Classification of Topological Insulators

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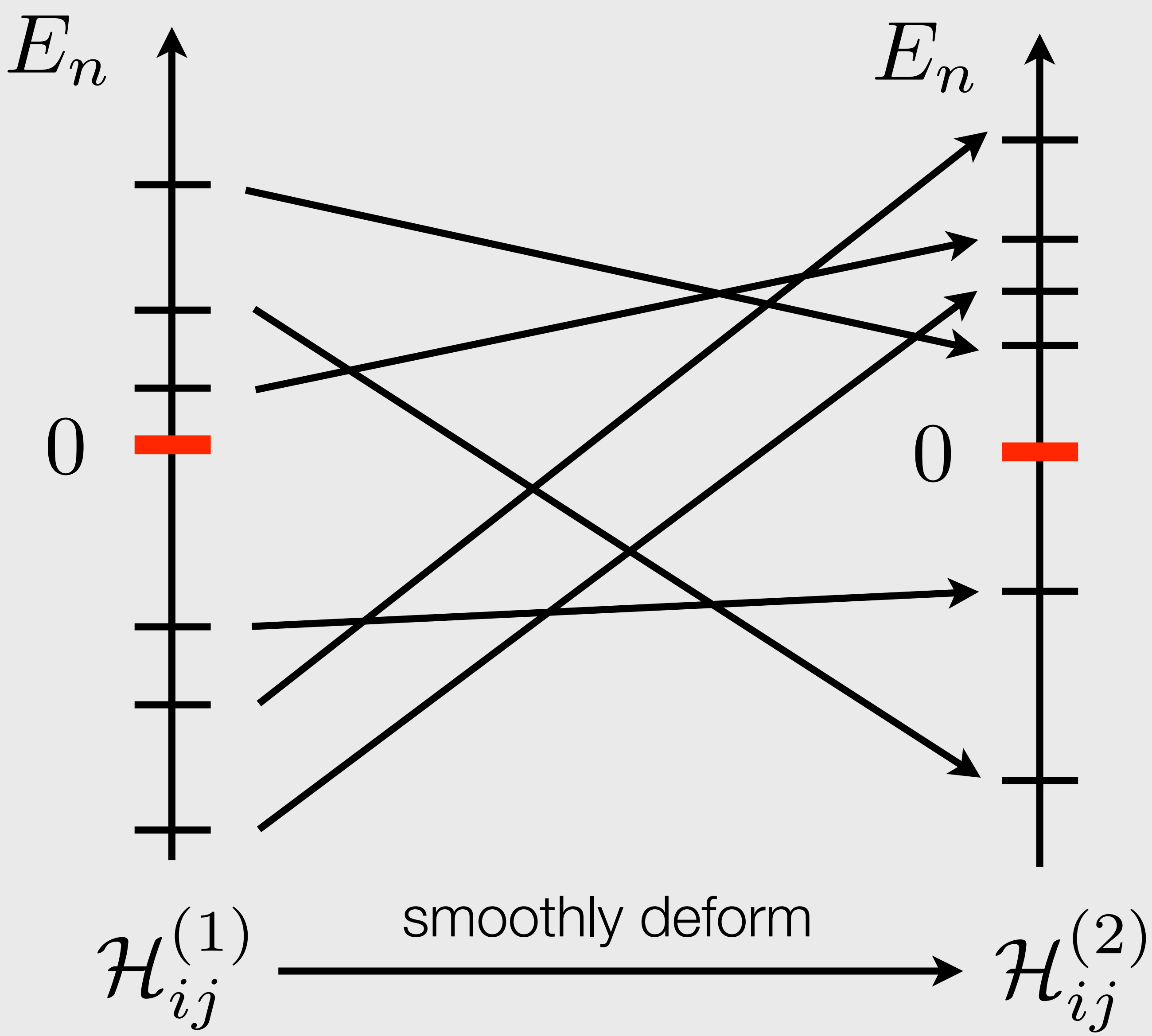
Classes of topologically distinct Hamiltonians

$\mathcal{H}_{ij}$ matrix (Hamiltonian) E_n eigenvalues

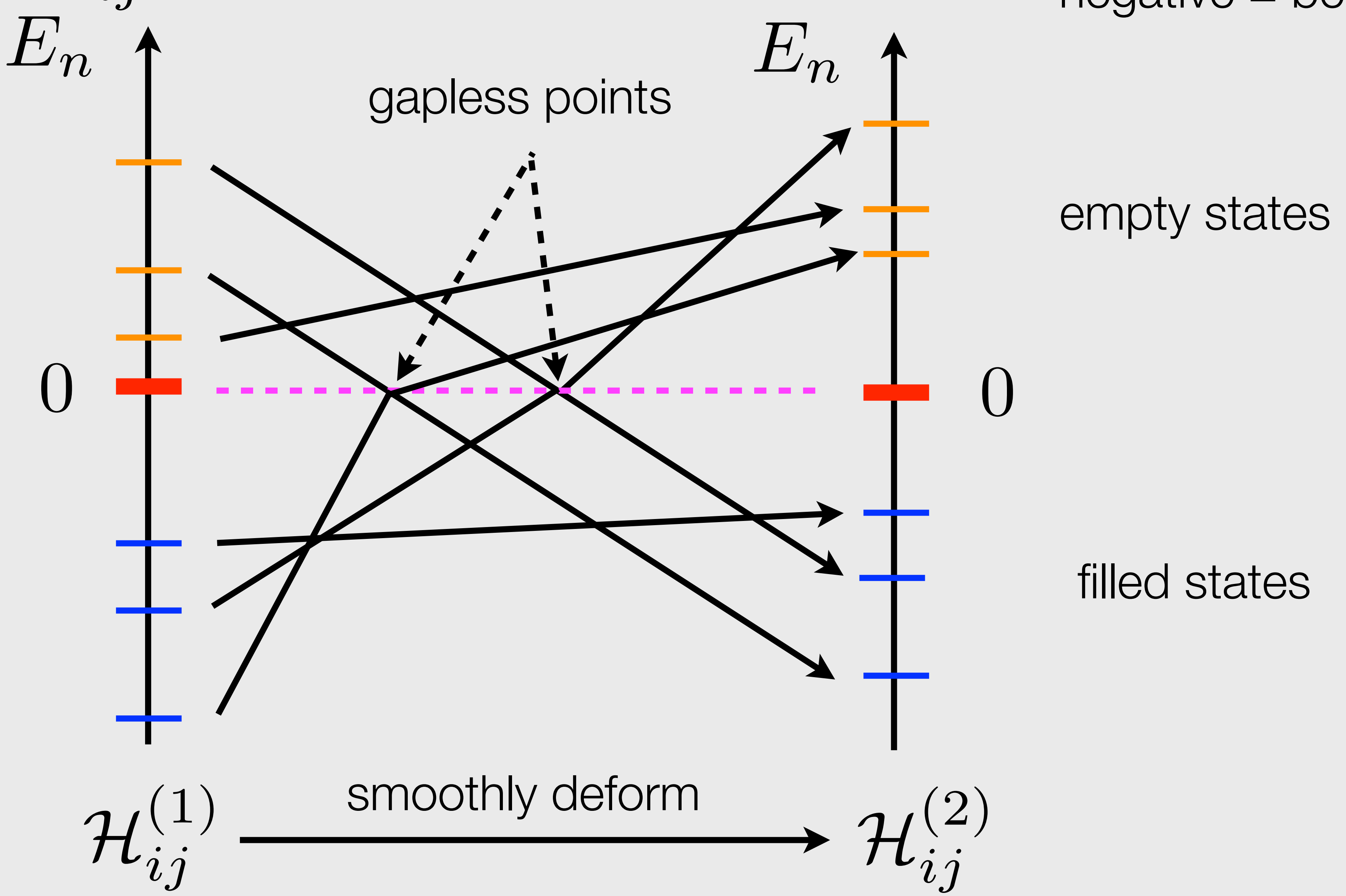


Two matrices are **topologically equivalent** if one can be deformed into another without any of its energy levels ever becoming equal to 0.

Application of topological classes

$$\hat{H} = \sum_{ij} \mathcal{H}_{ij}\, \hat{a}_i^\dagger \hat{a}_j$$ fill negative energy levels with fermions

negative = below the chemical potential



Topological invariants

Mathematical expressions which take integer values and change only if $\mathcal{H}$ acquires a zero eigenvalue (acquires zero energy)

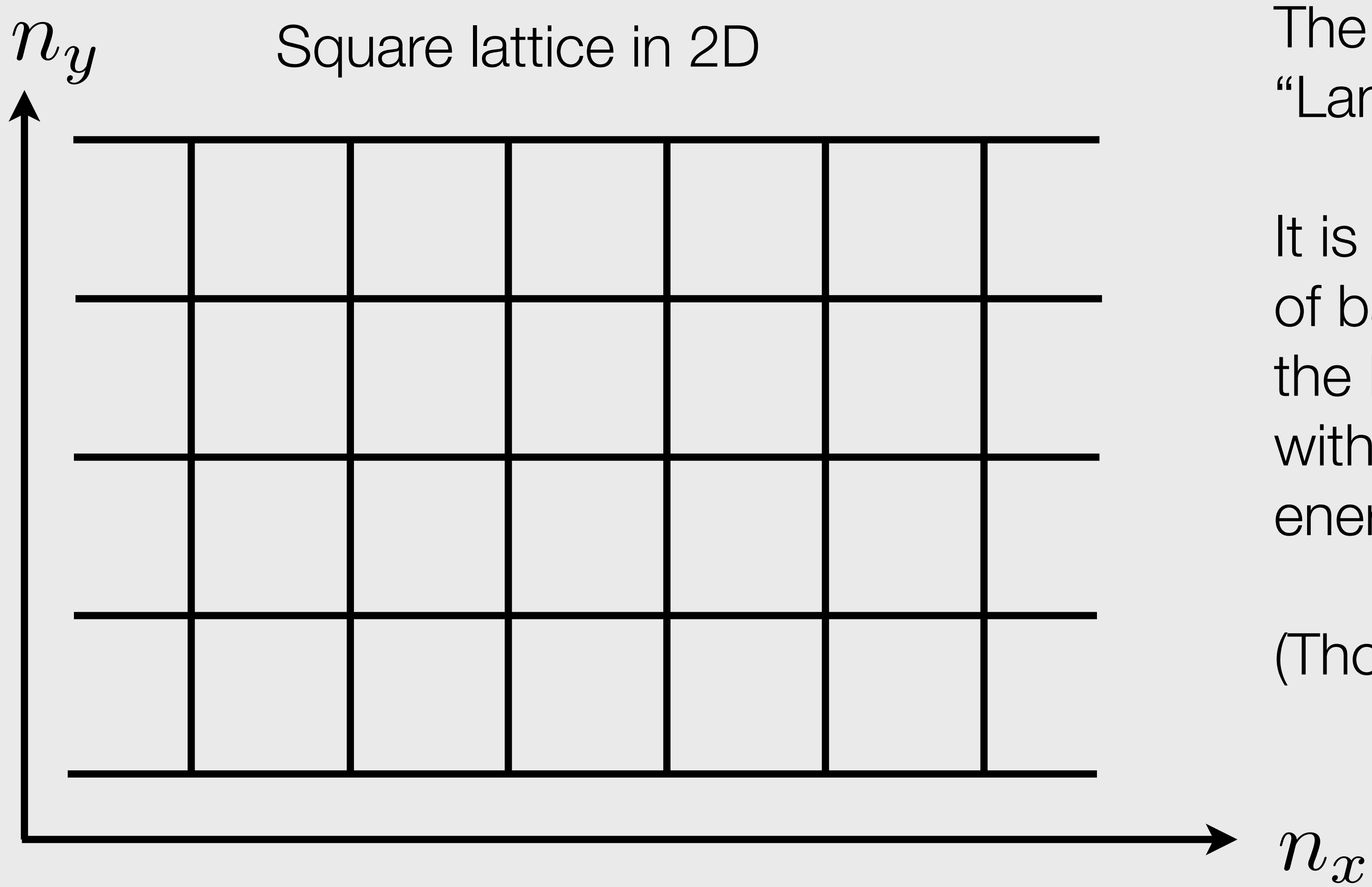
For example: $N_0 =$ # of levels below zero

very simple topological invariant

But there are many more less trivial invariants
(more on that later)

