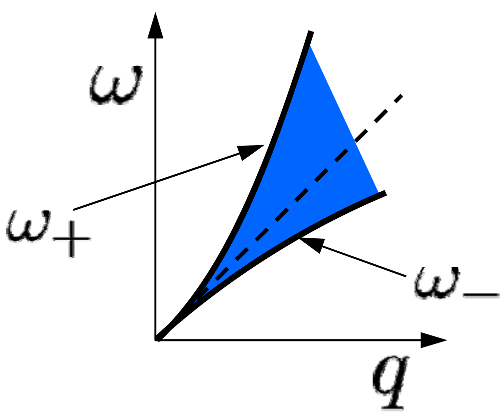
- **the peak is narrow** (recall $\frac{1}{\tau_\varepsilon} \propto \varepsilon^2/\epsilon_F$ in D=3)

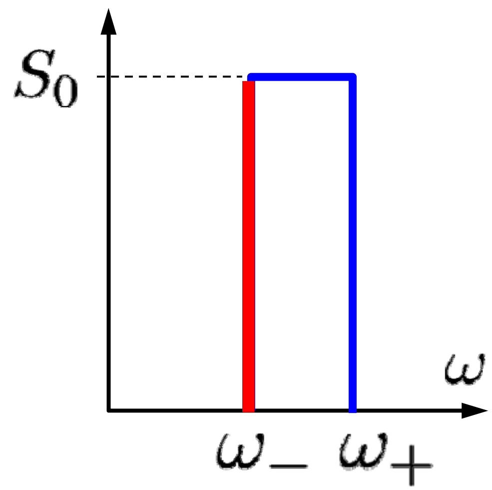
but…

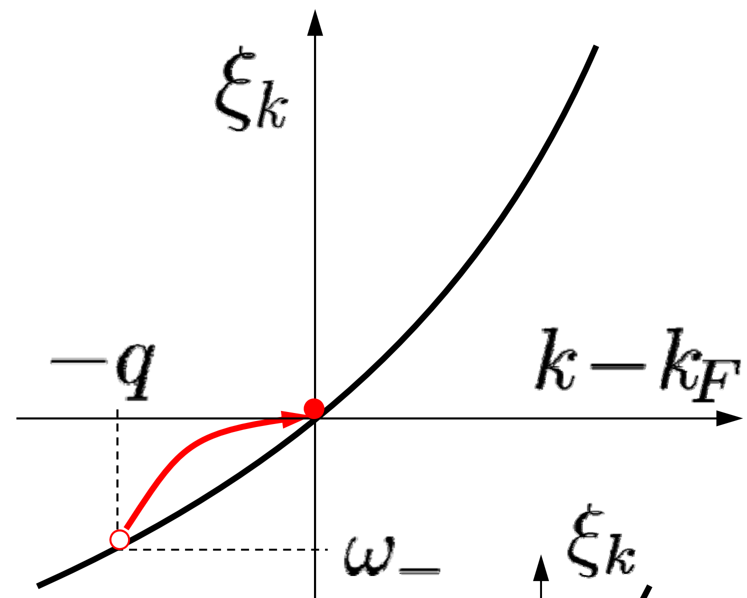
- **it is not a Lorentzian (non-analytical in ω)**
- $\delta\omega \propto 1/|m|$ **(non-analytical in curvature)**

Effect of interaction, $\omega \rightarrow \omega_-$

free electrons

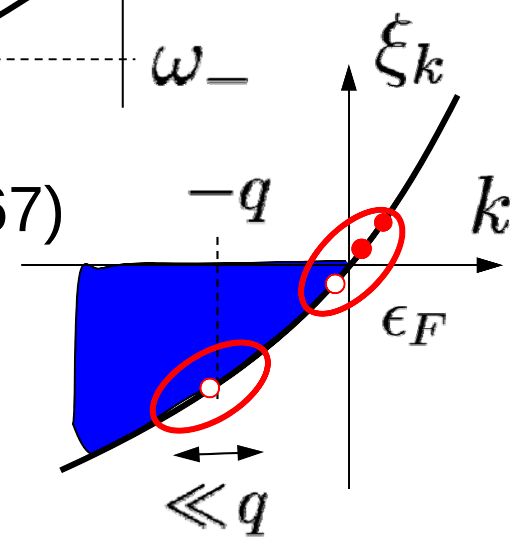






Analogy: Fermi-edge singularity (Mahan 1967)

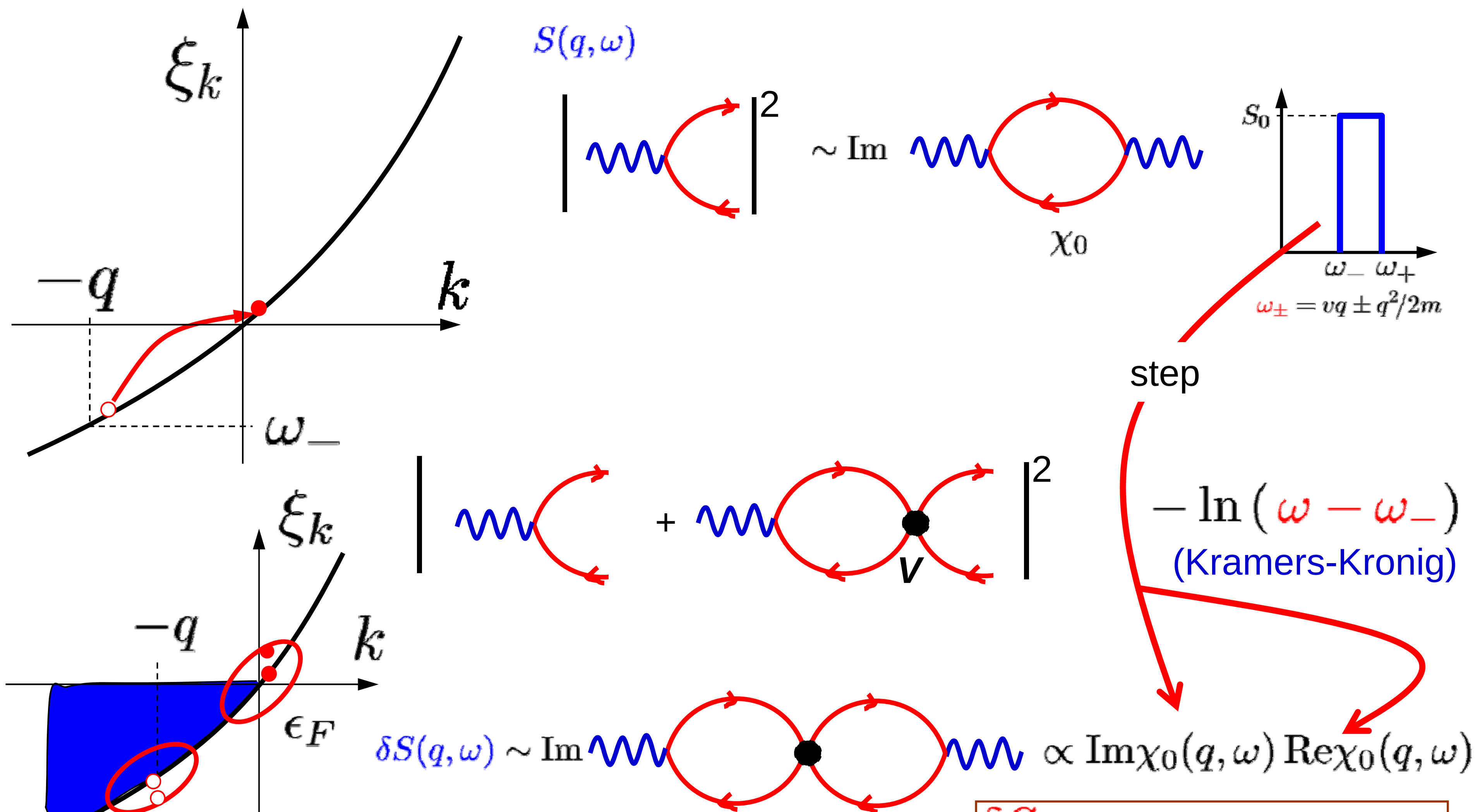
important states:



interaction with the “core hole”

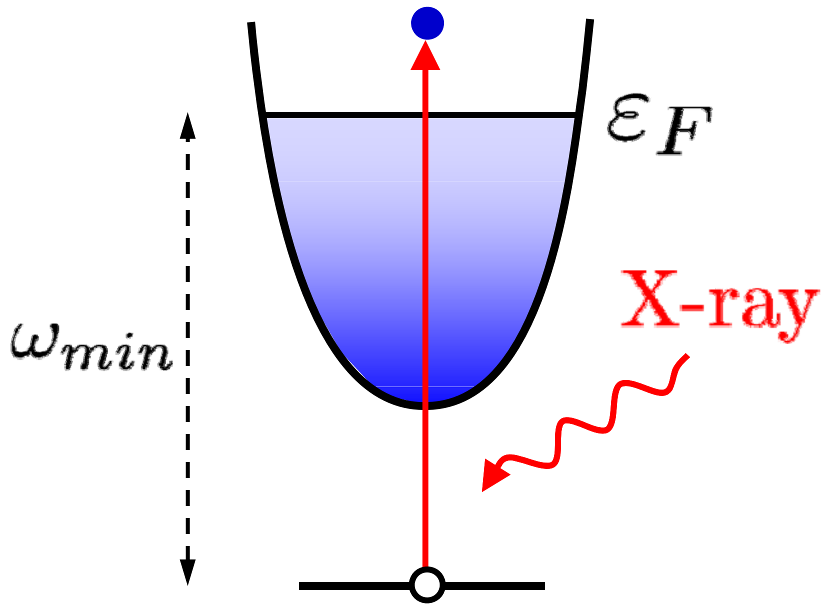
singularity $[\ln(\omega - \omega_-)]^n$ in each order of perturb. theory in V

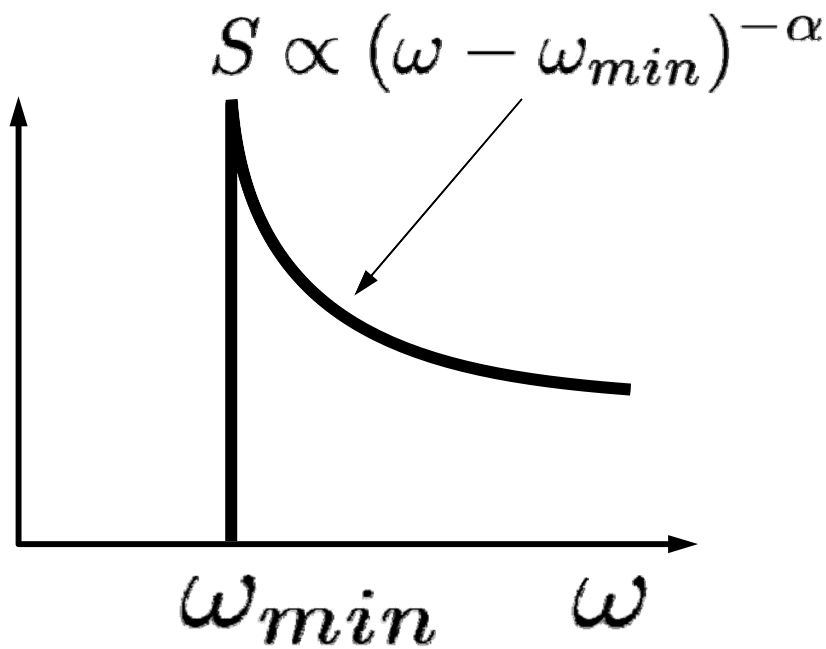
1-st order perturbation theory in interaction

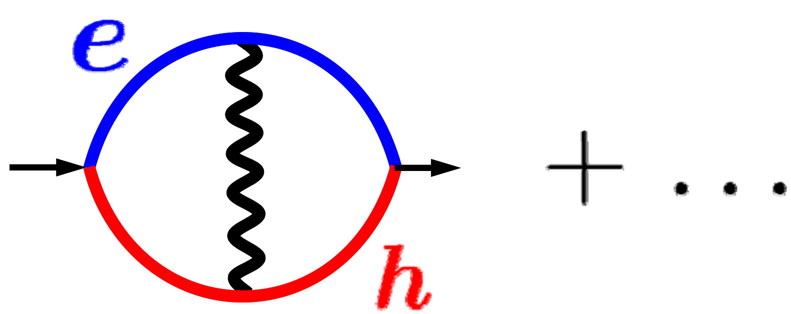


$$\frac{\delta S}{S_0} \propto -V \ln(\omega - \omega_-)$$

Fermi edge singularity in metals







Mahan 67
Nozieres, DeDominicis 69

threshold + interactions = power law

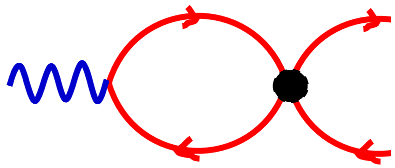
What is different in our case?

✓ *Hole is mobile.*

Does not spoil power-law singularity in D=1,
but rather modifies the exponent.