Example: particle in 2D in a magnetic field

$$\hat{H} = t \sum_{n_x, n_y} \left[\hat{a}^\dagger_{n_x+1, n_y} \hat{a}_{n_x, n_y} + e^{2\pi i q n_x} \hat{a}^\dagger_{n_x, n_y+1} \hat{a}_{n_x, n_y} + \text{h.c.} \right] - \mu \sum_{n_x, n_y} \hat{a}^\dagger_{n_x, n_y} \hat{a}_{n_x, n_y}$$



The spectrum consists of 1/*q*-bands (or "Landau levels") $E_n(k_x, k_y)$

It is not possible to change the number of bands below 0 by smoothly changing the Hamiltonian (including by changing **μ**) without tuning through a point with zero energy single-particle states

(Thouless et al, 1982)

$u(k_x, k_y; \vec{r})$ Bloch waves

$$\sigma_{xy} = \frac{ie^2}{2\pi h} \int d^2k \int d^2r \left(\frac{\partial u^*}{\partial k_x} \frac{\partial u}{\partial k_y} - \frac{\partial u^*}{\partial k_y} \frac{\partial u}{\partial k_x} \right)$$

topological invariant (Chern number)

Chern number in terms of Green's functions

$$G_{n_x,n_y} = [i\omega - \mathcal{H}]^{-1}$$

$$G_{ab\in \text{the basis}}(k_x, k_y) = \sum_{n_x,n_y \in \text{Bravais lattice}} e^{-i(n_x k_x + n_y k_y)} G(n_x, n_y)$$

Chern number (an alternative form)

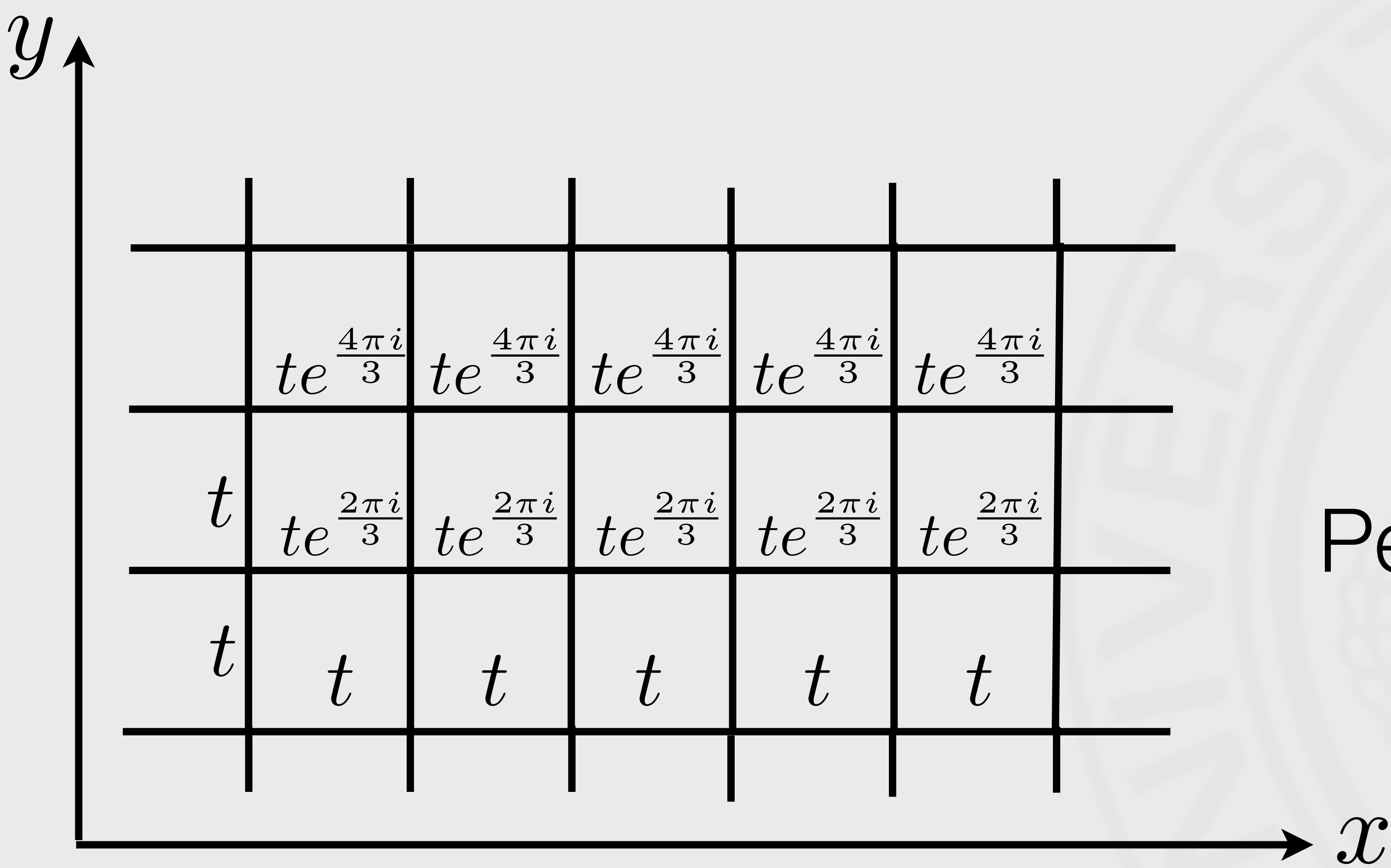
$$N_2 = \frac{1}{24\pi^2} \sum_{\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma} \int d\omega dk_x dk_y \, \text{tr} \left[G^{-1}\partial_\alpha G G^{-1}\partial_\beta G G^{-1}\partial_\gamma G\right]$$

α, β, γ take values ω, k_x, k_y

$$G \to G + \delta G \longrightarrow \delta N_2 = 0$$

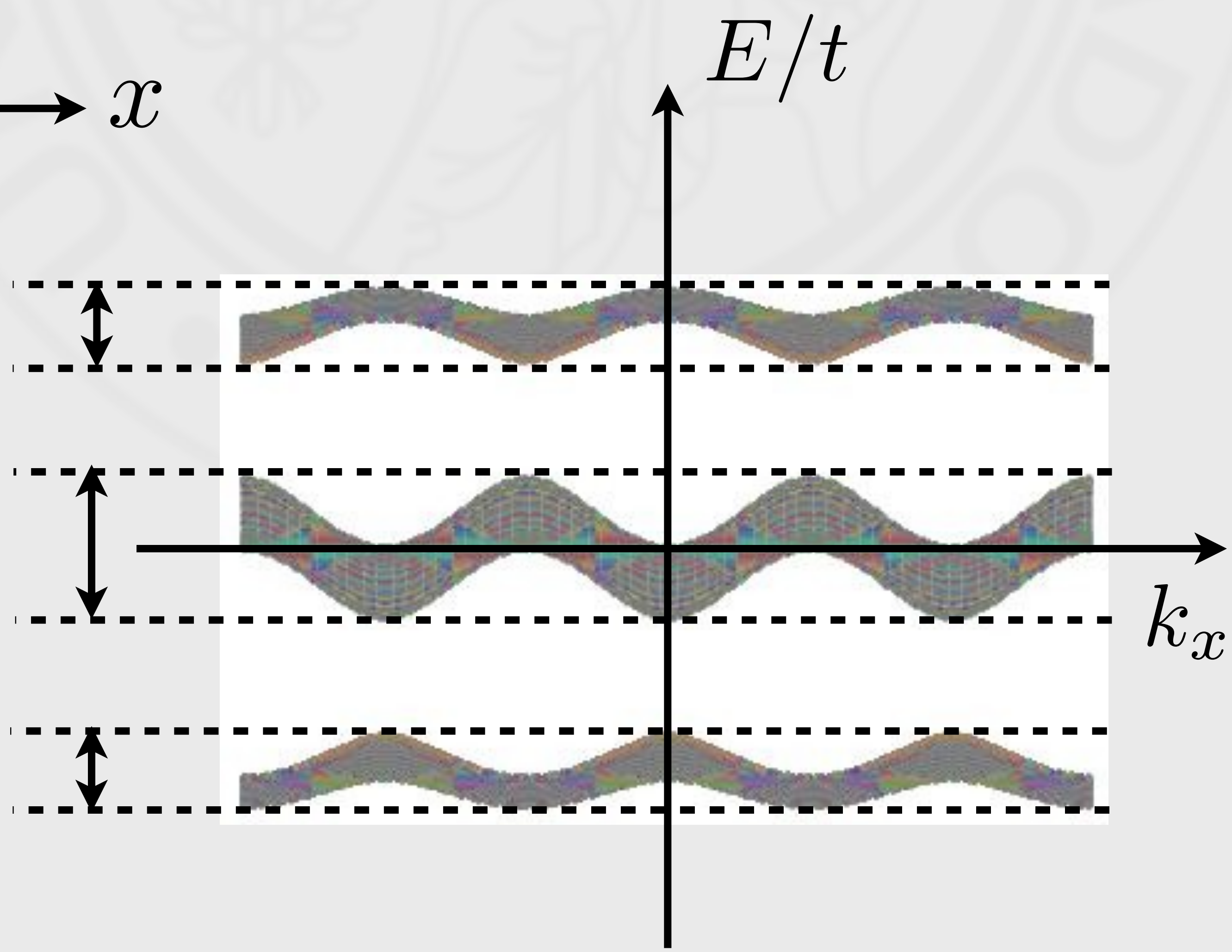
The only way to change N_2 is by making G singular.
That requires $\mathcal{H}$ to have zero energy eigenvalues.