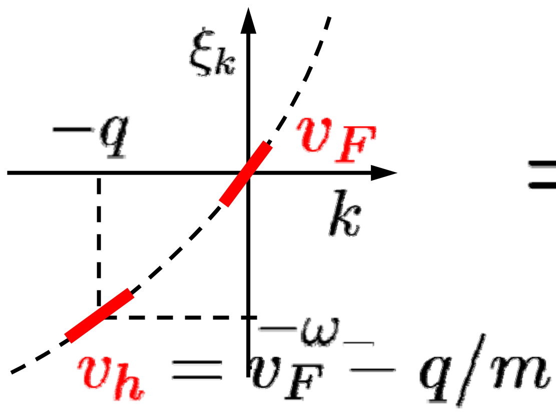
Ogawa, Furusaki, Nagaosa,1992
Nozieres, 1994; Balents, 2000

✓ *Hole belongs to the same band.*



Requires inclusion of exchange interaction

Absorption edge: $\omega \longrightarrow \omega_-$



$\Longrightarrow$

$$S \propto (\omega - \omega_-)^{-\mu}$$

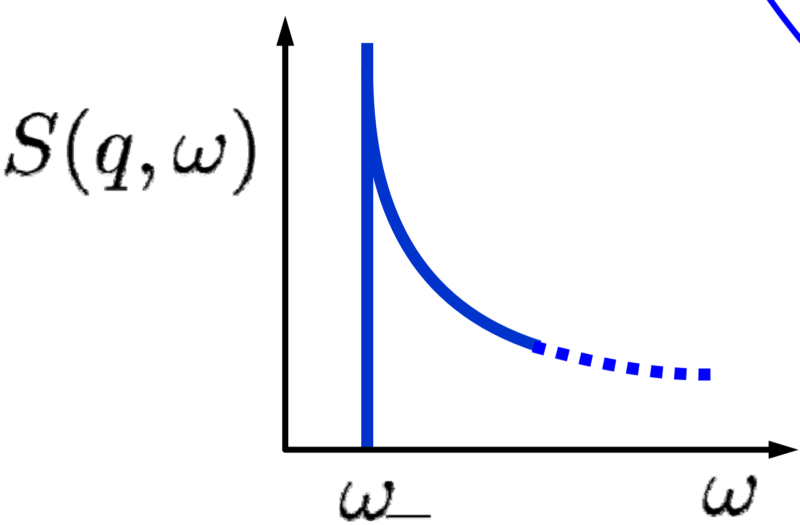
$$0 < \omega - \omega_- \ll \delta\omega$$

Linear order in *V*:

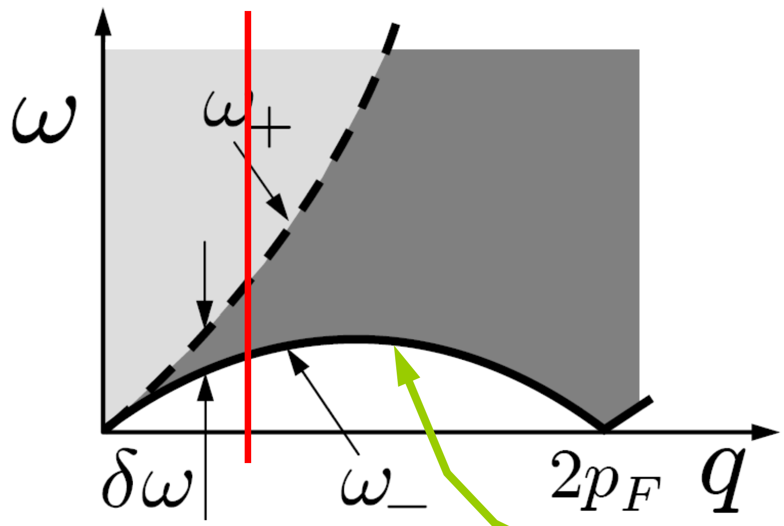
$$\mu = \frac{V_0 - V_q}{\pi(v_F - v_h)} = \frac{m}{\pi q}(V_0 - V_q)$$

✓ *Mobile hole*

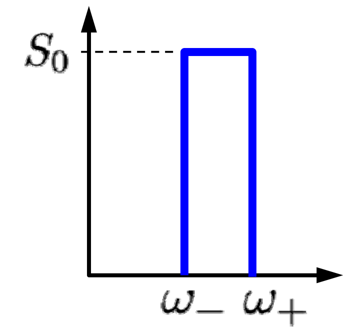
✓ *Exchange*



1D Fermions – Structure Factor

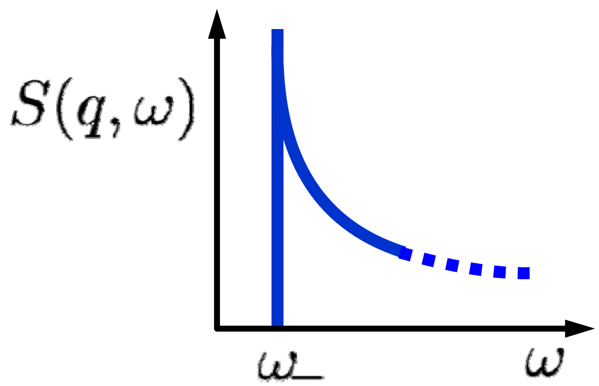


No interaction:



$\omega_\pm = vq \pm q^2/2m$

Repulsion: divergent DSF



$$\frac{S(q,\omega)}{S_0} = \left[\frac{\delta\omega}{\omega - \omega_-(q)}\right]^{\mu(q)}$$

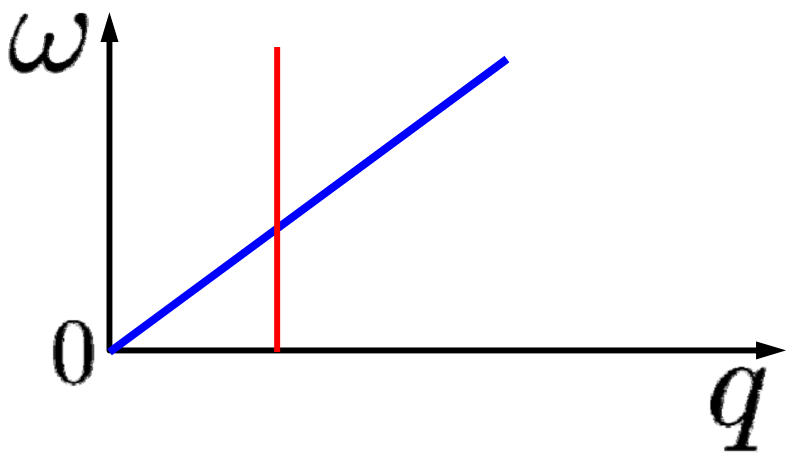
$$\mu = (V_0 - V_q)m/\pi q \ll 1$$

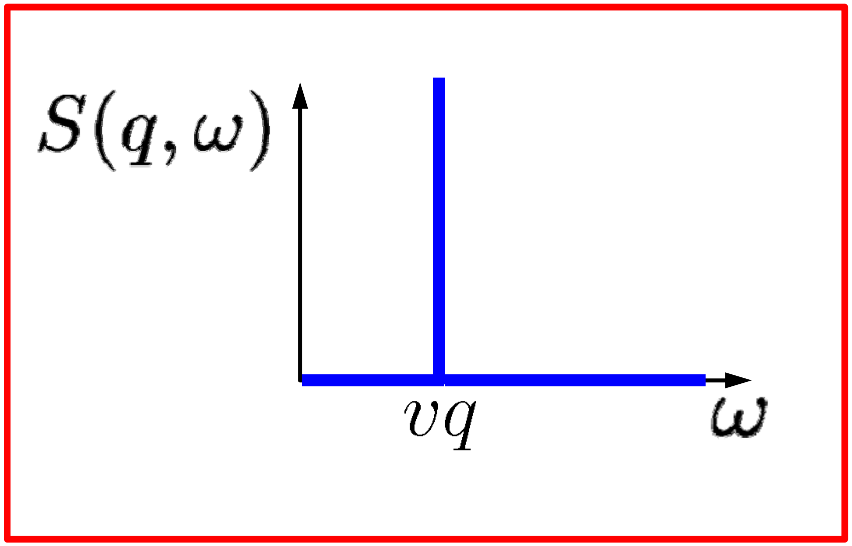
Struct. factor of a linear Luttinger liquid

$n(x) = \partial_x \varphi$ "acoustic phonons"

$$S(q,\omega) = \Big\langle n(q,\omega) n(-q,-\omega) \Big\rangle \propto \Big\langle \varphi(q,\omega)\varphi(-q,-\omega) \Big\rangle$$

$$\sim q\,\delta(\omega - vq)$$





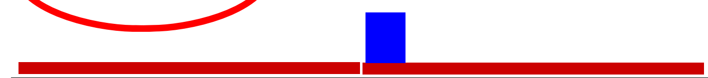
Spectral function of a linear Luttinger liquid-1

$$A(k,\omega) = -\mathrm{Im}\, G^R(k,\omega)$$

$$G^R(x,t) \propto \langle \hat{\Psi}^\dagger(x,t)\hat{\Psi}(0,0)\rangle\theta(t)$$

Fermionic field: $\Psi(x,t) \approx \hat{\Psi}_R(x,t)e^{ik_F x} + \hat{\Psi}_L(x,t)e^{-ik_F x}$

$$\Psi_R^\dagger(x,t) \propto e^{-i\varphi_R(x,t)} \propto e^{-i\varphi(x,t)}e^{i\theta(x,t)}$$



$$G^R(x,t) \to \langle e^{-i\varphi_R(x,t)}e^{i\varphi_R(0,0)}\rangle_H$$

$$H_{kin}(x) = \int_{k_L(x)}^{k_R(x)} \xi_k\, dk = \frac{v_F}{2}\left[(\partial_x\varphi_L)^2 + (\partial_x\varphi_R)^2\right]$$

$$H_{int}(x) = V_{LL}(\partial_x\varphi_L)^2 + V_{RR}(\partial_x\varphi_R)^2 + 2V_{LR}(\partial_x\varphi_L)(\partial_x\varphi_R)$$

Spectral function of a linear Luttinger liquid-2

$\langle e^{-i\varphi_R(x,t)} e^{i\varphi_R(0,0)} \rangle_H$

$$H_{kin}(x) = \frac{v_F}{2} \left[(\partial_x \varphi_L)^2 + (\partial_x \varphi_R)^2 \right]$$

$$H_{int}(x) = V_{LL}(\partial_x \varphi_L)^2 + V_{RR}(\partial_x \varphi_R)^2 + 2V_{LR}(\partial_x \varphi_L)(\partial_x \varphi_R)$$

$$H = H_{kin} + H_{int} = A[(\partial_x \varphi_L)^2 + (\partial_x \varphi_R)^2] + B\partial_x \varphi_L \partial_x \varphi_R$$

$\partial_x \varphi_L \pm \partial_x \varphi_R$ canonically conjugate

Diagonalization, re-scaling (canon. transf.): $\varphi_L, \varphi_R \rightarrow \tilde{\varphi}_L, \tilde{\varphi}_R$

$$\tilde{H} = \frac{v}{2} \left[(\partial_x \tilde{\varphi}_L)^2 + (\partial_x \tilde{\varphi}_R)^2 \right]$$

(looks like bosonized free fermions)

Spectral function of a linear Luttinger liquid-3

Diagonalization, re-scaling (canon. transf.): $\varphi_L, \varphi_R \to \tilde{\varphi}_L, \tilde{\varphi}_R$

$$\tilde{H} = \frac{v}{2}\left[(\partial_x \tilde{\varphi}_L)^2 + (\partial_x \tilde{\varphi}_R)^2\right]$$

$$\langle e^{-i\varphi_R(x,t)} e^{i\varphi_R(0,0)} \rangle_H$$

$$\to \langle e^{-i\alpha_L \tilde{\varphi}_L(x,t)} e^{i\alpha_L \tilde{\varphi}_L(0,0)} \rangle_{\tilde{H}} \cdot \langle e^{-i\alpha_R \tilde{\varphi}_R(x,t)} e^{i\alpha_R \tilde{\varphi}_R(0,0)} \rangle_{\tilde{H}}$$

Gaussian average

$$\langle \exp\{-i\alpha_R \tilde{\varphi}_R(x,t)\} \cdot \exp\{i\alpha_R \tilde{\varphi}_R(0,0)\} \rangle_{\tilde{H}}$$

$$= \exp\{-(1/2)\alpha_R^2 \langle (\tilde{\varphi}_R(x,t) - \tilde{\varphi}_R(0,0)) \tilde{\varphi}_R(0,0) \rangle_{\tilde{H}}\}$$

Evaluation of correlation functions

$$\exp\{-(1/2)\alpha_R^2\langle(\tilde{\varphi}_R(x,t)-\tilde{\varphi}_R(0,0))\tilde{\varphi}_R(0,0)\rangle_{\tilde{H}}\}$$

Equation of motion:

$$\left\{\frac{\partial}{\partial t}-v\frac{\partial}{\partial x}\right\}\langle\tilde{\varphi}_R(x,t)\tilde{\varphi}_R(0,0)\rangle_{\tilde{H}}\propto\delta(x-vt)$$

$$[\varphi_R(x,t),\varphi_R(0)]\propto\mathrm{sign}(x-vt)$$

$$\langle(\tilde{\varphi}_R(x,t)-\tilde{\varphi}_R(0,0))\tilde{\varphi}_R(0,0)\rangle_{\tilde{H}}\propto\ln[(x-vt)/x_0]$$

$A(\varepsilon, k)$ in the linear Luttinger Liquid

Linear spectrum $\xi_k = v_F k$

(k is measured from Fermi point k_F)

$$A(\varepsilon, k) \propto \frac{\theta(\varepsilon^2 - v^2 k^2)}{\varepsilon - vk} (\varepsilon^2 - v^2 k^2)^{\frac{1}{4}\left(K + \frac{1}{K}\right) - \frac{1}{2}}$$

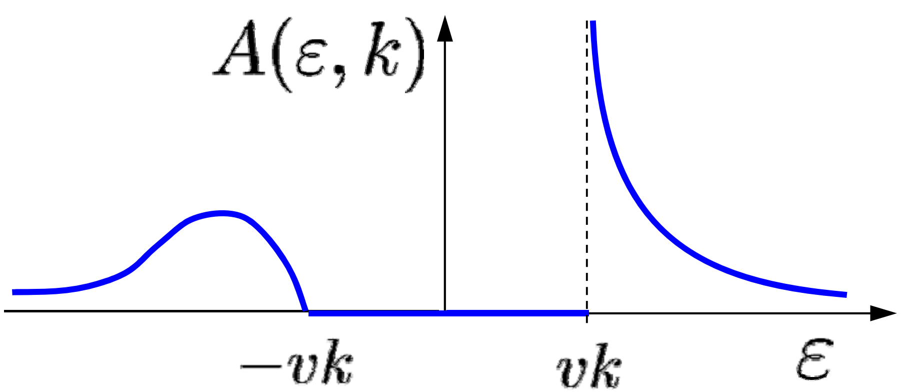
Tomonaga-Luttinger model: (1974) bosonisation (Luther, Peschel) or series summation for original fermions (Dzyaloshinskii, Larkin)

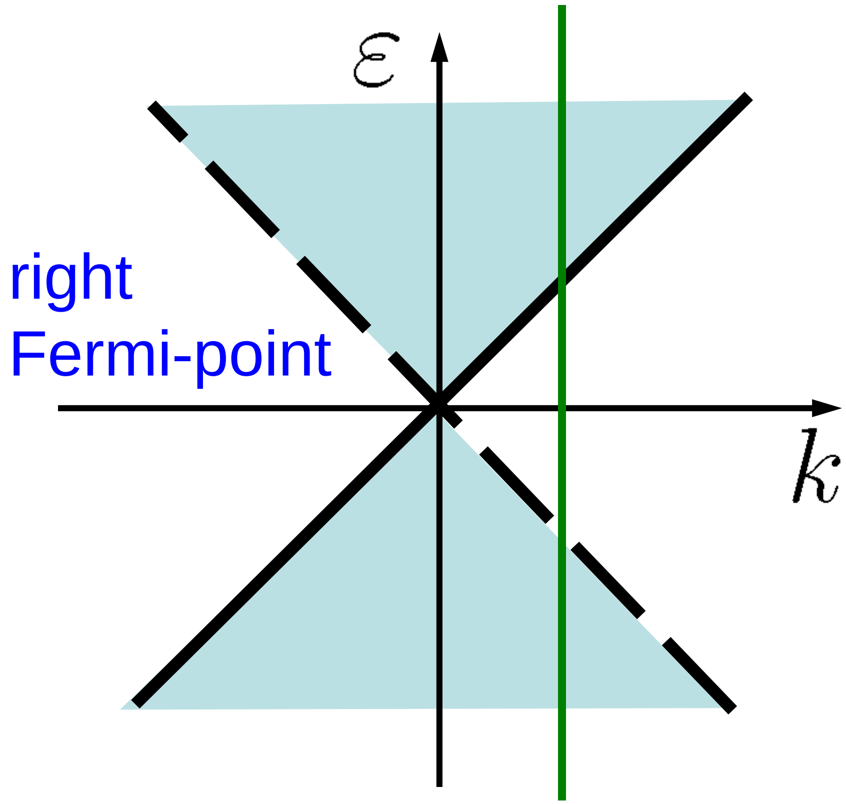
Linear Luttinger Liquid Phenomenology

Deemed adequate at arbitrarily small ξ_k

in the scaling limit $\varepsilon/vk = \text{finite}\,,\;\; k/k_F \to 0$

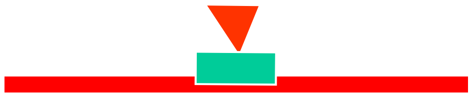
$$A(\varepsilon,k) \propto \frac{\theta(\varepsilon^2 - v^2k^2)}{\varepsilon - vk}(\varepsilon^2 - v^2k^2)^{\frac{1}{4}\left(K+\frac{1}{K}\right)-\frac{1}{2}}$$





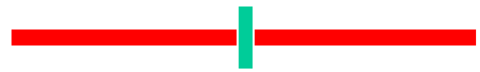
Point Tunneling: $dI/dV \propto \int A(\varepsilon = eV, k)\,dk$

Local tunneling density of states:



$$\left.\frac{dI}{dV}\right|_{\text{bulk}} \propto |V|^{\alpha_{\text{bulk}}}$$

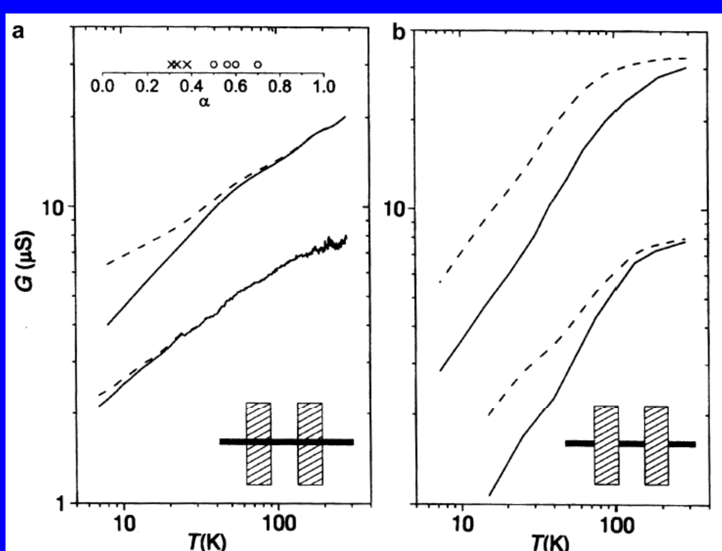
$$\alpha_{\text{bulk}} = \frac{1}{4}\left[K + \frac{1}{K} - 2\right]$$



$$\left.\frac{dI}{dV}\right|_{\text{end}} \propto |V|^{\alpha_{\text{end}}}$$

$$\alpha_{\text{end}} = \frac{1}{2}\left[\frac{1}{K} - 1\right]$$

Single-wall nanotubes – 4-mode (incl. spin) Luttinger liquids



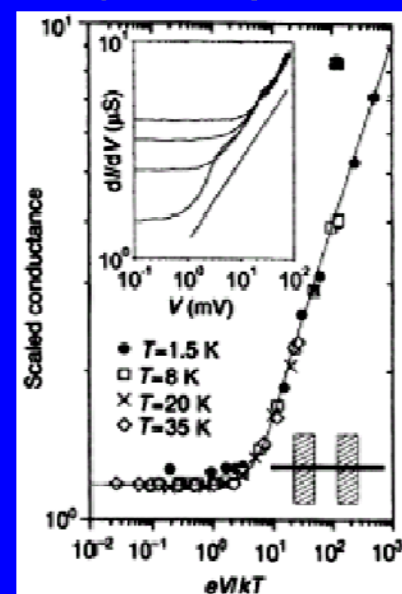
$\alpha_{\text{bulk}} \approx 0.3$

$\alpha_{\text{end}} \approx 0.6$

Bockrath *et al* 1999

+ data scaling

$$\frac{dI}{dV} = V^{\alpha} f\left(\frac{V}{T}\right)$$

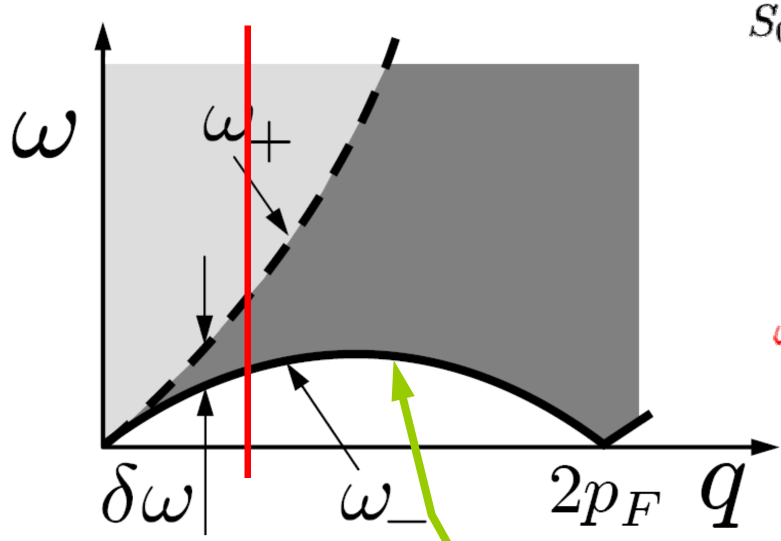


Outline

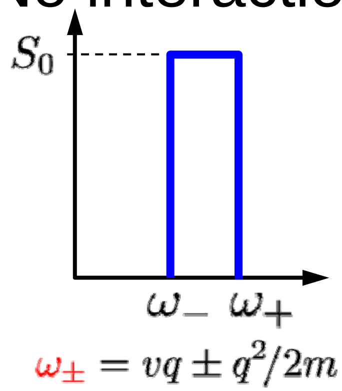
- Quasiparticle description of interacting fermions: D>1 vs. D=1
- Fermions with a generic spectrum: interaction as perturbation
- Linear Luttinger liquid: bosonization, full solution for interacting fermions with linear spectrum (long wavelength excitations)

- Arbitrary interactions and wavelengths: nonlinear Luttinger liquid
  - Nonlinear Luttinger liquid: new phenomenology
  - Fermions with spin, holons and spinons
  - Kinetics of a 1D quantum liquid
  - Dynamic Responses of 1D bosonic and spin liquids

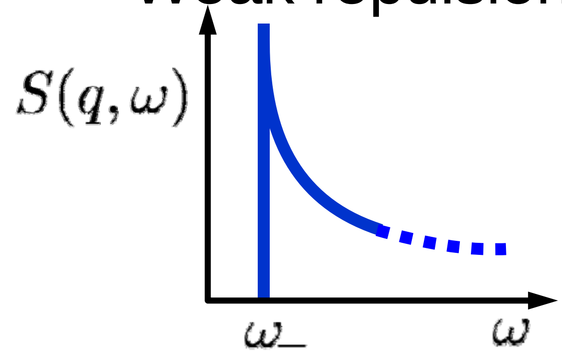
Back to generic 1D Fermions



No interaction:



Weak repulsion:



power-law DSF

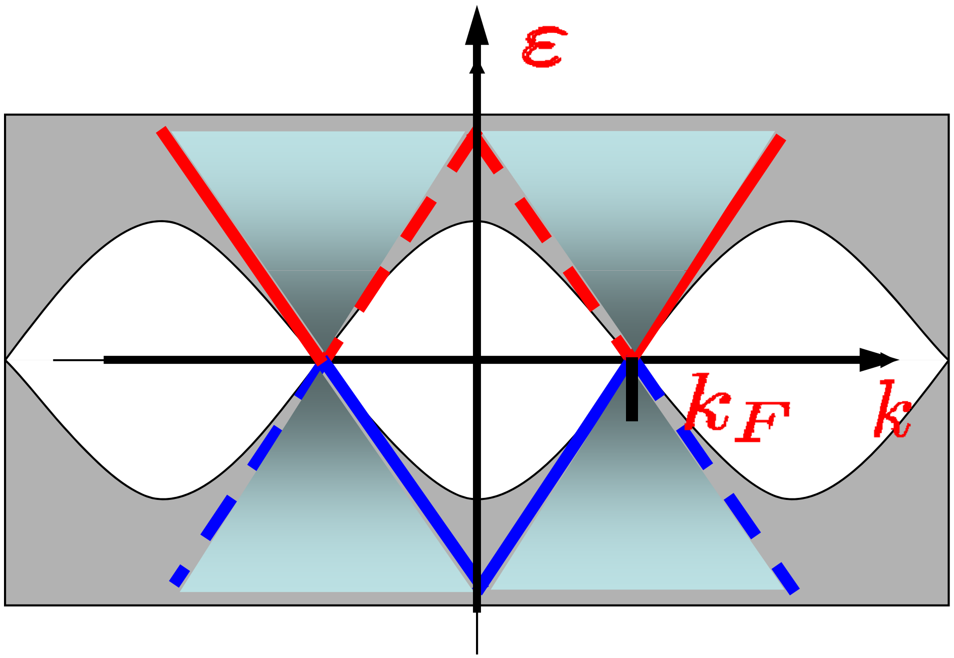
$$\frac{S(q,\omega)}{S_0} = \left[\frac{\delta\omega}{\omega-\omega_-(q)}\right]^{\mu(q)} \propto \sum_l \frac{1}{l!}\left[(V_q - V_0)\ln\frac{\delta\omega}{\omega-\omega_-}\right]^l$$

$$\mu = (V_0 - V_q)m/\pi q \ll 1$$

“Leading logarithm” series

Arbitrary interaction strength and momenta

Universality?