Edge states

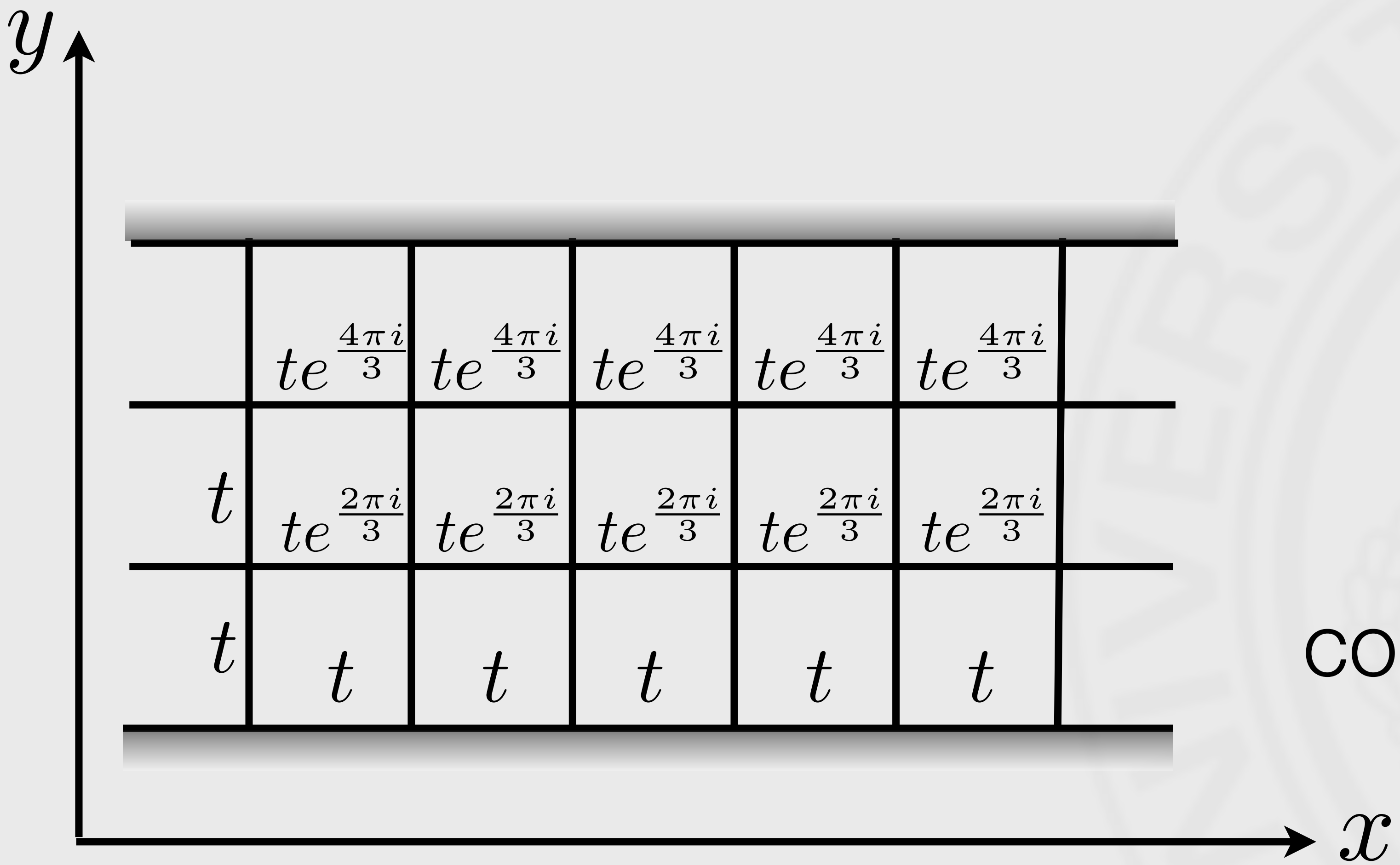


Periodic boundary conditions in the y-direction

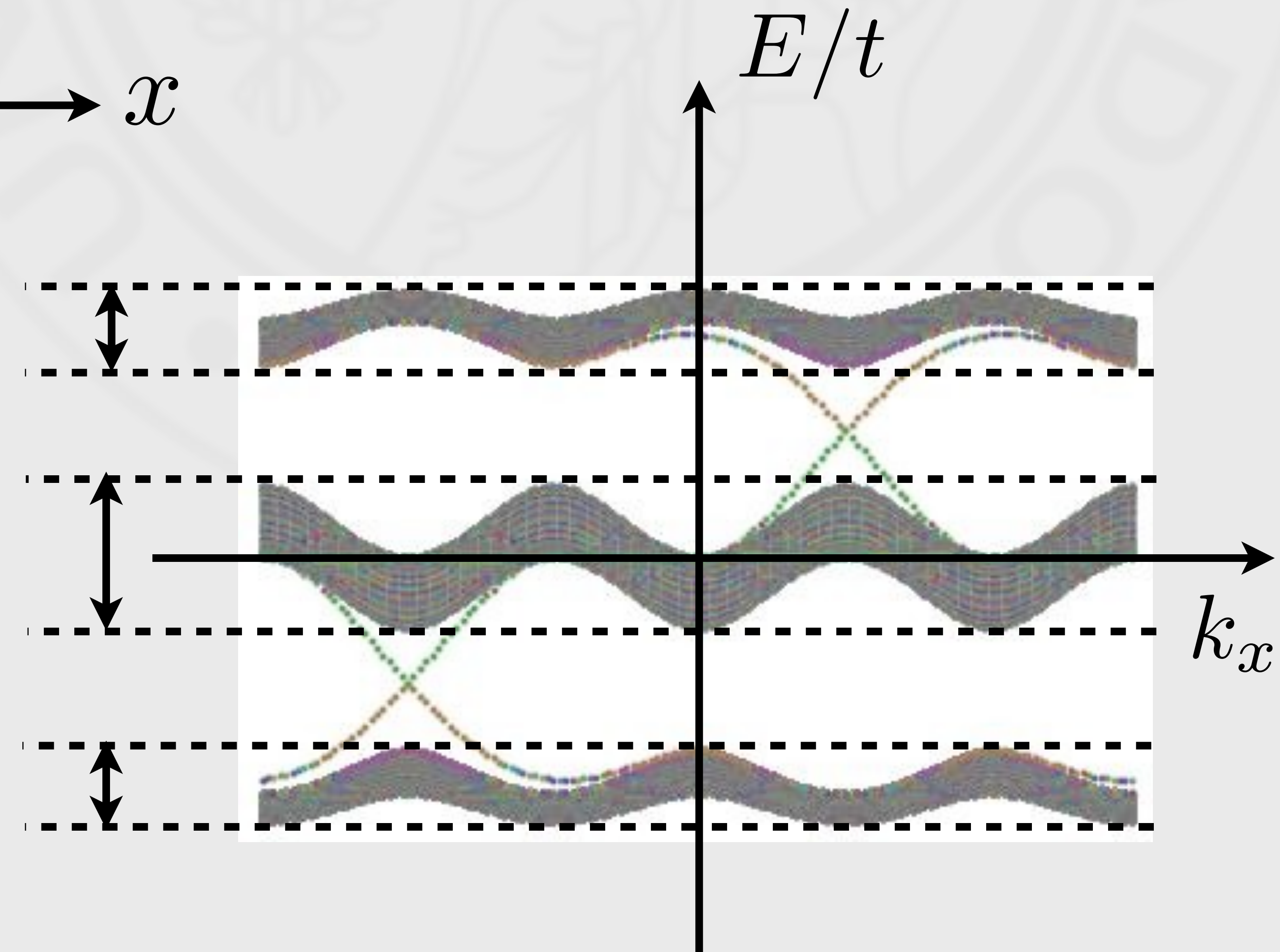
Particle hopping on a lattice with 2π/3 magnetic flux through each plaquette



Edge states

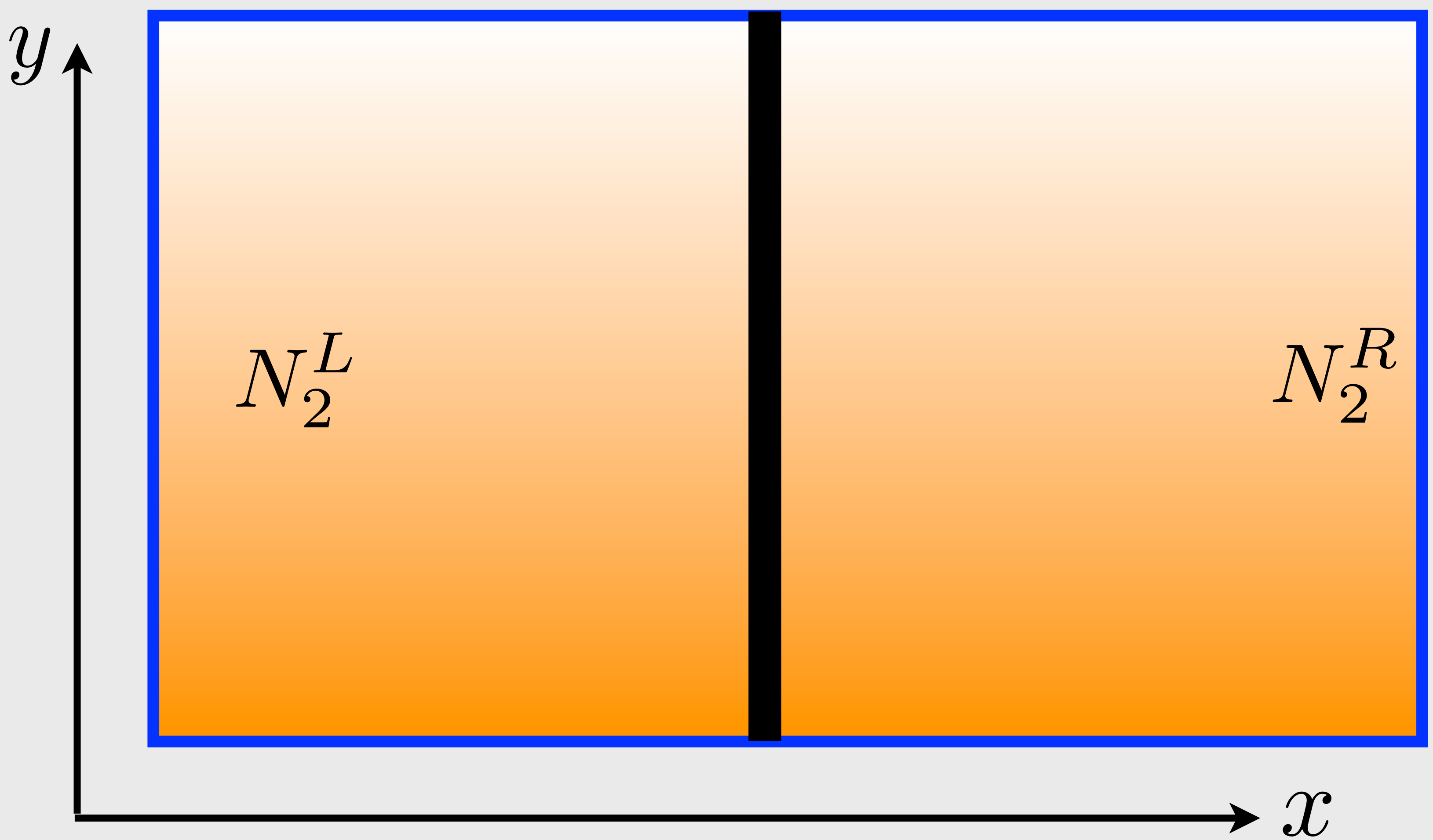


Hard wall boundary conditions in the y-direction



Particle hopping on a lattice with 2π/3 magnetic flux through each plaquette

Edge states as a result of topology



Zero energy states must
live in the boundary

$$\psi(x,y) \sim e^{-\frac{|x|}{\ell}} e^{ik_y y}$$

$$E(k_y) \sim k_y - k_0$$

Other topological classes?