1. Excitation energies at given (finite) momentum are finite – true at ANY interaction strength

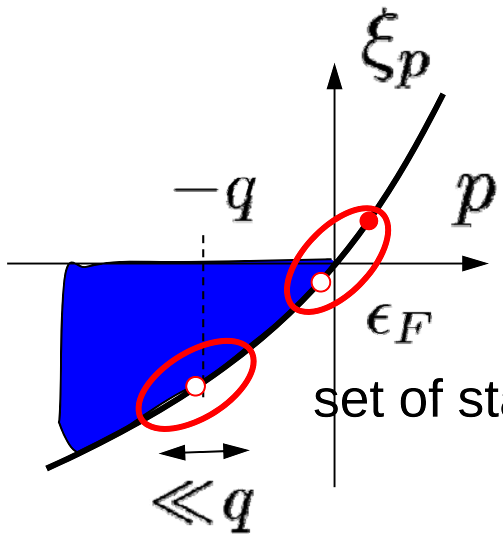
$$\varepsilon = |\xi_k| \text{ for } A(k,\varepsilon)$$
$$\omega = \omega_-(q) \text{ for } S(q,\omega)$$

2. Low-energy dynamics at arbitrary momentum – UNIVERSAL (power-law threshold singularities in the response functions), allows for a phenomenological description – nonlinear Luttinger liquid

3. Shape of the edges, ($\omega = \omega_-(q)$, $\varepsilon = \xi_k$)

are not universal (microscopics)

Phenomenology: a hint from perturbation theory

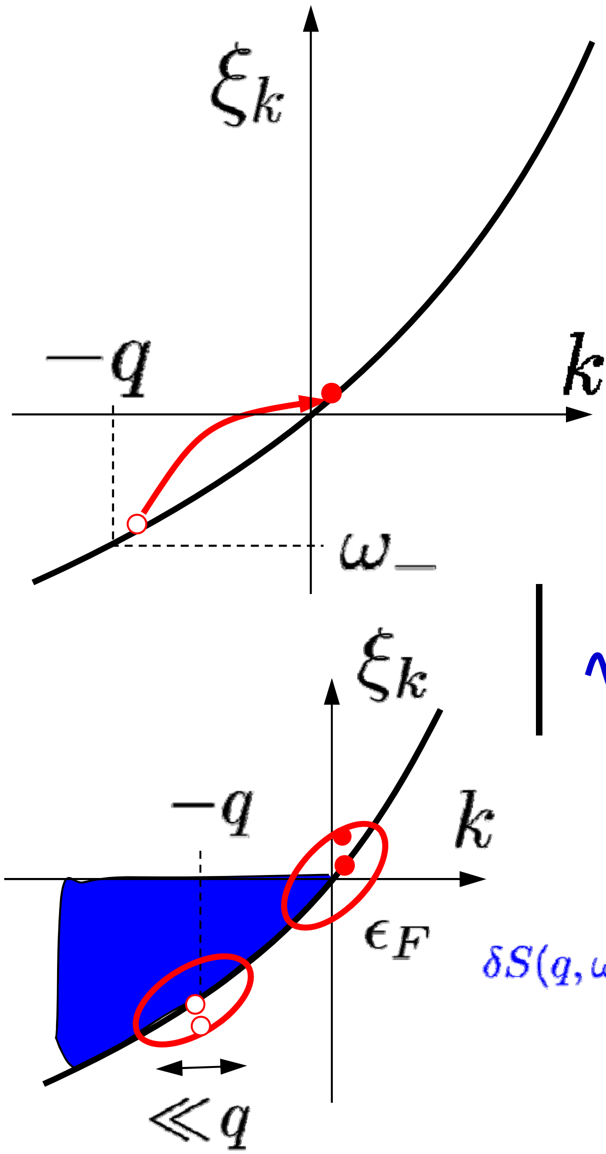


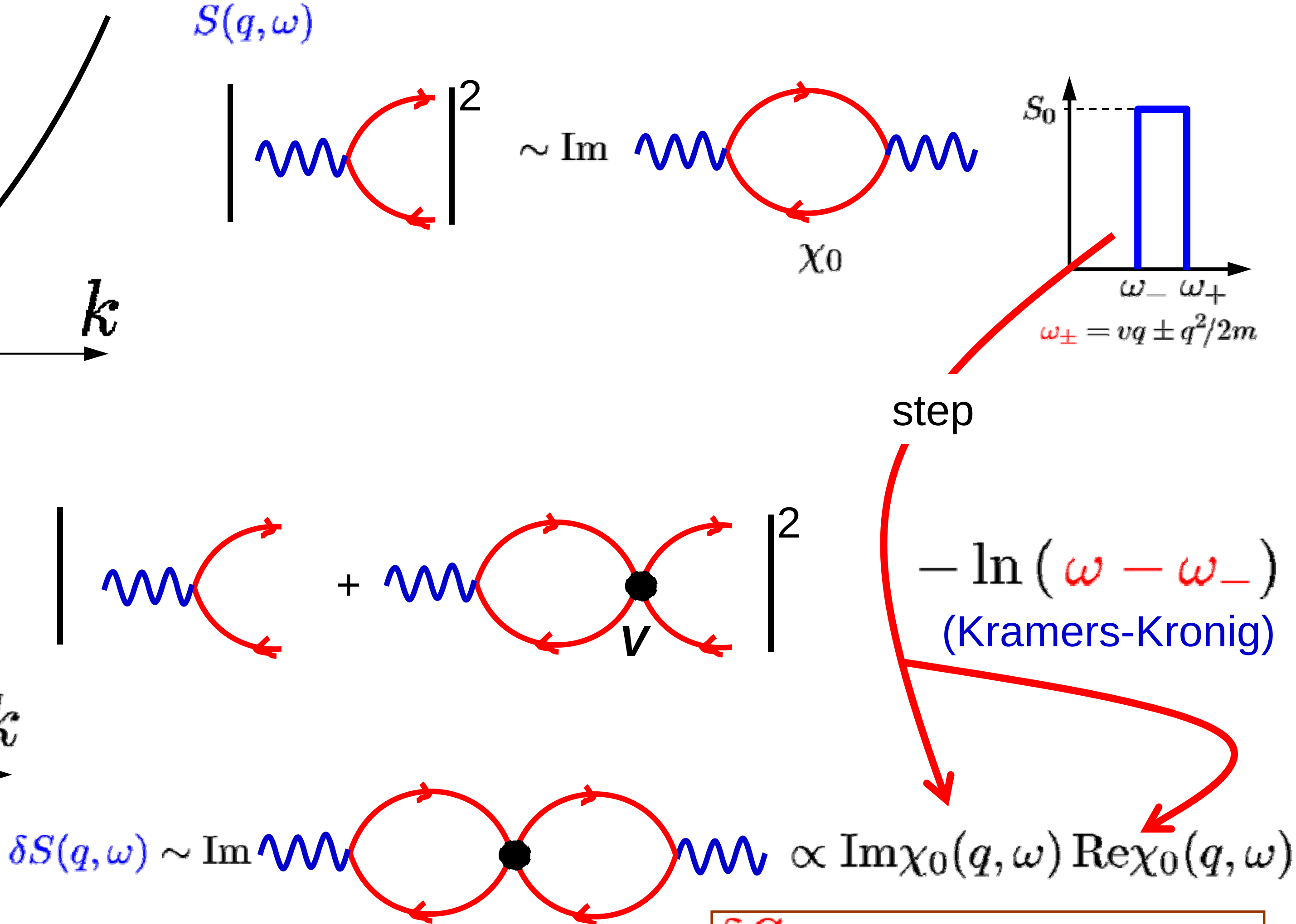
Fermi-edge singularity physics of responses

Shake-up pairs (narrow band containing Fermi level)

set of states admitting 1 hole $v_d \neq v_F$

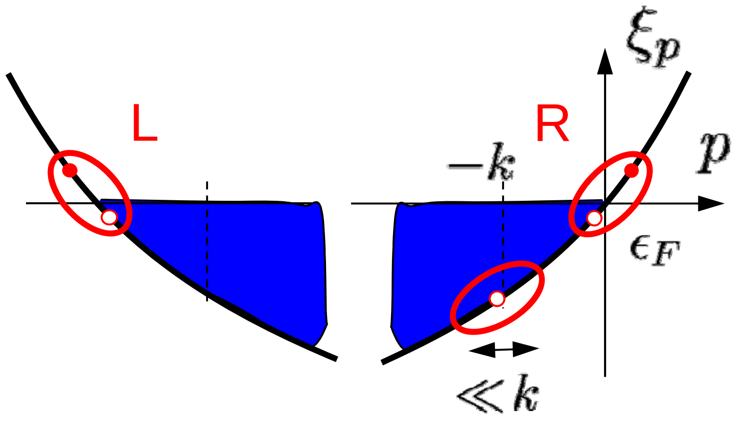
1-st order perturbation theory in interaction





$$\frac{\delta S}{S_0} \propto -V \ln(\omega - \omega_-)$$

Generalization: "quantum impurity"



Left and Right movers:

$$H_0 = \frac{v}{2\pi}\int dx \left(K(\nabla\vartheta)^2 + \frac{1}{K}(\nabla\varphi)^2 \right)$$

$$\varphi, \vartheta = \varphi_L \pm \varphi_R$$

$$K = \frac{\pi n_0}{mv}$$

d: $$H_d = \int dx\, d^\dagger(x)(\varepsilon(k) - i v_d \frac{\partial}{\partial x}) d(x)$$

$$v_d = \partial\varepsilon(k)/\partial k$$

$$H_{int} = \int dx \left(V_\varphi \nabla \frac{\varphi}{2\pi} - V_\theta \nabla \frac{\vartheta}{2\pi} \right) d(x) d^\dagger(x)$$

Phenomenology of interaction constants

$$H_0 = \frac{v}{2\pi}\int dx\left(K(\nabla\vartheta)^2 + \frac{1}{K}(\nabla\varphi)^2\right)$$

$$H_d = \int dx\; d^\dagger(x)(\underline{\varepsilon(k)} - iv_d\frac{\partial}{\partial x})d(x)$$

$$H_{int} = \int dx\left(V_\varphi \nabla\frac{\varphi}{2\pi} - V_\theta \nabla\frac{\vartheta}{2\pi}\right) d(x)d^\dagger(x)$$

$$\delta\rho \neq 0 \qquad\qquad \langle\nabla\varphi\rangle = -\pi\delta\rho$$

$$\frac{1}{2}V_\varphi = \frac{\partial\varepsilon(k)}{\partial\rho} + \frac{\partial\mu}{\partial\rho} = \frac{\partial\varepsilon(k)}{\partial\rho} + \frac{\pi v}{K}$$

Imambekov, 2007

Phenomenology of interaction constants

$$H_0 = \frac{v}{2\pi}\int dx \left(K(\nabla\vartheta)^2 + \frac{1}{K}(\nabla\varphi)^2\right)$$

$$H_d = \int dx\; d^\dagger(x)(\varepsilon(k) - iv_d\frac{\partial}{\partial x})d(x)$$

$$H_{int} = \int dx \left(V_\varphi \nabla\frac{\varphi}{2\pi} - V_\theta \nabla\frac{\vartheta}{2\pi}\right) d(x)d^\dagger(x)$$

$$\frac{1}{m}\frac{\partial\theta}{\partial x} \neq 0$$

$$\langle\nabla\theta\rangle \neq 0$$

use Galilean invariance, Baym&Ebner, 1967

$$\frac{1}{2}V_\theta = \frac{\partial\varepsilon(k)}{\partial k} - \frac{k}{m}$$

Mapping on free chiral fermions

$$\varphi, \vartheta \to \tilde{\varphi} \pm \tilde{\vartheta} \to \tilde{\varphi}_{L,R}$$

$$H_0 = \frac{v}{2\pi} \int dx \left((\nabla \tilde{\varphi}_L)^2 + (\nabla \tilde{\varphi}_R)^2 \right)$$ Free chiral (L,R) fermions

$$H_d = \int dx\ d^\dagger(x)(\varepsilon(k) - i v_d \frac{\partial}{\partial x}) d(x)$$ impurity

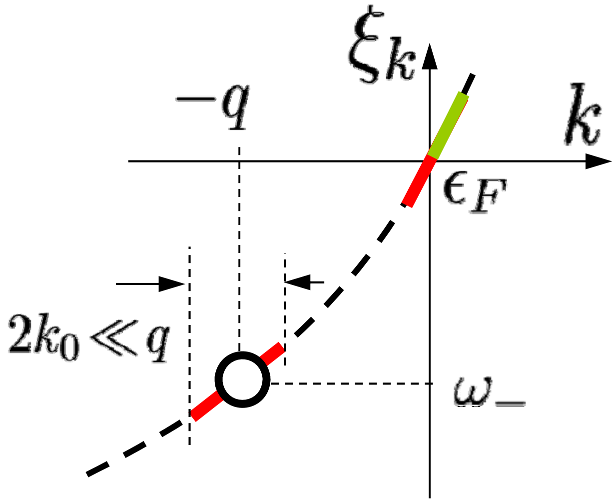
$$H_{int} = \int dx \left(\tilde{V}_L \nabla \frac{\tilde{\varphi}_L}{2\pi} - \tilde{V}_R \nabla \frac{\tilde{\varphi}_R}{2\pi} \right) d(x) d^\dagger(x)$$

Forward-scattering of L and R fermions off impurity

Scattering phase shifts of L and R off impurity: $\frac{\delta_\pm}{2\pi} = \frac{\tilde{V}_{R,L}}{v \mp v_d}$

$$V_\varphi, V_\theta \to \delta_+, \delta_-$$

Operators: Hole creation



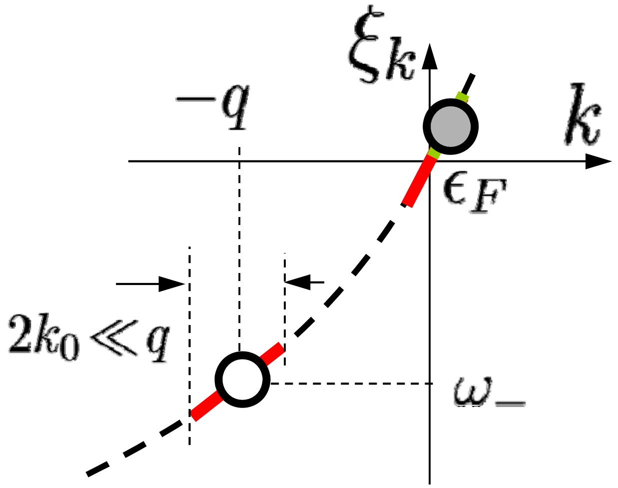
Creating a hole close to the threshold:

$$\psi(x) \rightarrow \hat{d}(x)$$

$$G^R(x,t) \propto \langle \hat{d}^\dagger(x,t)\hat{d}(0,0)\rangle\theta(t)$$

$$A(k,\omega) = -\mathrm{Im}\, G^R(k,\omega)$$

Operators: Hole density



$$\psi(x) \to \hat{d}(x)$$

$$\Psi_R^\dagger(x,t) \propto e^{-i\varphi_R(x,t)}$$

Density operator close to the threshold:

$$\hat{n}^\dagger(x) \to \psi_R^\dagger(x)\hat{d}(x) \propto e^{-i\varphi_R(x)}\hat{d}(x)$$

$$\chi(q,\omega) = \left\langle -i\theta(t)\left[\hat{n}(x,t), \hat{n}^\dagger(0,0)\right] \right\rangle_{q,\omega}$$