For a long time 2D particle in a magnetic field was considered to be the only example of topological classes of single-particle Hamiltonians

Generalizations to 4D, 6D, generally even *d*, was known, however

$$N_d = -\frac{\left(\frac{d}{2}\right)!}{(2\pi i)^{\frac{d}{2}+1}(d+1)!} \sum_{\alpha_0 \alpha_1 \ldots \alpha_d} \epsilon_{\alpha_0 \alpha_1 \ldots \alpha_d} \int d\omega d^d k \, \mathrm{tr} \, \left[G^{-1} \partial_{\alpha_0} G G^{-1} \partial_{\alpha_1} G \ldots G^{-1} \partial_{\alpha_d} G \right]$$

d must be even   all **α** take values **ω**, k_1, k_2, ..., k_d

existence of this topological invariant reflects the homotopy class $\pi_{d+1}\left(GL(\mathcal{N}, \mathbb{C}\right) = \mathbb{Z}$

if *d* is even

Topological classes in high dimensions - perhaps not very physical

Fortunately, it turns out these are not the only topological insulators

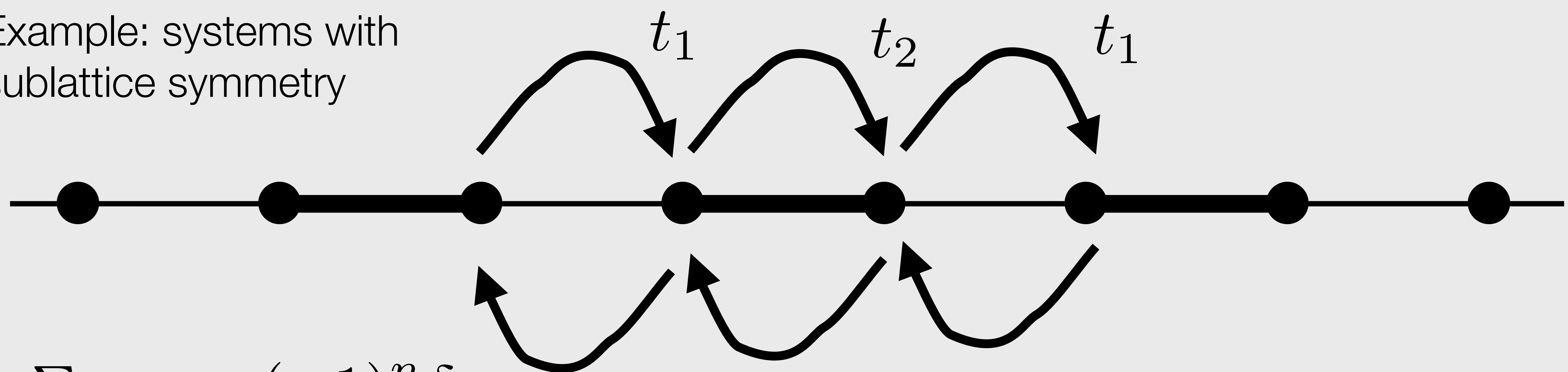
Chiral systems

What if we have Hamiltonians with a special symmetry

$$\Sigma \mathcal{H} \Sigma^\dagger = -\mathcal{H}$$
$$\Sigma \Sigma^\dagger = 1$$

It follows $\Sigma^2 = 1$

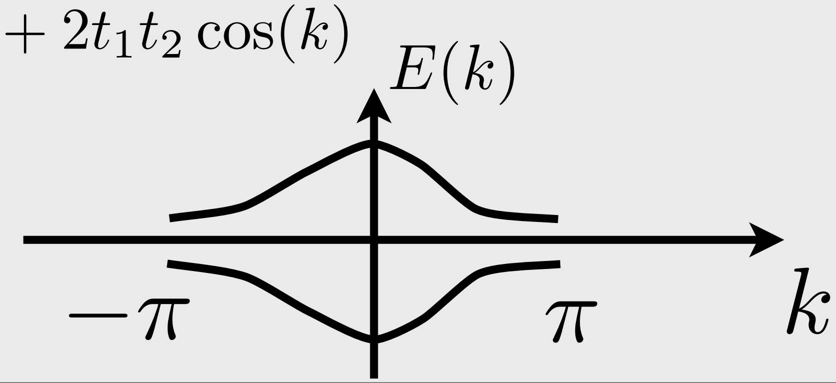
Example: systems with sublattice symmetry



$$\Sigma_{n,n'} = (-1)^n \delta_{nn'}$$

$$\hat{H} = \sum_n \left[t_1\, \hat{a}^\dagger_{2n+1} \hat{a}_{2n} + t_2\, \hat{a}^\dagger_{2n+2} \hat{a}_{2n+1} + \text{h.c.} \right] = \sum_{n_1,n_2} \mathcal{H}_{n_1 n_2}\, \hat{a}^\dagger_{n_1} \hat{a}_{n_2}$$

$$E(k) = \pm\sqrt{t_1^2 + t_2^2 + 2 t_1 t_2 \cos(k)}$$



$$\mathcal{H} = \begin{pmatrix} 0 & t_1 & 0 & \dots & 0 & 0 \\ t_1 & 0 & t_2 & \dots & 0 & 0 \\ 0 & t_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & t_1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & t_1 \\ 0 & 0 & 0 & \dots & t_1 & 0 \end{pmatrix}$$

Topological invariant for chiral systems

$$\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \mathcal{H} = \begin{pmatrix} 0 & V \\ V^\dagger & 0 \end{pmatrix}$$

basis $\begin{pmatrix} \hat{a}_{\mathrm{odd}} \\ \hat{a}_{\mathrm{even}} \end{pmatrix}$

General topological invariant

But here *d* is **odd**

$$N_d = -\frac{\left(\frac{d-1}{2}\right)!}{(2\pi i)^{\frac{d+1}{2}} d!} \sum_{\alpha_1 \ldots \alpha_d} \epsilon_{\alpha_1 \ldots \alpha_d} \int d^d k \, \mathrm{tr} \left[V^{-1} \partial_{\alpha_1} V \ldots V^{-1} \partial_{\alpha_d} V \right]$$

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Example: d=1

$$\mathcal{H} = \begin{pmatrix} 0 & t_1 + t_2 e^{ik} \\ t_1 + t_2 e^{-ik} & 0 \end{pmatrix}$$

With some algebra, one can show

$$N_1 = \int_{-\pi}^{\pi} \frac{dk}{2\pi i} \partial_k \ln\left(t_1 + t_2 e^{ik}\right)$$

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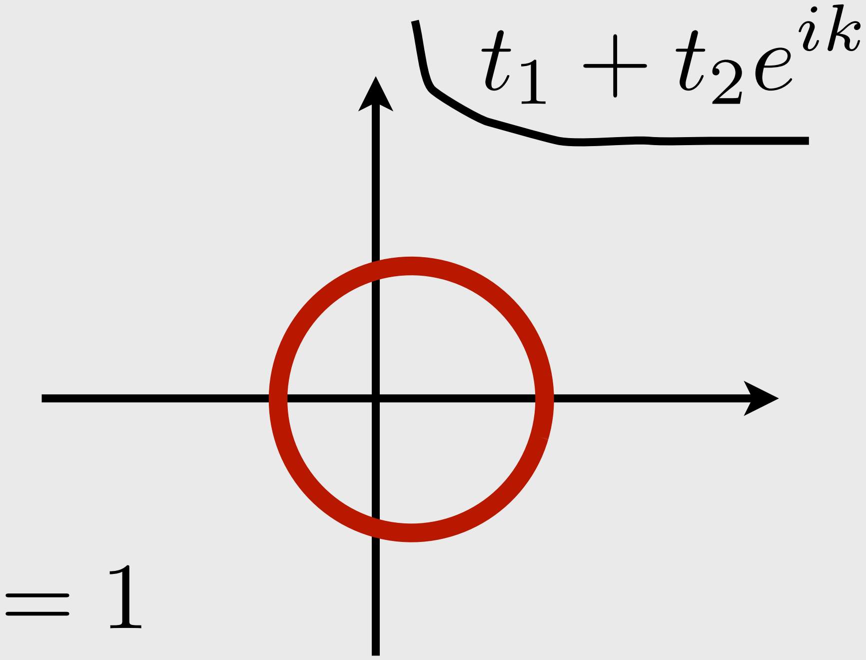

$N_1 = 1$

$t_1 < t_2$

Topological invariant for chiral systems

$$\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \mathcal{H} = \begin{pmatrix} 0 & V \\ V^\dagger & 0 \end{pmatrix}$$

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General topological invariant

But here *d* is **odd**

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Example: d=1

$$\mathcal{H} = \begin{pmatrix} 0 & t_1 + t_2 e^{ik} \\ t_1 + t_2 e^{-ik} & 0 \end{pmatrix}$$

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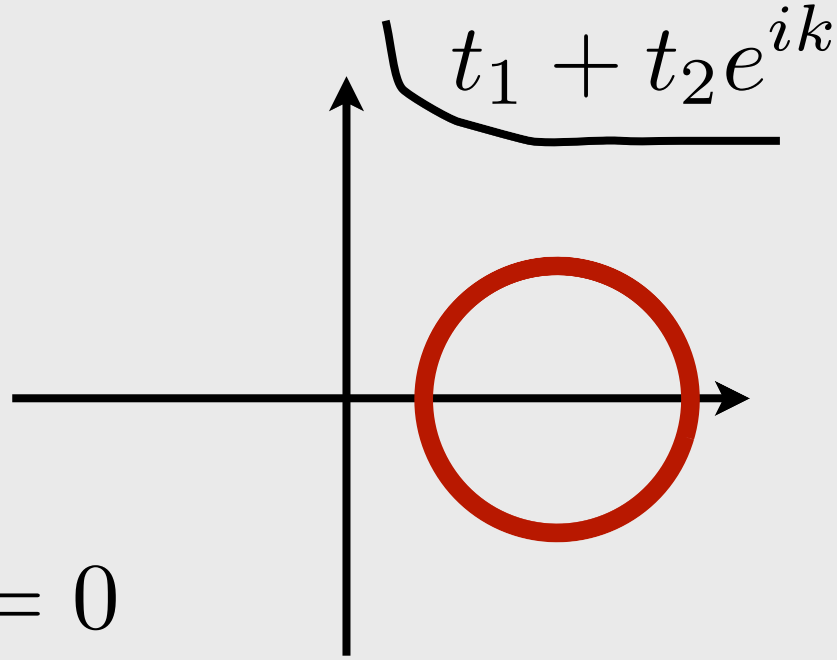
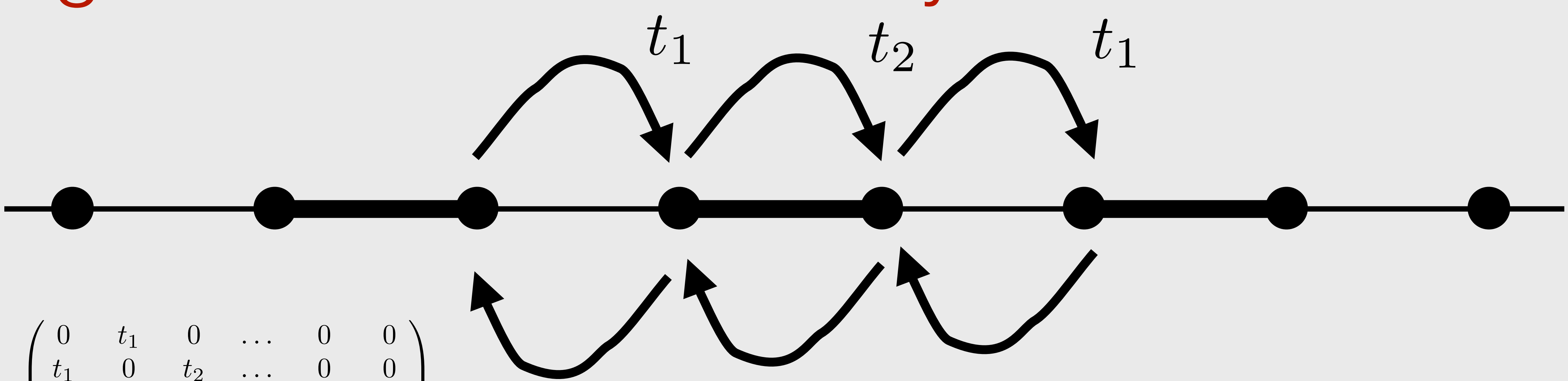
$$N_1 = \int_{-\pi}^{\pi} \frac{dk}{2\pi i} \partial_k \ln \left(t_1 + t_2 e^{ik} \right)$$



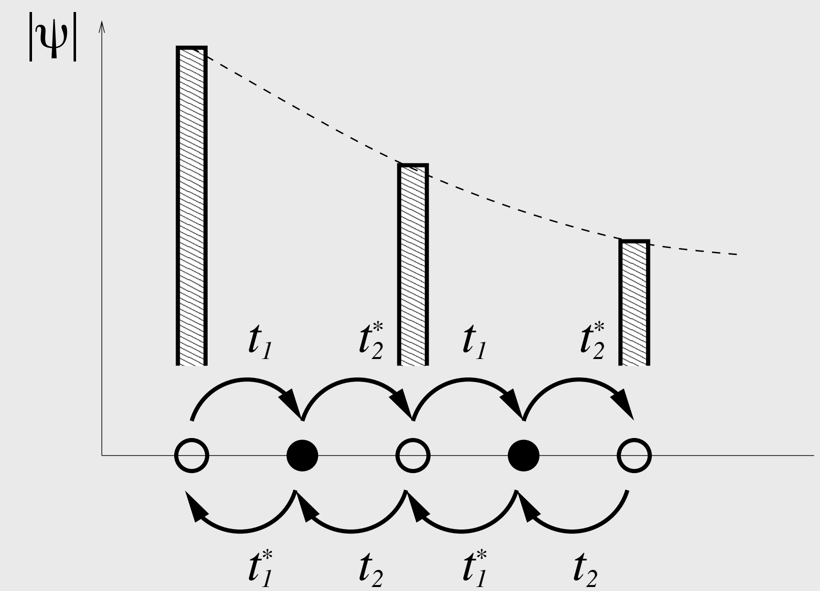
$N_1 = 0$

$t_1 > t_2$

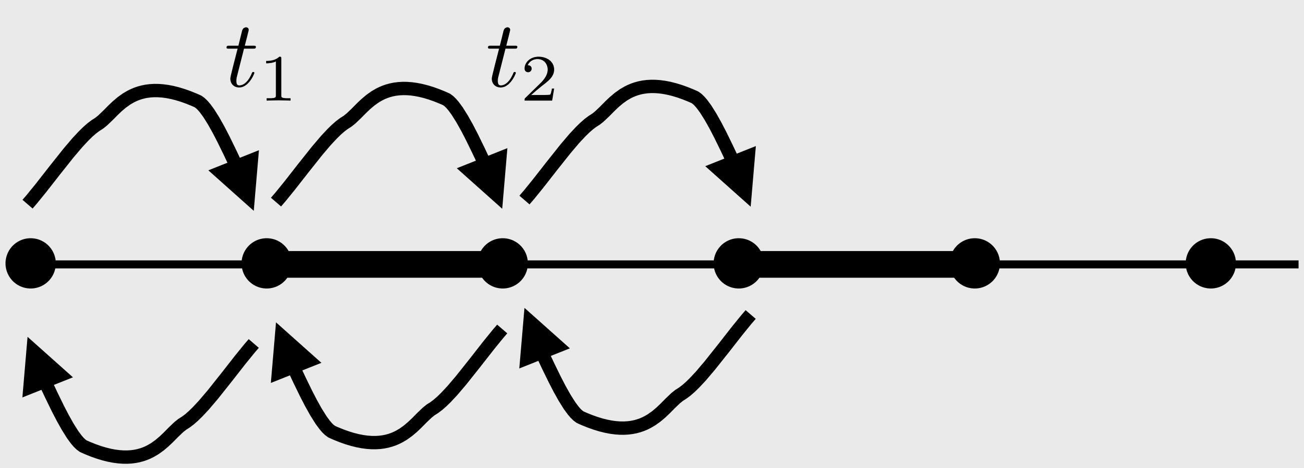
Edge states for 1D chiral systems



$$\mathcal{H} = \begin{pmatrix} 0 & t_1 & 0 & \dots & 0 & 0 \\ t_1 & 0 & t_2 & \dots & 0 & 0 \\ 0 & t_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & t_1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & t_1 \\ 0 & 0 & 0 & \dots & t_1 & 0 \end{pmatrix}$$



$$\mathcal{H}\psi = 0 \qquad t_1\psi_{2n-1} + t_2\psi_{2n+1} = 0$$



$$\psi_{2n+1} \sim \left(-\frac{t_1}{t_2}\right)^n$$

This works (decays for n>0) only if $t_1<t_2$

That is, if $N_1=1$ (not when it is zero)

Symmetry classes

space dimension	1	2	3	4	5	6
Class A (no symmetry)		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$
Class AIII (chiral symmetry)	$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$	

Altland-Zirnbauer
nomenclature

Other relevant symmetries

Time reversal $U_T^\dagger \mathcal{H}^* U_T = \mathcal{H}$ $\quad U_T^* U_T = \begin{cases} \text{either} & +1 \\ \text{or} & -1 \end{cases}$

Particle-hole conjugation $U_C^\dagger \mathcal{H}^* U_C = -\mathcal{H}$ $\quad U_C^* U_C = \begin{cases} \text{either} & +1 \\ \text{or} & -1 \end{cases}$

If both symmetries are present, chiral symmetry is automatically present, with $\Sigma = U_T^* U_C$

This leads to 10 "symmetry classes", introduced by Altland and Zirnbauer

Cartan label	T	C	S
A (unitary)	0	0	0
AI (orthogonal)	+1	0	0
AII (symplectic)	−1	0	0
AIII (ch. unit.)	0	0	1
BDI (ch. orth.)	+1	+1	1
CII (ch. sympl.)	−1	−1	1
D (BdG)	0	+1	0
C (BdG)	0	−1	0
DIII (BdG)	−1	+1	1
CI (BdG)	+1	−1	1

From: Ryu, Schnyder, Furusaki, Ludwig, 2010

Classes with time reversal invariance only

Time reversal $U_T^\dagger \mathcal{H}^*(-k) U_T = \mathcal{H}(k)$ $\quad U_T^* U_T = \begin{cases} \text{either} & +1 \\ \text{or} & -1 \end{cases}$

Class AI: time reversal for spinless particles or spin rotation invariant Hamiltonians

Example: $\mathcal{H}_{\alpha\beta}(k) = \frac{k^2}{2m}\delta_{\alpha\beta}$ $\quad \alpha, \beta = \uparrow, \downarrow$

Class AII: time reversal for spin-dependent spin-1/2 Hamiltonians (usually implies spin-orbit coupling)

$$\mathcal{H}_{\alpha\beta}(k) = \frac{k^2}{2m}\delta_{\alpha\beta} + g_{SO}\sum_\mu k_\mu \sigma^\mu_{\alpha,\beta}$$

$\sigma^y \mathcal{H}^*_{\alpha\beta}(-k)\sigma^y = \mathcal{H}(k)$ $\quad U_T = \sigma^y$ $\quad U_T U_T^* = -1$

Only time-reversal is present

These are classes AI, AII

Green's function transposed

$$G = [i\omega - \mathcal{H}]^{-1} \longrightarrow U_T^\dagger G^T U_T = G$$

$$G^T_{ab}(k) = G_{ba}(-k)$$

Applying the symmetry to G, we can show that the invariant is identically zero if $d = 2 + 4n$

$$N_d = -\frac{\left(\frac{d}{2}\right)!}{(2\pi i)^{\frac{d}{2}+1}(d+1)!} \sum_{\alpha_0\alpha_1\ldots\alpha_d} \epsilon_{\alpha_0\alpha_1\ldots\alpha_d} \int d\omega d^d k \,\mathrm{tr}\, \left[G^{-1}\partial_{\alpha_0} G G^{-1}\partial_{\alpha_1} G \ldots G^{-1}\partial_{\alpha_d} G\right]$$

space dimension	1	2	3	4	5	6	7	8
Class A (no symmetry)		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$
Class AI (time reversal)				$\mathbb{Z}$				$\mathbb{Z}$
Class AII (time reversal with spin-1/2)				$\mathbb{Z}$				$\mathbb{Z}$

Consequence: no topological band structure for time reversal invariant systems in 2D.
This is not quite true, however - there is a different topological invariant we haven't yet looked at.

Only particle-hole is present

These are classes D, C

$$U_C^\dagger \mathcal{H}^* U_C = -\mathcal{H}$$
$$G = [i\omega - \mathcal{H}]^{-1}$$
$$\longrightarrow \quad U_C^\dagger G^T(\omega) U_C = -G(-\omega)$$

Applying the symmetry to G, we can show that the invariant is identically zero if $d = 4n$

$$N_d = -\frac{(\frac{d}{2})!}{(2\pi i)^{\frac{d}{2}+1}(d+1)!} \sum_{\alpha_0\alpha_1\ldots\alpha_d} \epsilon_{\alpha_0\alpha_1\ldots\alpha_d} \int d\omega d^d k \,\mathrm{tr}\, \left[G^{-1}\partial_{\alpha_0} G G^{-1}\partial_{\alpha_1} G \ldots G^{-1}\partial_{\alpha_d} G\right]$$

space dimension	1	2	3	4	5	6	7	8
Class A (no symmetry)		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$
Class D (p-h, $U_C U_C^* = 1$)		$\mathbb{Z}$				$\mathbb{Z}$		
Class C (p-h, $U_C U_C^* = -1$)		$\mathbb{Z}$				$\mathbb{Z}$		

The origin of p-h symmetry

$$\Delta = -\Delta^T \quad h = h^\dagger$$

BCS superconductor

$$\hat{H} = \sum_{ij} \left[2h_{ij}\hat{a}_i^\dagger \hat{a}_j + \Delta_{ij}\hat{a}_i^\dagger \hat{a}_j^\dagger + \Delta_{ij}^\dagger \hat{a}_i \hat{a}_j \right] = \sum_{ij} \begin{pmatrix} \hat{a}_i^\dagger & \hat{a}_i \end{pmatrix} \begin{pmatrix} h_{ij} & \Delta_{ij} \\ \Delta_{ij}^\dagger & -h_{ij}^T \end{pmatrix} \begin{pmatrix} \hat{a}_i \\ \hat{a}_j^\dagger \end{pmatrix}$$

$$\mathcal{H} = \begin{pmatrix} h_{ij} & \Delta_{ij} \\ \Delta_{ij}^\dagger & -h_{ij}^T \end{pmatrix} \qquad \sigma_x \mathcal{H}^* \sigma_x = -\mathcal{H} \qquad U_C \equiv \sigma_x$$

This is class D

This describes Bogoliubov quasiparticles

Famous example: $p_x + i\,p_y$ spin-polarized superconductor (important: it breaks time reversal)

$$\mathcal{H} = \begin{pmatrix} \frac{p^2}{2m} - \mu & \Delta(p_x + ip_y) \\ \Delta(p_x - ip_y) & -\frac{p^2}{2m} + \mu \end{pmatrix} \qquad N_2 = 1$$

$$E_p = \pm\sqrt{\left(\frac{p^2}{2m} - \mu\right)^2 + \Delta^2 p^2}$$

$$\mathcal{H} = \sum_\mu n_\mu \sigma^\mu \qquad N_2 = \frac{1}{8\pi} \sum_{\alpha\beta\gamma} \sum_{\mu\nu} \epsilon_{\mu\nu\gamma} \epsilon_{\alpha\beta} \int d^2p \, \frac{n^\mu \partial_\alpha n^\nu \partial_\beta n^\gamma}{n^3}$$

This superconductor has edge states, just like a particle in a magnetic field

The origin of p-h symmetry

BCS superconductor

$$\Delta = -\Delta^T \quad h = h^\dagger$$

$$\hat{H} = \sum_{ij} \left[2h_{ij}\hat{a}_i^\dagger \hat{a}_j + \Delta_{ij}\hat{a}_i^\dagger \hat{a}_j^\dagger + \Delta_{ij}^\dagger \hat{a}_i \hat{a}_j \right] = \sum_{ij} \begin{pmatrix} \hat{a}_i^\dagger & \hat{a}_i \end{pmatrix} \begin{pmatrix} h_{ij} & \Delta_{ij} \\ \Delta_{ij}^\dagger & -h_{ij}^T \end{pmatrix} \begin{pmatrix} \hat{a}_i \\ \hat{a}_j^\dagger \end{pmatrix}$$

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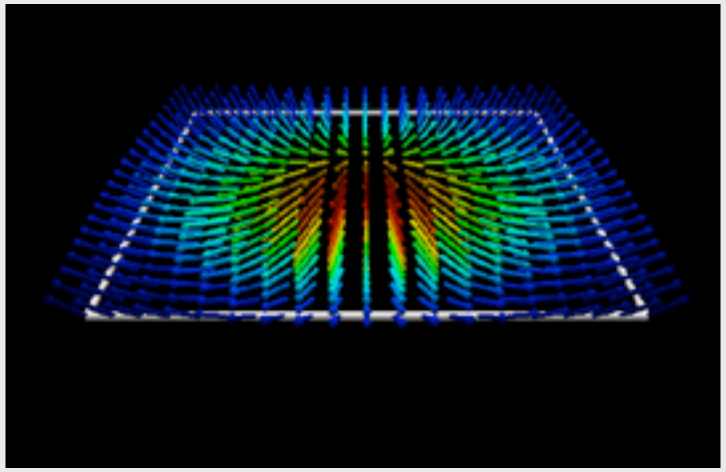