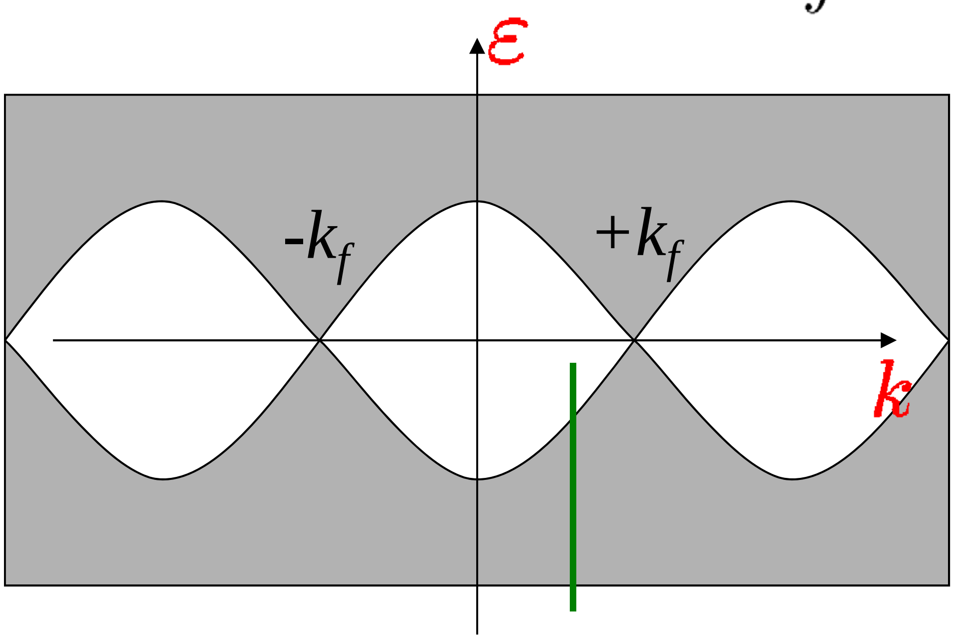
$$S(q,\omega) = -2\mathrm{Im}\chi(q,\omega)$$

Observables: Spectral function $A(k,\omega)$

$$A(k,\omega) \propto \theta(\varepsilon(k)-\omega)\left|\frac{1}{\varepsilon(k)-\omega}\right|^{1-\left(\frac{\delta_+(k)}{2\pi}\right)^2-\left(\frac{\delta_-(k)}{2\pi}\right)^2}$$

$$\delta_+,\delta_- \leftrightarrow V_\varphi, V_\theta$$

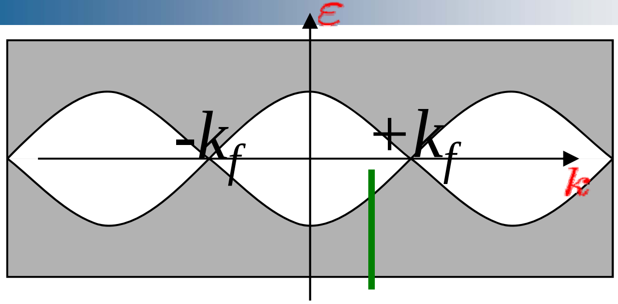
$$H_{int} = \int dx \left(V_\varphi \nabla \frac{\varphi}{2\pi} - V_\theta \nabla \frac{\theta}{2\pi}\right) d(x) d^\dagger(x)$$



$$\frac{1}{2}V_\varphi = \frac{\partial \varepsilon(k)}{\partial \rho} + \frac{\pi v}{K}$$

$$\frac{1}{2}V_\theta = \frac{\partial \varepsilon(k)}{\partial k} - \frac{k}{m}$$

Observables



$$A(k,\omega) \propto \theta(\varepsilon(k)-\omega)\left|\frac{1}{\varepsilon(k)-\omega}\right|^{1-\left(\frac{\delta_+(k)}{2\pi}\right)^2-\left(\frac{\delta_-(k)}{2\pi}\right)^2}$$

$$\frac{\delta_\pm(k)}{2\pi} = \frac{\frac{1}{\sqrt{K}}\left(\frac{k}{m}-\frac{\partial\varepsilon(k)}{\partial k}\right) \pm \sqrt{K}\left(\frac{1}{\pi}\frac{\partial\varepsilon(k)}{\partial\rho}+\frac{v}{K}\right)}{2\left(\pm\frac{\partial\varepsilon(k)}{\partial k}-v\right)}$$

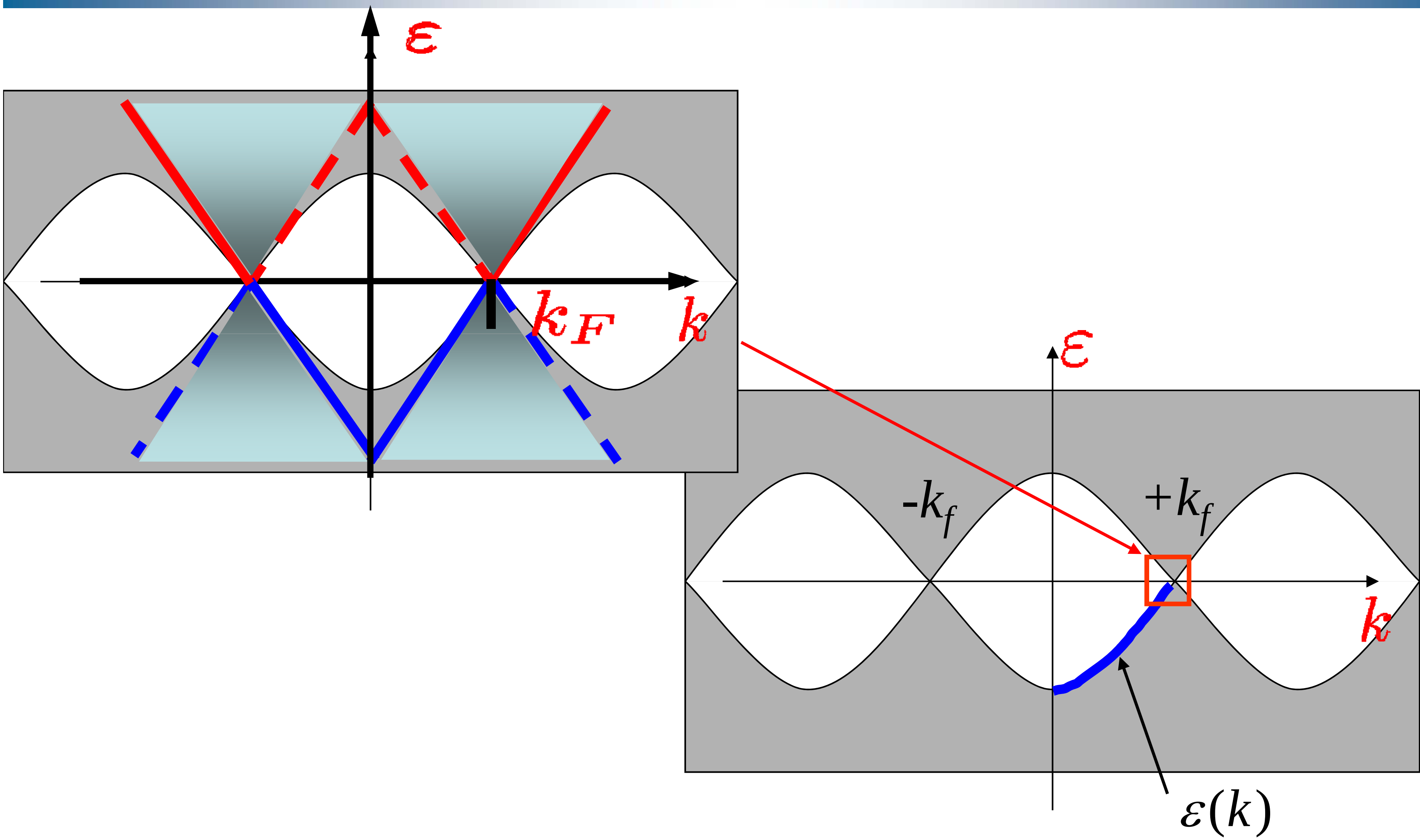
similar: structure factor $S(q,\omega)$

$$K = \frac{\pi n_0}{mv}$$

s=1/2 fermions: 4 modes (L,R; s,c), edge=*spinon* spectrum:

$\delta^c_\pm$ similar to $\delta_\pm(k)$, $\delta^s_\pm = 0$ due to SU(2)

Crossover to linear Luttinger liquid

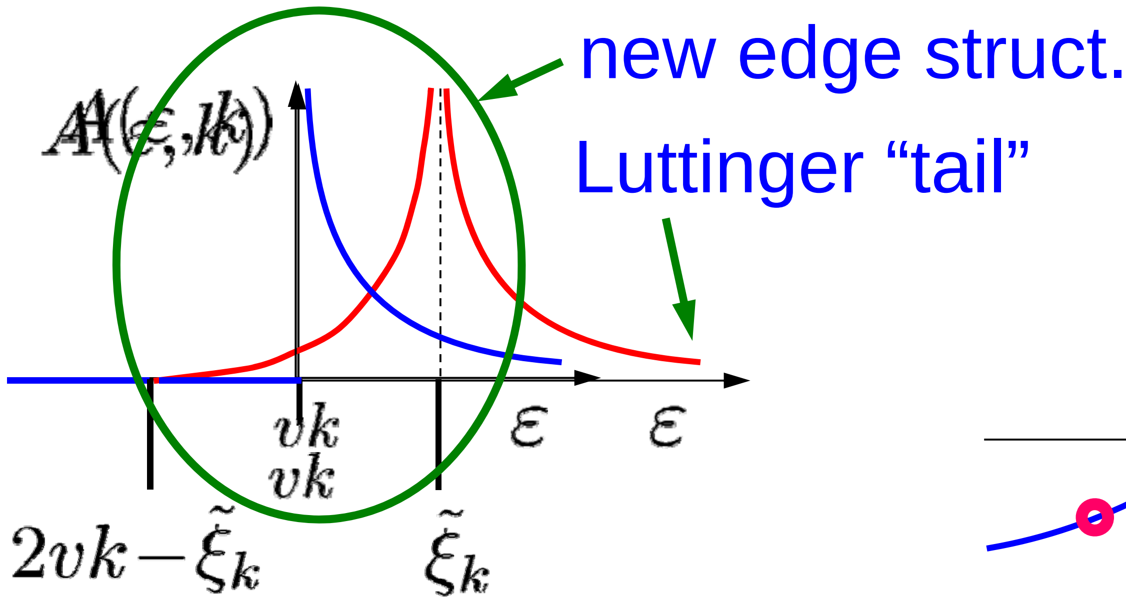


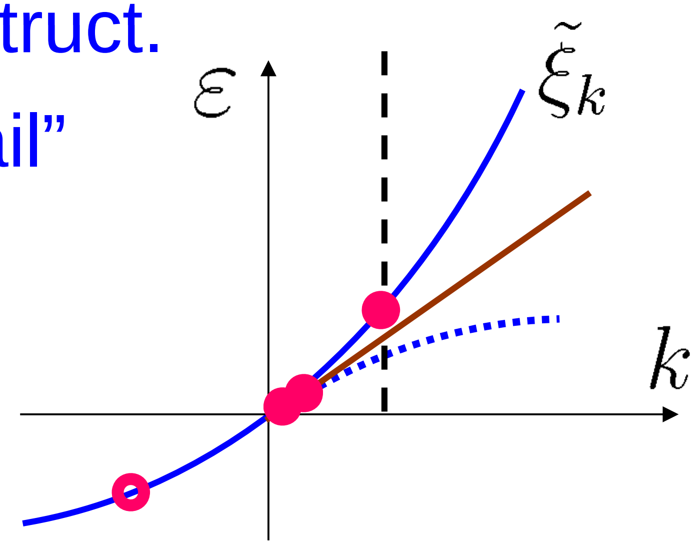
Position of the edge completely defines the singularities!