This superconductor has edge states, just like a particle in a magnetic field

Class C

BCS spin-singlet superconductor

$$\Delta = \Delta^T \qquad h = h^\dagger$$

$$\hat{H} = \sum_{ij} \left[\sum_{\sigma=\uparrow,\downarrow} 2h_{ij}\hat{a}^\dagger_{i\sigma}\hat{a}_{j\sigma} + \Delta_{ij}(\hat{a}^\dagger_{i\uparrow}\hat{a}^\dagger_{j,\downarrow} - \hat{a}^\dagger_{i,\downarrow}\hat{a}^\dagger_{j\uparrow}) + \Delta^\dagger_{ij}\left(\hat{a}_{i,\downarrow}\hat{a}_{j,\uparrow} - \hat{a}_{j,\downarrow}\hat{a}_{i,\uparrow}\right) \right]$$

$$= 2\sum_{ij} \begin{pmatrix} \hat{a}^\dagger_{i,\uparrow} & \hat{a}_{j,\downarrow} \end{pmatrix} \begin{pmatrix} h_{ij} & \Delta_{ij} \\ \Delta^\dagger_{ij} & -h^T_{ij} \end{pmatrix} \begin{pmatrix} \hat{a}_{j,\uparrow} \\ \hat{a}^\dagger_{j,\downarrow} \end{pmatrix}$$

$$\mathcal{H} = \begin{pmatrix} h_{ij} & \Delta_{ij} \\ \Delta^\dagger_{ij} & -h^T_{ij} \end{pmatrix} \qquad \sigma_y \mathcal{H}^* \sigma_y = -\mathcal{H} \qquad U_C = \sigma_y$$

$$U_C U_C^* = -1$$

Example: d-wave superconductor with the order parameter $d_{x^2-y^2} + id_{xy}$

Classes with both TR and PH

Automatically have chiral symmetry. Topological invariant in odd dimensional space.

$$N_d = -\frac{\left(\frac{d-1}{2}\right)!}{(2\pi i)^{\frac{d+1}{2}} d!} \sum_{\alpha_1 \ldots \alpha_d} \epsilon_{\alpha_1 \ldots \alpha_d} \int d^d k \, \mathrm{tr} \left[V^{-1} \partial_{\alpha_1} V \ldots V^{-1} \partial_{\alpha_d} V \right]$$

$$N_d = -\frac{1}{2} \frac{\left(\frac{d-1}{2}\right)!}{(2\pi i)^{\frac{d+1}{2}} d!} \sum_{\alpha_1 \ldots \alpha_d} \int d^d k \, \mathrm{tr} \, \left[\Sigma \mathcal{H}^{-1} \partial_{\alpha_1} \mathcal{H} \ldots \mathcal{H}^{-1} \partial_{\alpha_d} \mathcal{H} \right]$$

$$U_C U_C^* = \epsilon_C \qquad U_T U_T^* = \epsilon_T$$

$N_d = 0$ if

$$\epsilon_C \epsilon_T = 1 \qquad d = 3 + 4n$$

$$\epsilon_C \epsilon_T = -1 \qquad d = 1 + 4n$$

Example: class DIII

$$U_T U_T^* = -1$$

$$U_C U_C^* = 1$$

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AIII (ch. unit.)	0	0	1
BDI (ch. orth.)	+1	+1	1
CII (ch. sympl.)	−1	−1	1
D (BdG)	0	+1	0
C (BdG)	0	−1	0
DIII (BdG)	−1	+1	1
CI (BdG)	+1	−1	1

Example: class DIII

$$U_T U_T^* = -1$$

$$U_C U_C^* = 1$$

1. can be a superconductor

Cartan label	T	C	S
A (unitary)	0	0	0
AI (orthogonal)	+1	0	0
AII (symplectic)	−1	0	0
AIII (ch. unit.)	0	0	1
BDI (ch. orth.)	+1	+1	1
CII (ch. sympl.)	−1	−1	1
D (BdG)	0	+1	0
C (BdG)	0	−1	0
DIII (BdG)	−1	+1	1
CI (BdG)	+1	−1	1

Example: class DIII

$$U_T U_T^* = -1$$

$$U_C U_C^* = 1$$

1. can be a superconductor
2. has to be a spin-triplet (p-wave) superconductor

Cartan label	T	C	S
A (unitary)	0	0	0
AI (orthogonal)	+1	0	0
AII (symplectic)	−1	0	0
AIII (ch. unit.)	0	0	1
BDI (ch. orth.)	+1	+1	1
CII (ch. sympl.)	−1	−1	1
D (BdG)	0	+1	0
C (BdG)	0	−1	0
DIII (BdG)	−1	+1	1
CI (BdG)	+1	−1	1

Example: class DIII

$$U_T U_T^* = -1$$

$$U_C U_C^* = 1$$

1. can be a superconductor
2. has to be a spin-triplet (p-wave) superconductor
3. has to have spin-orbit coupling

Cartan label	T	C	S
A (unitary)	0	0	0
AI (orthogonal)	+1	0	0
AII (symplectic)	−1	0	0
AIII (ch. unit.)	0	0	1
BDI (ch. orth.)	+1	+1	1
CII (ch. sympl.)	−1	−1	1
D (BdG)	0	+1	0
C (BdG)	0	−1	0
DIII (BdG)	−1	+1	1
CI (BdG)	+1	−1	1

Example: class DIII

$$U_T U_T^* = -1$$

$$U_C U_C^* = 1$$

1. can be a superconductor
2. has to be a spin-triplet (p-wave) superconductor
3. has to have spin-orbit coupling

This is ^{3}He phase B.

Cartan label	T	C	S
A (unitary)	0	0	0
AI (orthogonal)	+1	0	0
AII (symplectic)	−1	0	0
AIII (ch. unit.)	0	0	1
BDI (ch. orth.)	+1	+1	1
CII (ch. sympl.)	−1	−1	1
D (BdG)	0	+1	0
C (BdG)	0	−1	0
DIII (BdG)	−1	+1	1
CI (BdG)	+1	−1	1

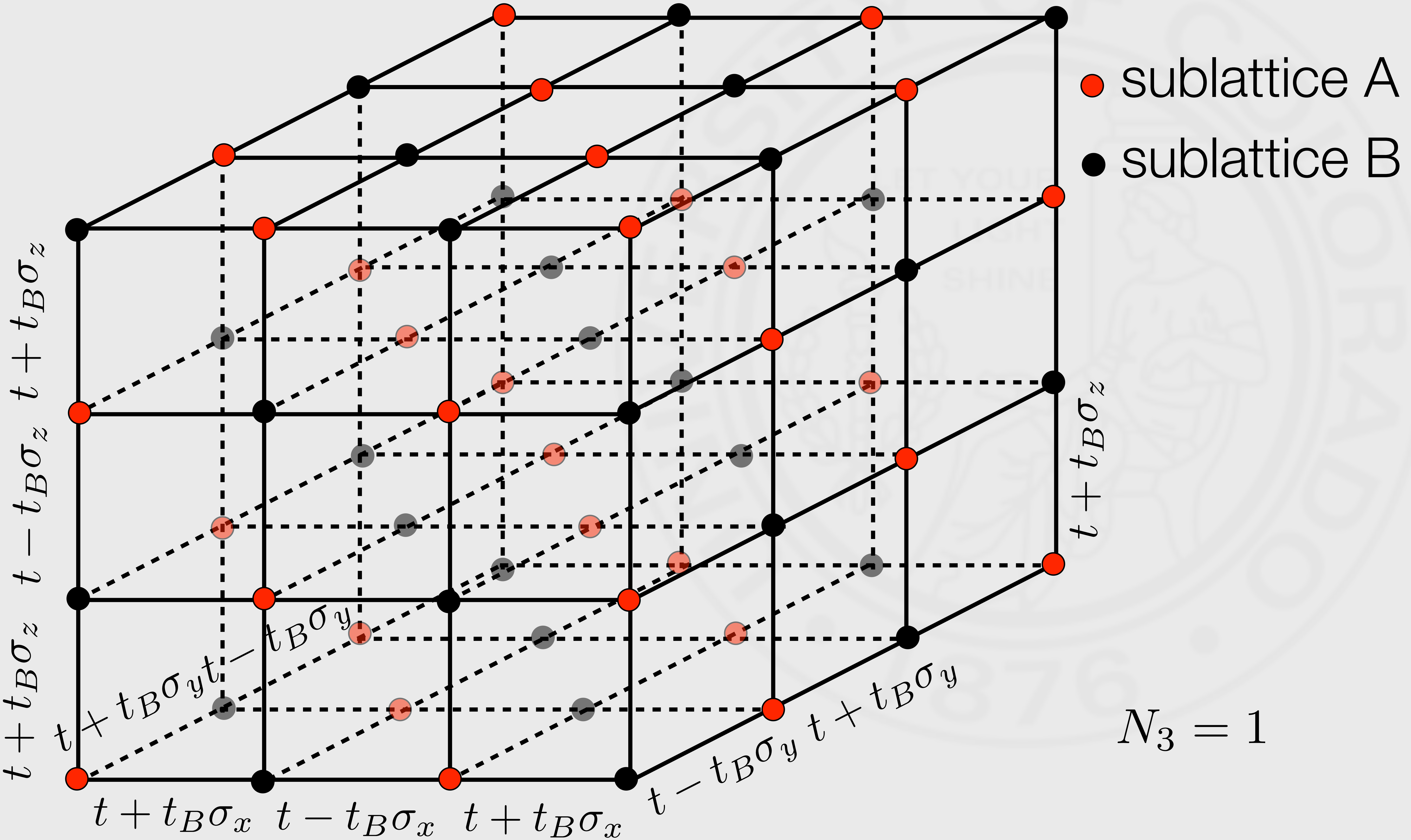
$$\hat{H} = \sum_{p,\alpha=\uparrow,\downarrow} \left(\frac{p^2}{2m} - \mu\right) \hat{a}^\dagger_{p\alpha}\hat{a}_{p\alpha} + \Delta \sum_{p,\alpha,\beta,\gamma} p_\mu\, \sigma^y_{\alpha\beta}\sigma^\mu_{\beta\gamma}\, \hat{a}_{p\alpha}\hat{a}_{-p\gamma} + \Delta \sum_{p,\alpha,\beta,\gamma} p_\mu\, \sigma^\mu_{\alpha\beta}\sigma^y_{\beta\gamma}\, \hat{a}^\dagger_{-p\alpha}\hat{a}^\dagger_{p\gamma}$$

$$\mathcal{H} = \begin{pmatrix} \frac{1}{2}\left(\frac{p^2}{2m} - \mu\right)\delta_{\alpha\beta} & \Delta \sum_{\gamma\mu} p_\mu \sigma^\mu_{\alpha\gamma}\sigma^y_{\gamma\beta} \\ \Delta \sum_{\gamma\mu} p_\mu \sigma^y_{\alpha\gamma}\sigma^\mu_{\gamma\beta} & -\frac{1}{2}\left(\frac{p^2}{2m} - \mu\right)\delta_{\alpha\beta} \end{pmatrix}$$

this must be chirally symmetric.
Its invariant is $N_3=1$.

^{3}He is topological and has edge states (discovered only in ~2008 by Ludwig et al)

Another example of a 3D DIII insulator



Example: Class CI

$$U_T U_T^* = 1$$

$$U_C U_C^* = -1$$

Spin-singlet time-reversal invariant superconductor

This is a conventional s-wave spin-singlet superconductor.