Spectral function threshold at $p > k_F$

$|\varepsilon - vk| \ll k\,,\ k/k_F \to 0$ (here *k* is measured from k_F)

Finite mass of fermion – new energy scale $\delta\omega = \dfrac{k^2}{2m_*}$





Universal crossover function

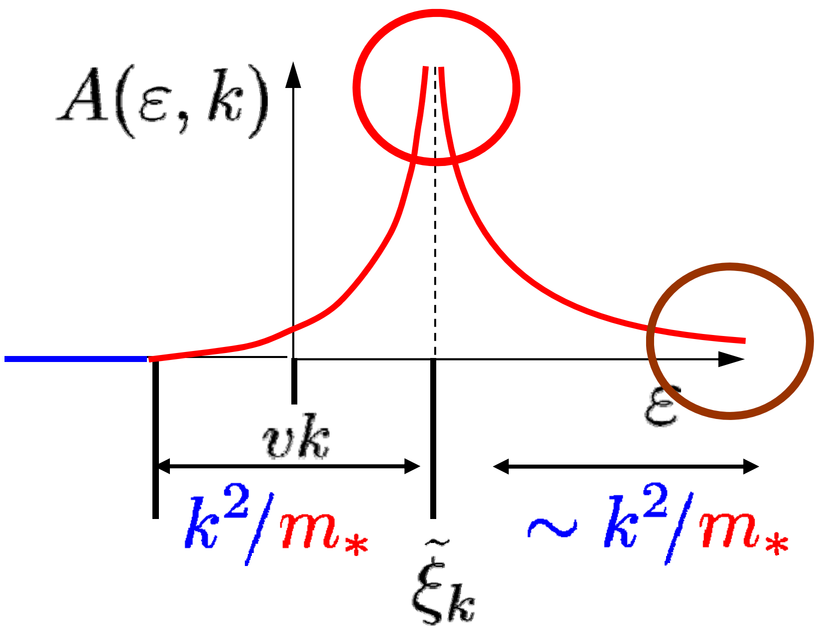
$$A(\varepsilon, k) = A\left(\frac{\varepsilon - vk}{\delta\omega}\right)$$

True *vs.* Luttinger liquid exponents

$$A(\varepsilon, k) \propto |\varepsilon - \tilde{\xi}_k|^{\gamma_{\text{true}}}$$

$$K = \frac{\pi n_0}{mv}$$

$$\gamma_{\text{true}} = \frac{1}{2}\left[\left(1 - \frac{1}{\sqrt{K}}\right)^2 + \left(1 - \sqrt{K}\right)^2\right] - 1$$



$$A(\varepsilon, k) \propto (\varepsilon - vk)^{\gamma_{\text{L}}}$$

$$\gamma_{\text{L}} = \frac{1}{4}\left(\frac{1}{K} + K\right) - \frac{3}{2}$$

Luttinger liquid of *s=1/2* fermions

spin&charge

$$\varphi_{\uparrow,\downarrow} = \frac{1}{2}(\varphi_c \pm \varphi_s)$$

rigidity: Fermi+Coulomb

φ_c + - + - + - + - + - + - + - + - + - + -

$$v_c > v_F$$

rigidity: Fermi only

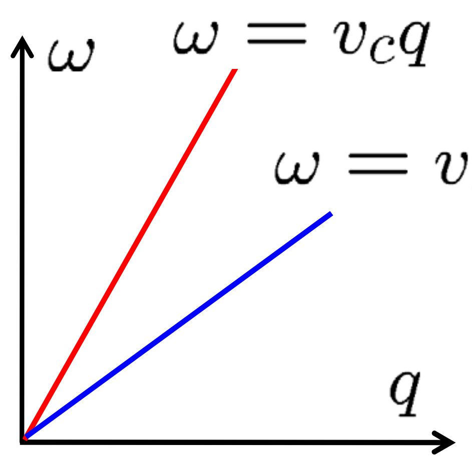
φ_s ↑↓↑↓↑↓↑↓↑↓↑↓↑↓↑↓

$$v_s = v_F$$

$$H_\nu = \frac{v_\nu}{2\pi}\int dx \left[K_\nu(\nabla\theta_\nu)^2 + \frac{1}{K_\nu}(\nabla\phi_\nu)^2\right]$$

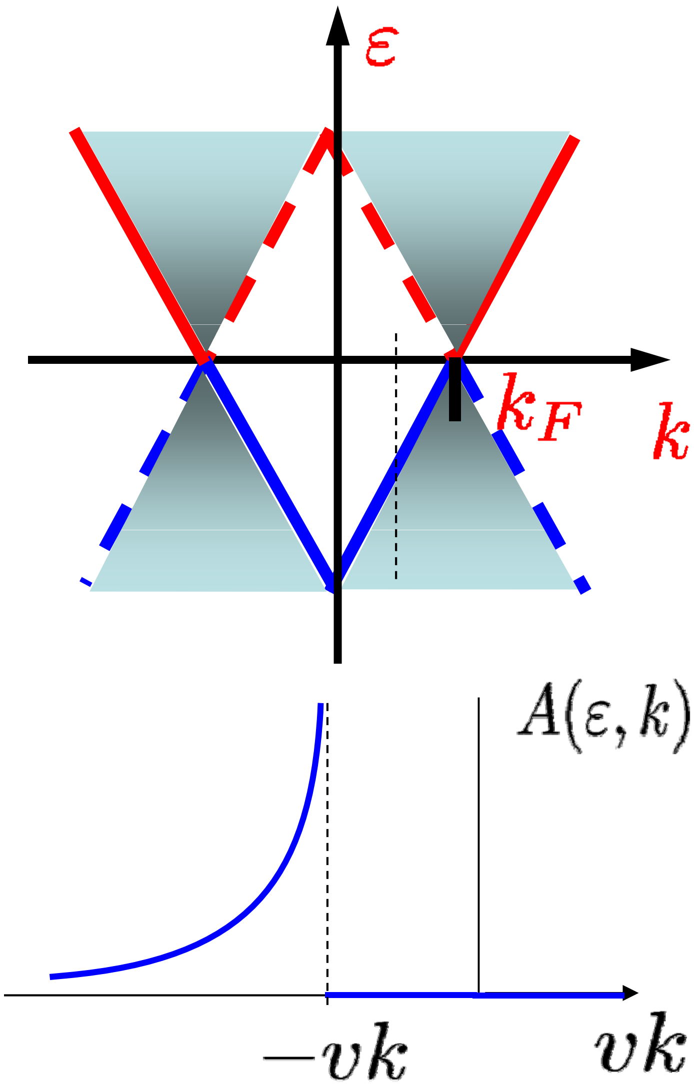
$$K_s \to 1$$

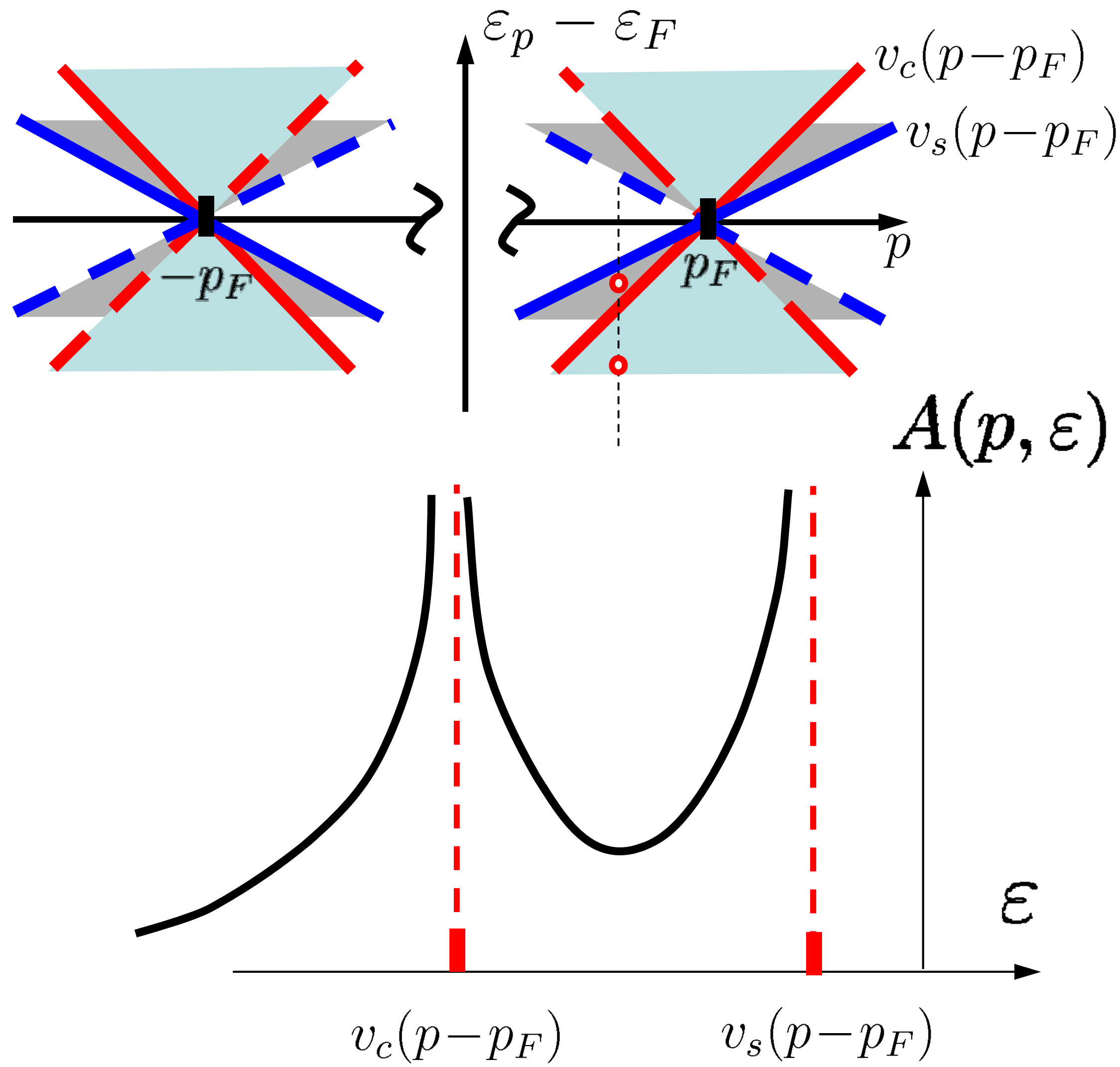
$K_c < 1$ (repulsion)



Getting closer to experiments: *s=1/2*

Tunneling rate at given (p,ε) → spectral function $A(p,\varepsilon)$

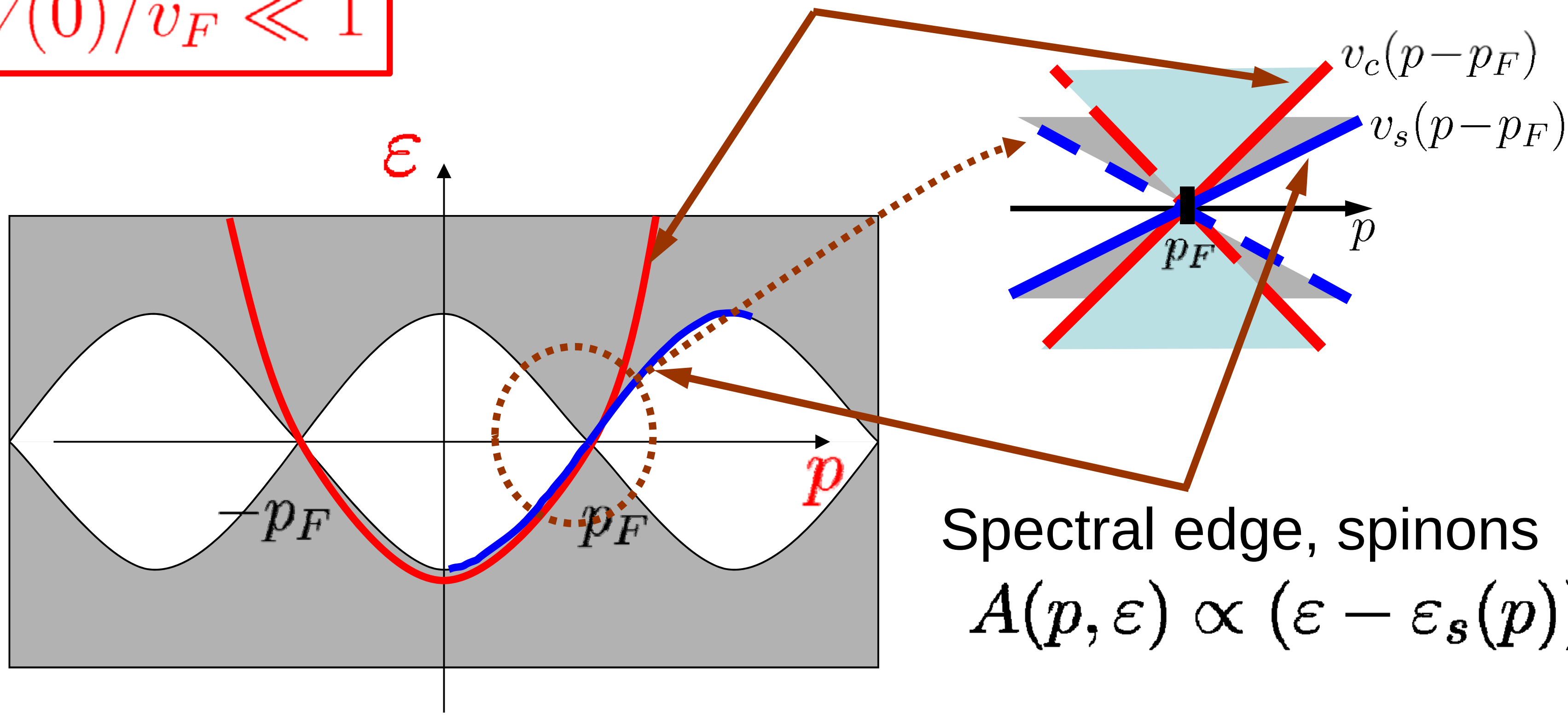




Spinon and holon modes away from $\pm p_F$

$V(0)/v_F \ll 1$

Broadened mass shell states, holons



Spectral edge, spinons

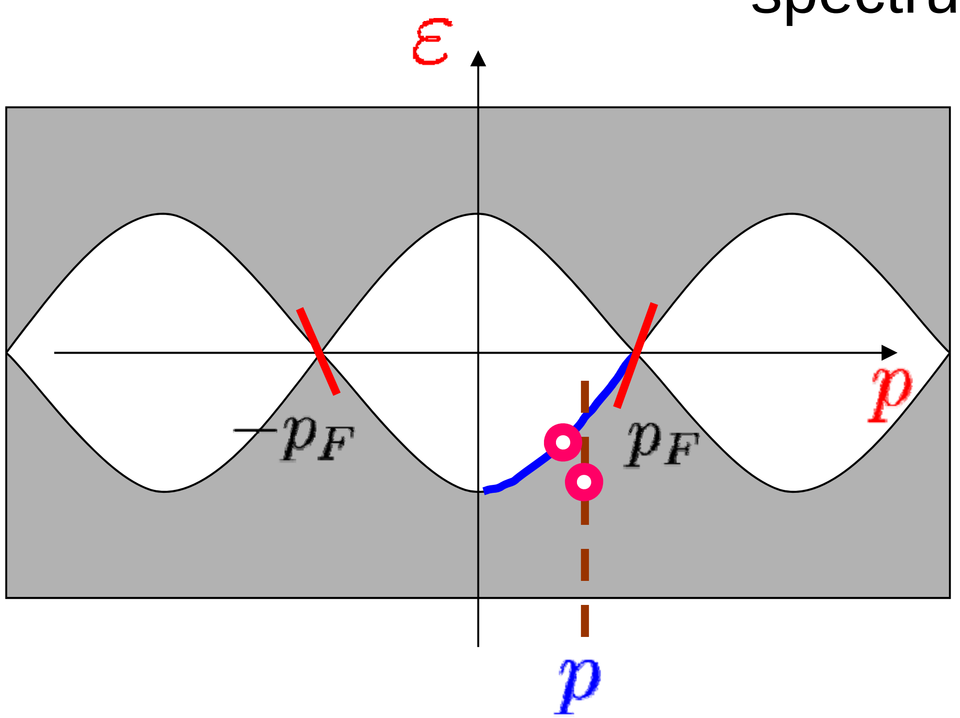
$$A(p,\varepsilon) \propto (\varepsilon - \varepsilon_s(p))^\gamma$$