Can be topological in 3D

Conventional superconductors are not topological, but an example of a 3D CI topological superconductor is known (Ludwig et al)

Full classification table

White - nonchiral
Grey - chiral

Table from Ryu, Schnyder, Furusaki, Ludwig, 2010

Cartan	d space dimensionality 0	1	2	3	4	5	6	7	8	9	10	11	…
Complex case:													
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	…
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	…
Real case:													
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	…
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	…
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	…
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	…
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	…
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	…
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	…
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	…

symmetry classes

Kitaev, 2009;
Ludwig, Ryu, Schnyder, Furusaki, 2009.

Full classification table

White - nonchiral
Grey - chiral

Table from Ryu, Schnyder, Furusaki, Ludwig, 2010

Cartan	0	1	2	3	4	5	6	7	8	9	10	11	...
	d space dimensionality												
Complex case:													
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	...
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	...
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BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	...
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	...
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	...
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	...
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	...
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	...
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	...

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symmetry classes

Kitaev, 2009;
Ludwig, Ryu, Schnyder, Furusaki, 2009.

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AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	…
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	…
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	…
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	…
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	…
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	…
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	…
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	…

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Su, Schrieffer, Heeger

symmetry classes

Kitaev, 2009;
Ludwig, Ryu, Schnyder, Furusaki, 2009.

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Real case:													
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	...
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	...
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	...
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	...
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	...
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	...
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	...
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	...

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Su, Schrieffer, Heeger

2D p-wave supercond uctor

symmetry classes

Kitaev, 2009;
Ludwig, Ryu, Schnyder, Furusaki, 2009.

Full classification table

White - nonchiral
Grey - chiral

Table from Ryu, Schnyder, Furusaki, Ludwig, 2010

Cartan	0	1	2	3	4	5	6	7	8	9	10	11	...
Complex case:													
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	...
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	...
Real case:													
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	...
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	...
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	...
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	...
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	...
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	...
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	...
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	...

d space dimensionality

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2D p-wave superconductor

symmetry classes

^{3}He, phase B

Kitaev, 2009;
Ludwig, Ryu, Schnyder, Furusaki, 2009.

Full classification table

White - nonchiral
Grey - chiral

Table from Ryu, Schnyder, Furusaki, Ludwig, 2010

Cartan	d space dimensionality											
	0	1	2	3	4	5	6	7	8	9	10	11	…
Complex case:													
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	…
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Real case:													
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	…
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	…
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	…
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	…
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	…
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	…
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	…
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	…

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Su, Schrieffer, Heeger

2D p-wave supercond uctor

symmetry classes

^{3}He, phase B

New Kane-Mele topological insulators

Kitaev, 2009;
Ludwig, Ryu, Schnyder, Furusaki, 2009.

New crucial feature - Z_2 invariant

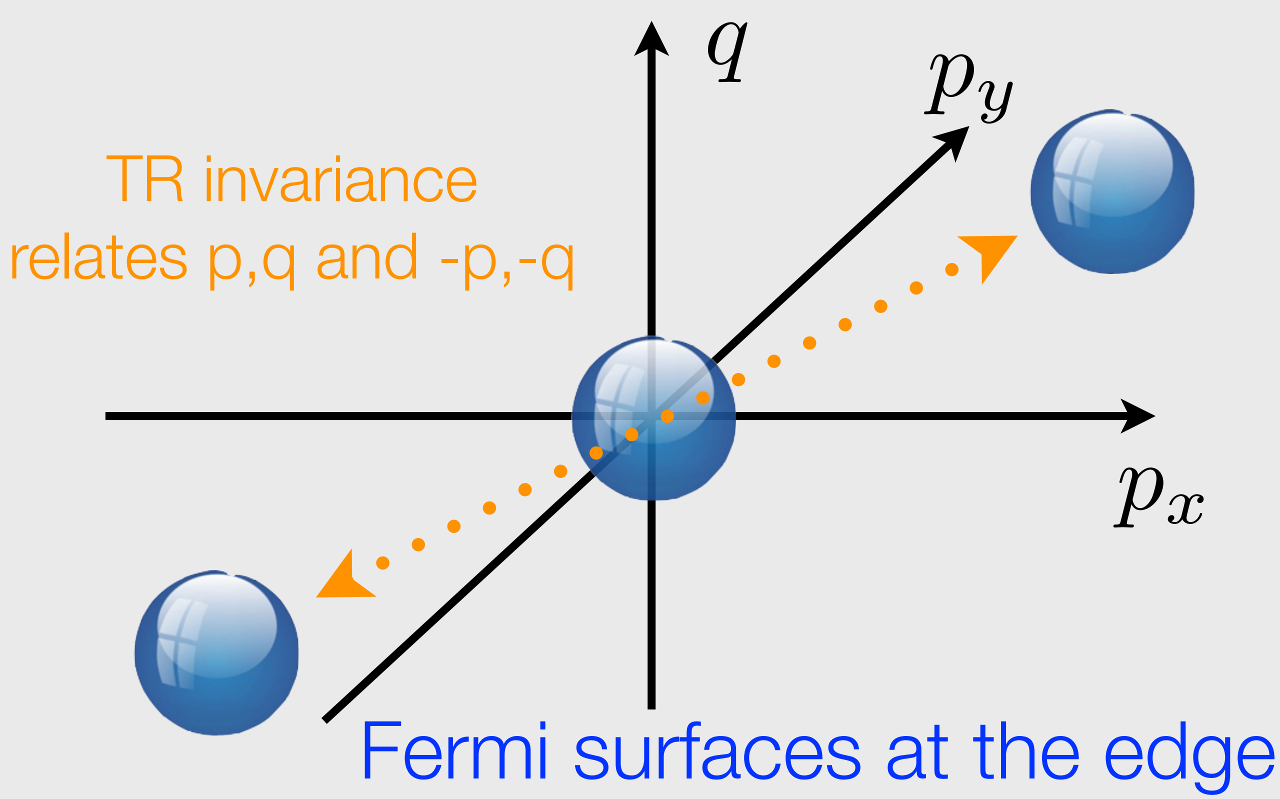
Take class AII: time reversal invariance with spin-1/2 (with spin-orbit coupling)

$$U_T^\dagger G^T(\omega, -\mathbf{k}) U^T = G(\omega, \mathbf{k})$$

In 4D it can be topological

Its 3D boundary has gapless excitations. These generally form a Fermi spheres.

3D boundary of a 4D insulator



Declare q "unphysical" and reduce dimensions to 3D insulator with 2D boundary

$$G_{\rm phys}(\omega, p_x, p_y) = G(\omega, p_x, p_y, q)|_{q=0}$$

If the number of Fermi spheres was odd, the physical 3D insulator has gapless excitations.
Otherwise, it does not.

$\mathbb{Z}_2$

Full classification table

White - nonchiral
Grey - chiral

Cartan	d space dimensionality											
	0	1	2	3	4	5	6	7	8	9	10	11	…
Complex case:													
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	…
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	…
Real case:													
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	…
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	…
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	…
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	…
AII	$2\mathbb{Z}$	0	**$\mathbb{Z}_2$**	**$\mathbb{Z}_2$**	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	…
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	…
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	…
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	…

New Kane-Mele topological insulators

symmetry classes

Edge excitations

Take nonchiral insulator in d-dimensions (d even)
It has a d-1 dimensional edge with gapless excitations

$$n_{\alpha_0} = -\frac{\left(\frac{d}{2}-1\right)!}{(2\pi i)^{\frac{d}{2}}(d-1)!} \sum_{\alpha_1 \ldots \alpha_{d-1}} \epsilon_{\alpha_0 \alpha_1 \ldots \alpha_{d-1}} \int d\omega d^{d-2}k \, \mathrm{tr} \left[G^{-1}\partial_{\alpha_1} G G^{-1} \partial_{\alpha_2} G \ldots G^{-1} \partial_{\alpha_{d-1}} G\right]$$

Here **α** go over $\omega, k_1, \ldots, k_{d-1}$

$$\sum_\alpha \partial_\alpha n_\alpha = 0$$

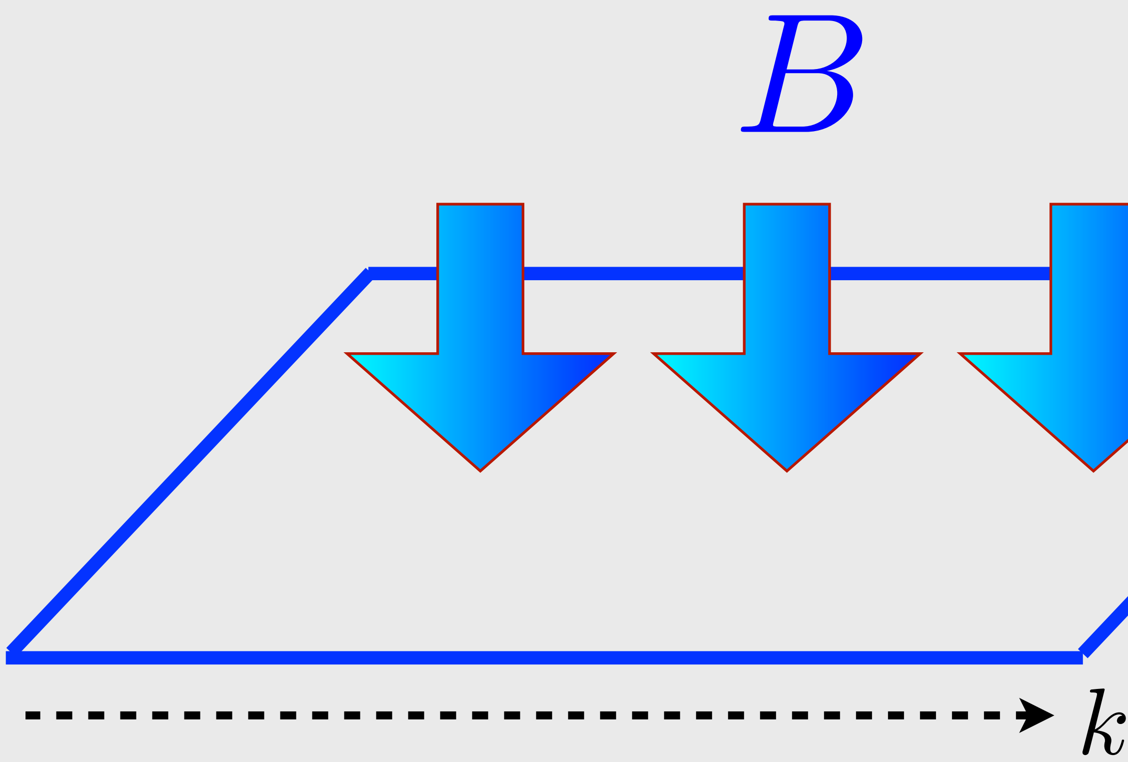
except where G is singular or where there are zero-energy excitations