$p < p_F$	$\lvert p-p_F\rvert \lesssim mV(0)$	$p > p_F$
$\gamma \to -1$	$\gamma = -1/2$	$\gamma \to 0$

Spin-charge separation at arbitrary *p*

Any interaction strength

Threshold in $A(p,\varepsilon)$ at the spinon spectrum $\varepsilon_s(p)$



Tunneling creates one spinon with energy $\varepsilon_s(p)$ and low-energy “shake-up” holons (but not spinons)

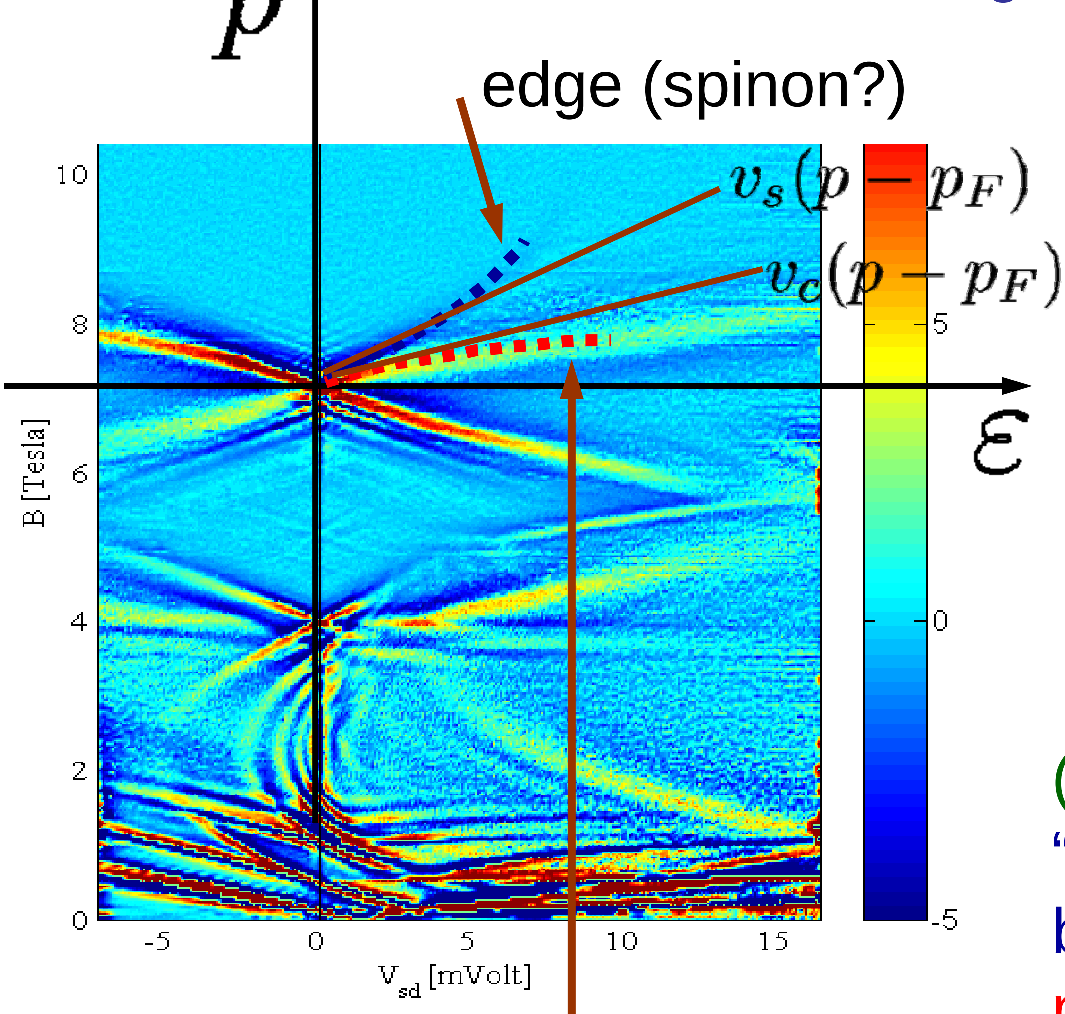
$$A(p,\varepsilon) \propto (\varepsilon - \varepsilon_s(p))^{\gamma}$$

$\gamma(p)$ can be expressed in terms of $\frac{\partial \varepsilon_s(p,\rho)}{\partial p}, \frac{\partial \varepsilon_s(p,\rho)}{\partial \rho}$

Schmidt, Imambekov, LG, PRL **104**, 116403 (2010)

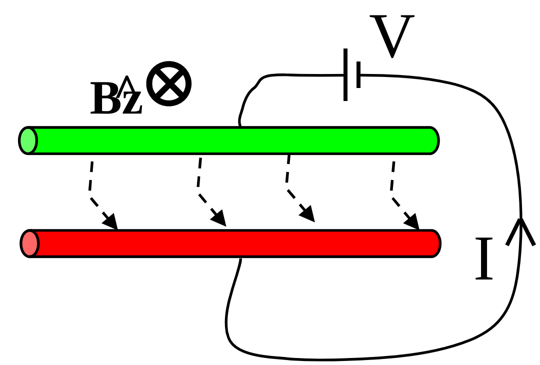
Experiment: spectrum of excitations

Momentum-conserving tunneling



(broadened) particle mass shell

Auslaender *et al.*, *Science* **308**, 88–92 (2005)

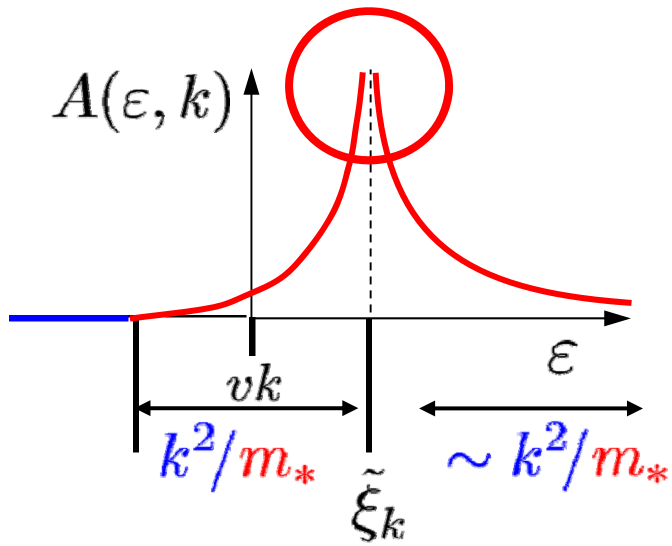


momentum “boost” $\Delta p = edB/c$

(1) opposite curvatures of “spinon” and “particle” branches; (2) particle peak vs spinon threshold

Kinetics of 1D quantum liquid

• **Broadening of singular responses within continuum at zero temperature, relaxation of finite-energy excitations**



•• **Finite-temperature relaxation:**

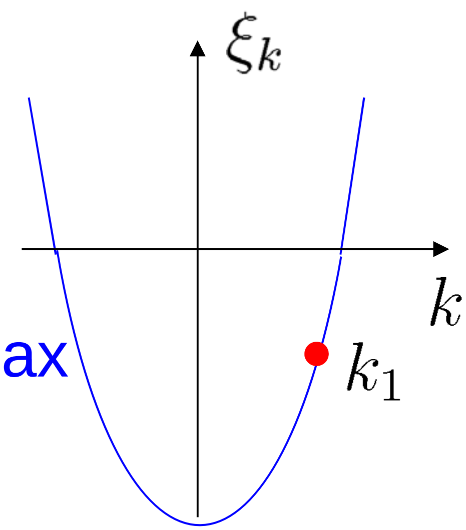
(1) low-energy processes;

(2) processes with a finite activation energy

••• **Thermalization**

Spinless fermions: $1/\tau(k)$ at $T=0$

Perturbation in interaction

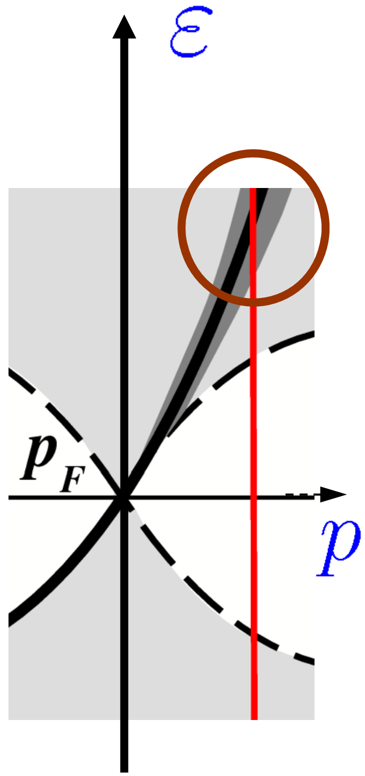


holes do not relax

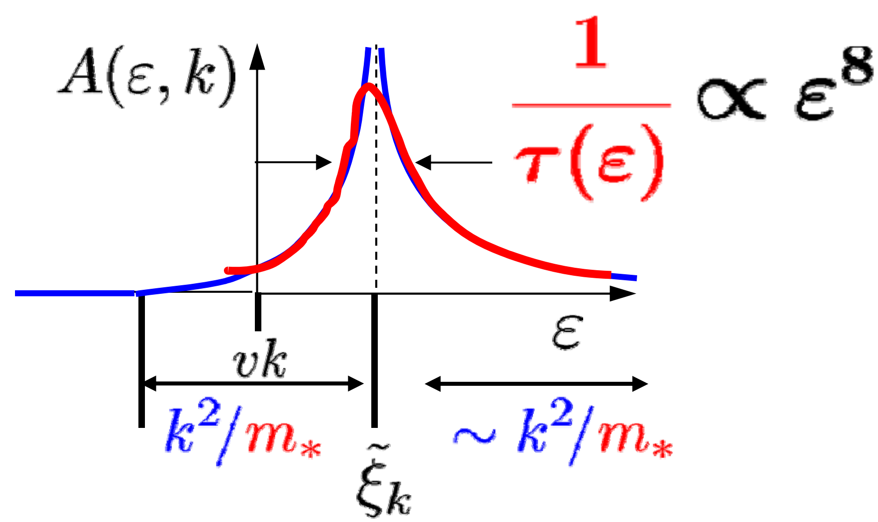
$v_h \leq v_F$

Leading Corrections to Nonlinear Luttinger

Spinless particle relaxation, generic interaction

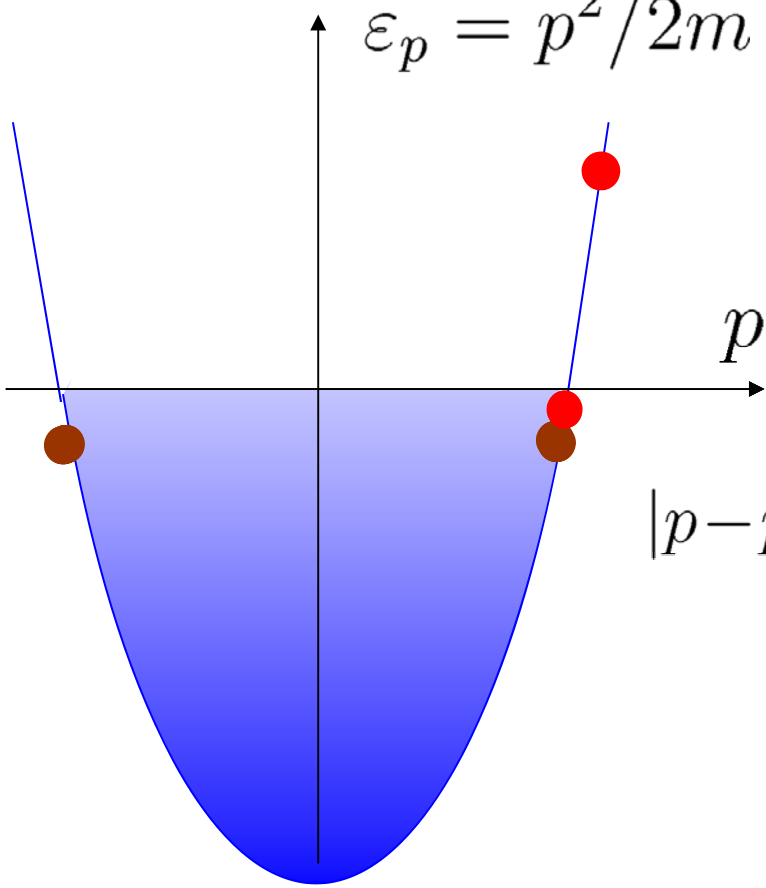


mass shell $\neq$ edge of $A(\varepsilon, p)$



Smearing of the spectral function's singularity at the mass shell: apparently $\propto (k/k_F)^8$ (Matveev 2012, private comm)

Particles (s=1/2) : finite lifetimes



2-particle collisions ineffective

$v > v_F$, “Cherenkov radiation” of two *p-h* pairs (“bosons”)

$$|p - p_F| \gtrsim mV(0) \qquad V(0)/v_F \ll 1$$

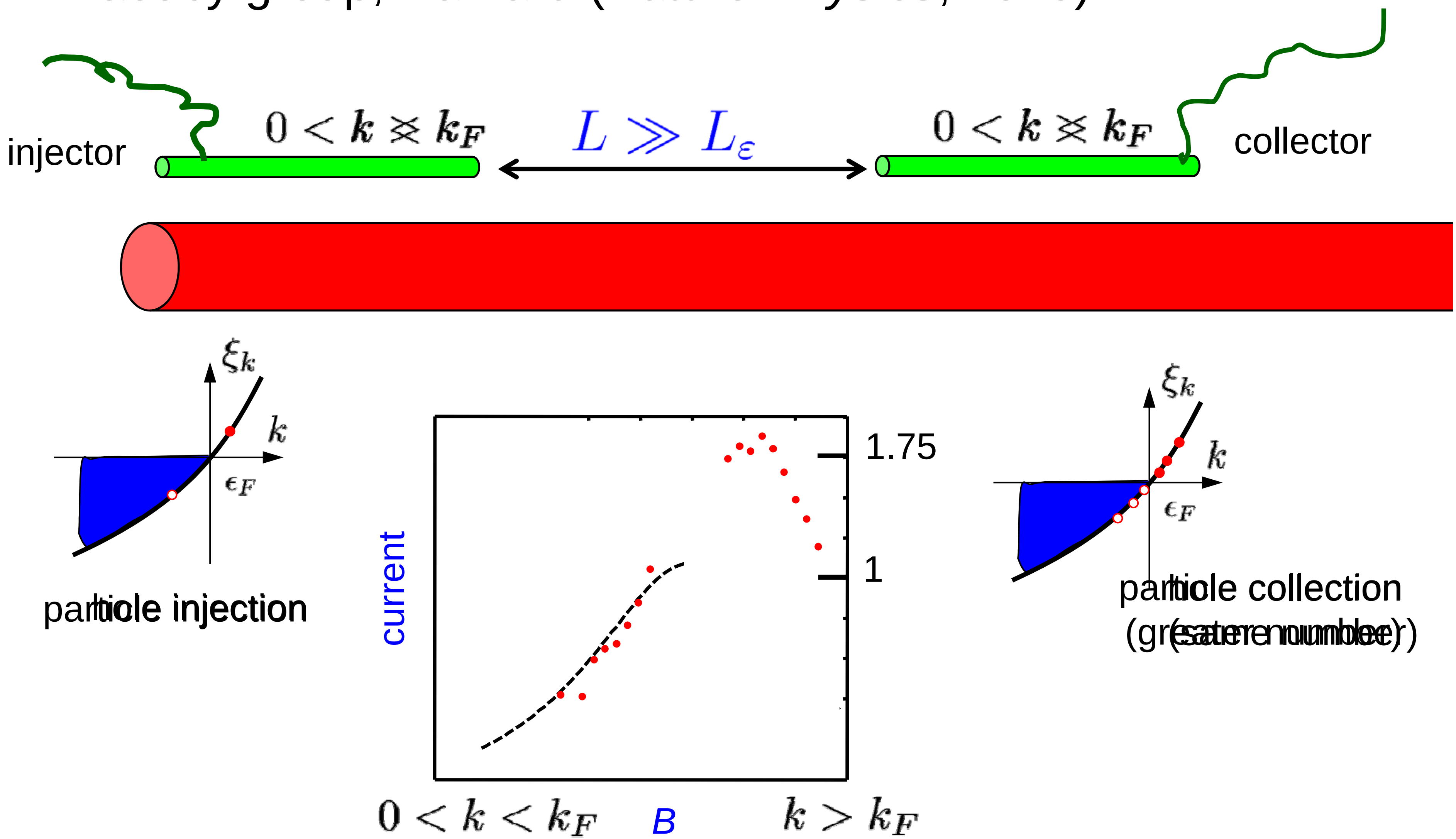
$$\frac{1}{\tau} \propto V^2(0)V^2(2p_F) \cdot (p - p_F)^2$$

3-particle collisions

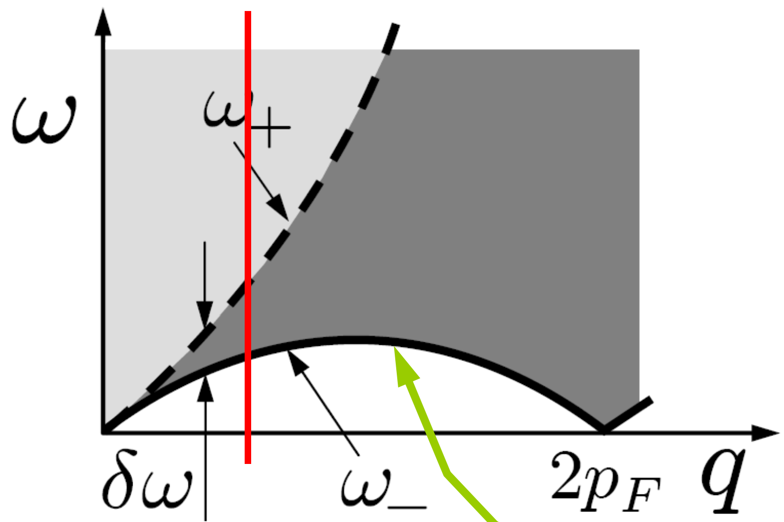
(*s=1/2* scattering channel, non-integrable potential, *T=0*)

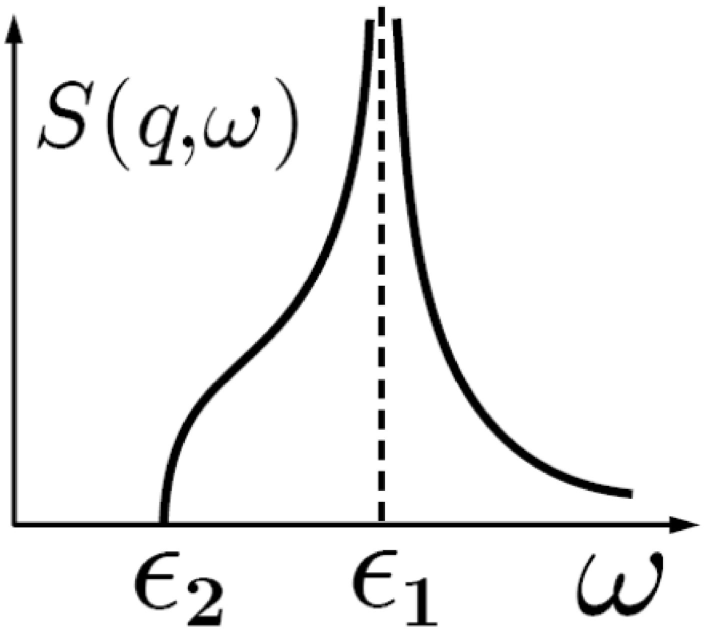
Particles relax, holes do not

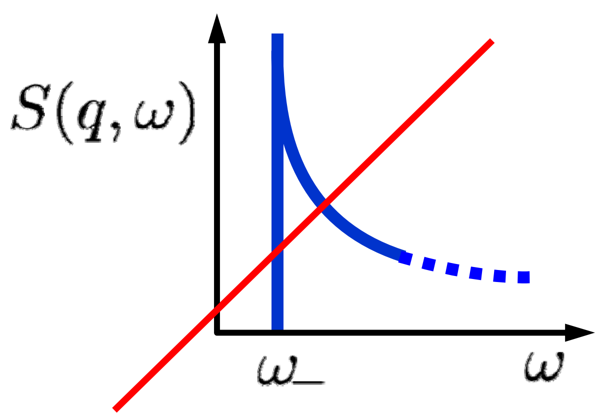
A. Yacoby group, Harvard (*Nature Physics*, 2010)



1D Fermions 1D Bosons – Structure Factor







$$\frac{S(q,\omega)}{S_0} = \left[\frac{\delta\omega}{\omega - \omega_+(q)}\right]^{\mu(q)}$$

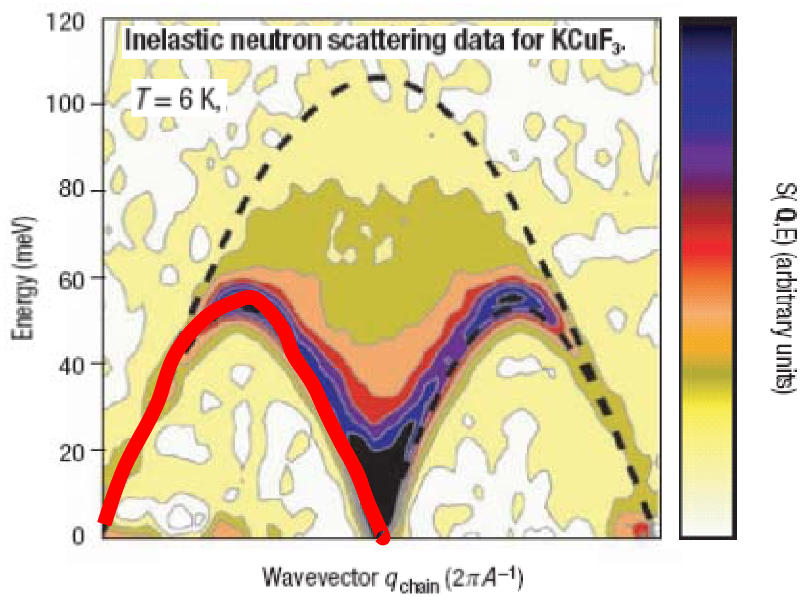
Divergence in continuum
(integrable Lieb-Liniger model)

Spin structure factor exponents

Jordan-Wigner transformation

AFM spin chain $\longleftrightarrow$ 1D spinless fermions

1D spinless fermions $\longleftrightarrow$ Quantum impurity

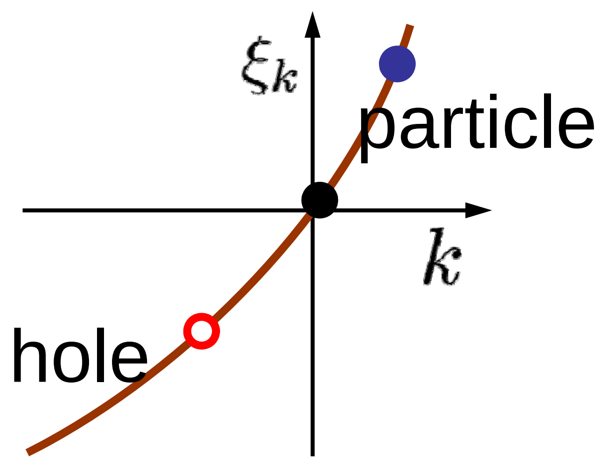


S. Nagler, et al 2005

$S(q,\omega) \propto (\omega - \omega_{-}(q))^{-1/2}$ Any *q* !!!

Conclusions

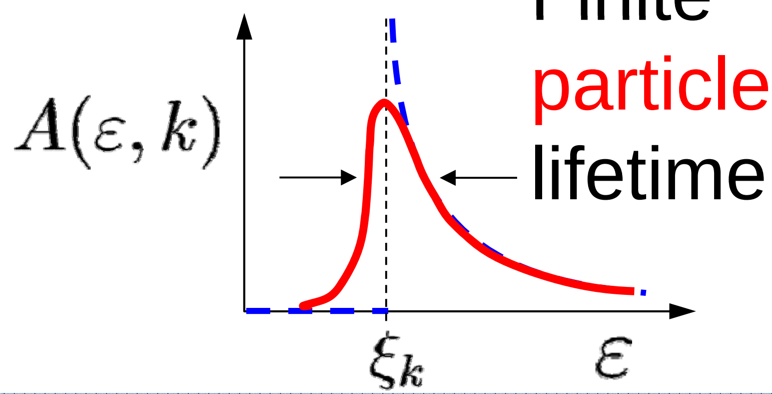
curvature+interaction in 1D:



particle-hole asymmetry

$$v_h < v_F < v_p$$

- **New singular behavior**
- **Asymmetry in lifetimes**



Conclusions

$$A(k,\omega) \propto \theta(\varepsilon(k)-\omega)\left|\frac{1}{\varepsilon(k)-\omega}\right|^{1-\left(\frac{\delta_+(k)}{2\pi}\right)^2-\left(\frac{\delta_-(k)}{2\pi}\right)^2}$$

Momentum dependent exponents

can be related to

$$\delta_\pm(k) \longleftrightarrow \frac{\partial\varepsilon(k,\rho)}{\partial k}, \frac{\partial\varepsilon(k,\rho)}{\partial\rho}$$

Direct relation between exponents and energy spectrum

More applications: electron (s=1/2) liquid, spinor Bose liquid, 1D magnets [use of SU(2) symmetry]; other responses

New use of TBA: dynamics of integrable models

Nonlinear Luttinger Liquids

collaborations:

Adilet Imambekov (Rice University), Alex Kamenev (U. of Minnesota), Thomas Schmidt, Shina Tan (Yale University), Michael Pustilnik (Georgia Tech), Maxim Khodas (Iowa Univ), Felix von Oppen (Berlin Free University)

discussions:

I. Affleck, R. Pereira (UBC), F. Essler (Oxford U), …

reading:

- One-dimensional quantum liquids: Beyond the Luttinger liquid paradigm
  Adilet Imambekov, Thomas L. Schmidt, and Leonid I. Glazman
  Accepted Thursday Feb 23, 2012 Rev. Mod. Phys.; arXiv:1110.1374