this integral must be equal to the bulk topological invariant of the insulator

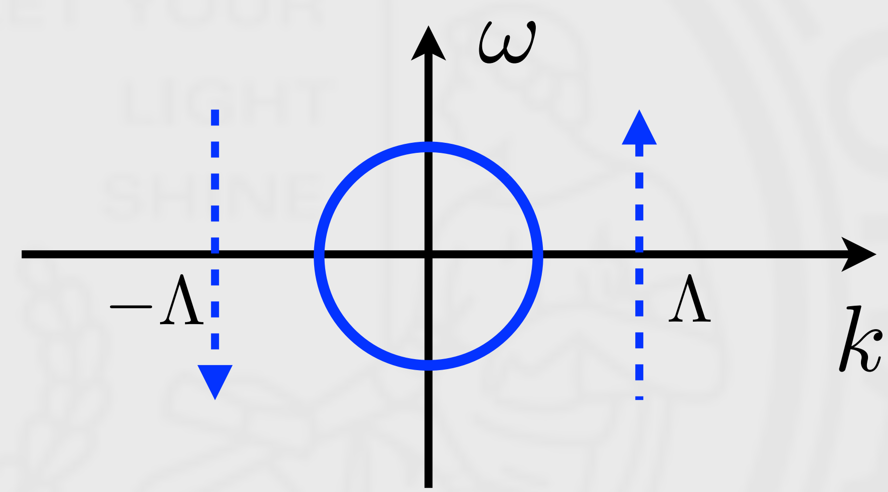
$$N_d = \oint ds^\alpha n_\alpha$$

similarly for d odd and chiral insulators

Example: quantum hall edge

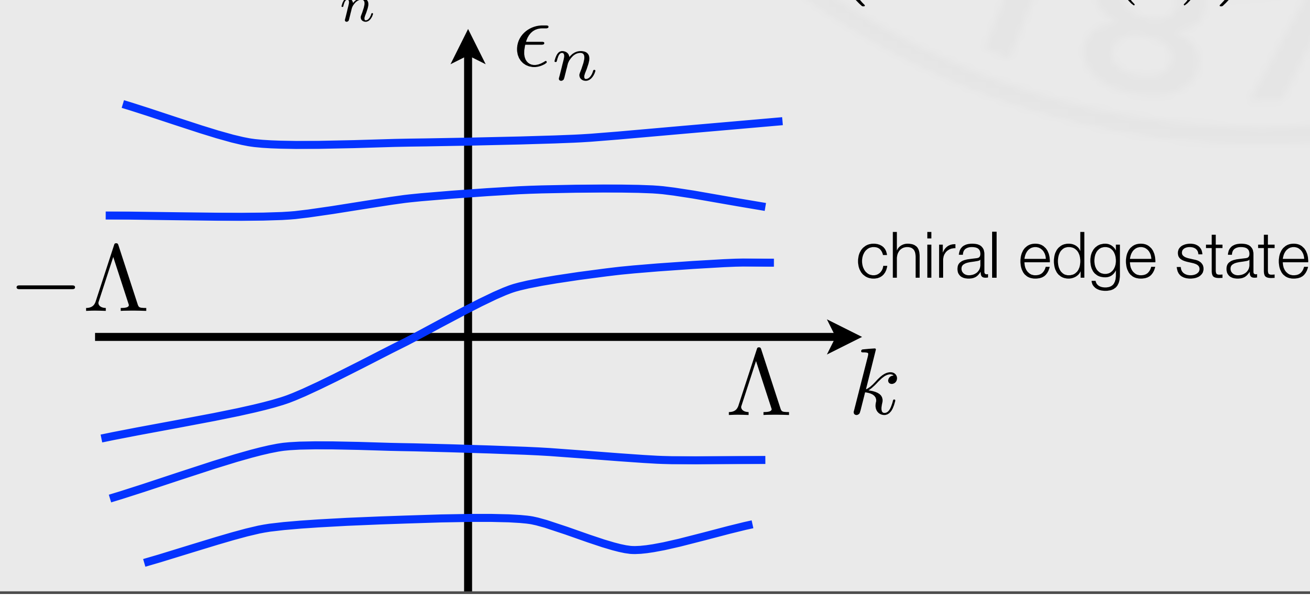


$$n_{\alpha_0} = \frac{1}{2\pi i} \sum_{\alpha_1} \epsilon_{\alpha_0 \alpha_1} \operatorname{tr} \left[G^{-1} \partial_{\alpha_1} G \right]$$



$$\oint ds^\alpha n_\alpha = N_0(\Lambda) - N_0(-\Lambda)$$

$$N_0(k) = \operatorname{tr} \int \frac{d\omega}{2\pi i} G^{-1} \partial_\omega G = \sum_n \int \frac{d\omega}{2\pi i} \partial_\omega \ln \left(\frac{1}{i\omega - \epsilon_n(k)} \right) = \frac{1}{2} \sum_n \operatorname{sign} \epsilon_n(k)$$



Example: an edge of a 3D DIII insulator

This is ^{3}He

$$\mathcal{H} = \sigma^x k_x + \sigma^y k_y$$

$$\sigma_x \mathcal{H}^*(-k)\sigma_x = -\mathcal{H}(k) \quad \text{p.h.}$$

$$\sigma_y \mathcal{H}^*(-k)\sigma_y = \mathcal{H}(k) \quad \text{t.r.}$$

$$\sigma_z \mathcal{H}(k)\sigma_z = \mathcal{H}(k) \quad \text{chiral}$$

$$\int \sum_{\alpha=x,y} ds^\alpha \, n_\alpha = 1.$$

Example: an edge of a 3D DIII insulator

This is ^{3}He

$$\mathcal{H} = \sigma^x k_x + \sigma^y k_y$$

$$\sigma_x \mathcal{H}^*(-k) \sigma_x = -\mathcal{H}(k) \quad \text{p.h.}$$

$$\sigma_y \mathcal{H}^*(-k) \sigma_y = \mathcal{H}(k) \quad \text{t.r.}$$

$$\sigma_z \mathcal{H}(k) \sigma_z = \mathcal{H}(k) \quad \text{chiral}$$

$$\int \sum_{\alpha=x,y} ds^\alpha \, n_\alpha = 1.$$

$$\mathcal{H} = \begin{pmatrix} 0 & k_x - i k_y \\ k_x + i k_y & 0 \end{pmatrix}$$

$$V = k_x - i k_y$$

$$\int ds^\alpha n_\alpha = \frac{1}{2\pi i} \oint dk^\mu \partial_{k_\mu} \ln\left(k_x - i k_y\right)$$

Edge theory of All 3D topological insulators

Literature states that its 2D edge is a Dirac fermion theory

$$H = v\left(\sigma_x p_x + \sigma_y p_y\right) - \mu$$

because it is

1. *linear in momenta*
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$$H(p) = \sigma_y H^*(-p)\sigma_y$$

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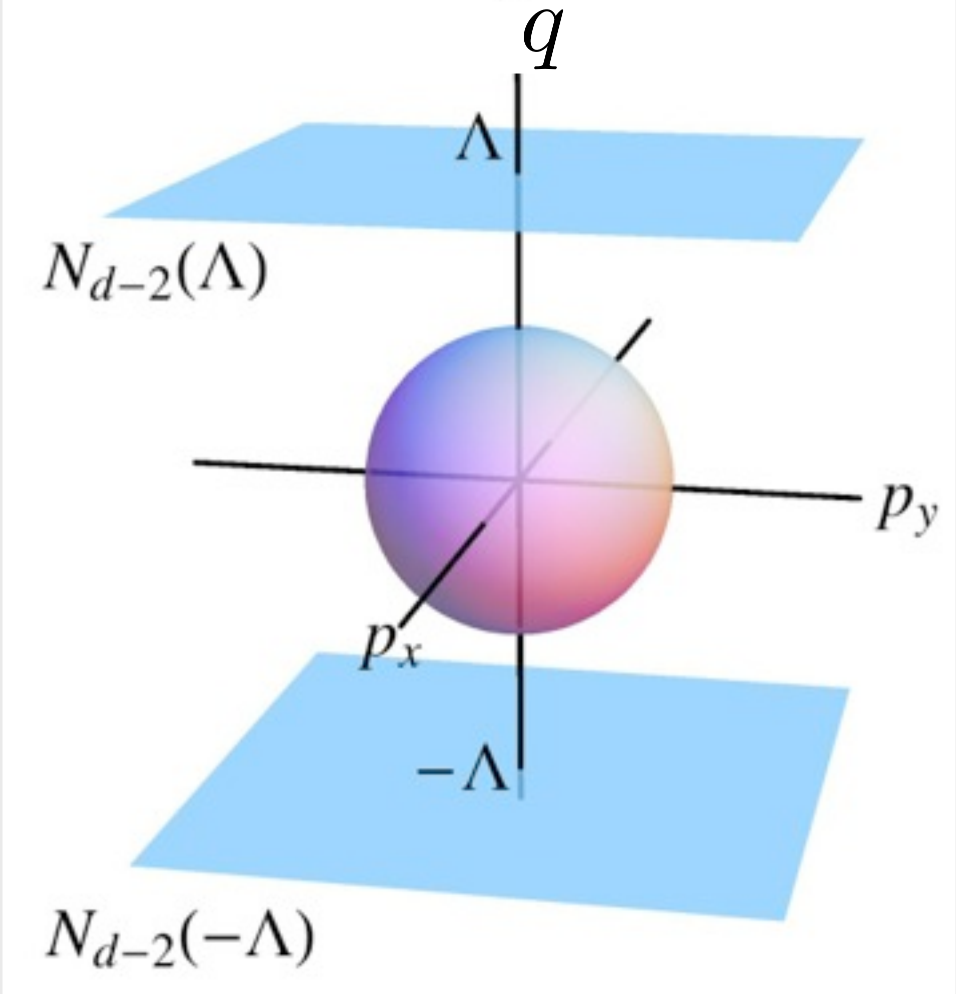

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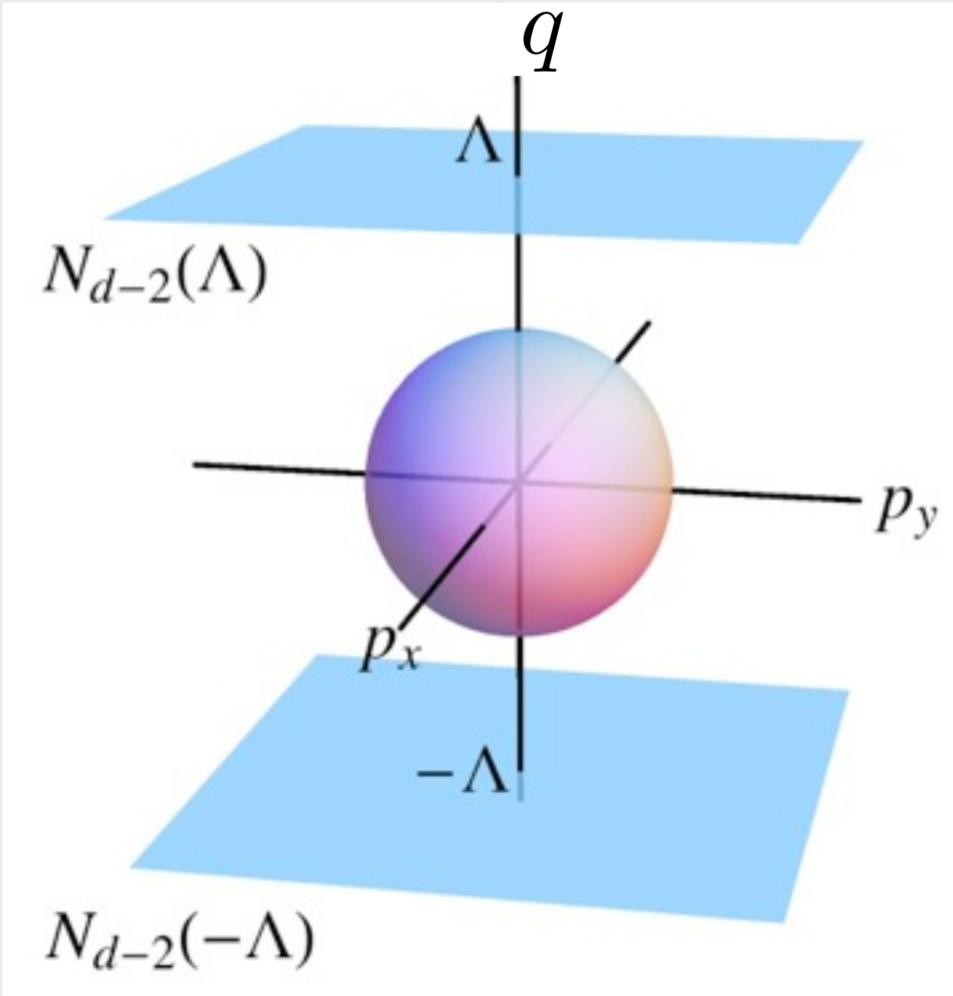

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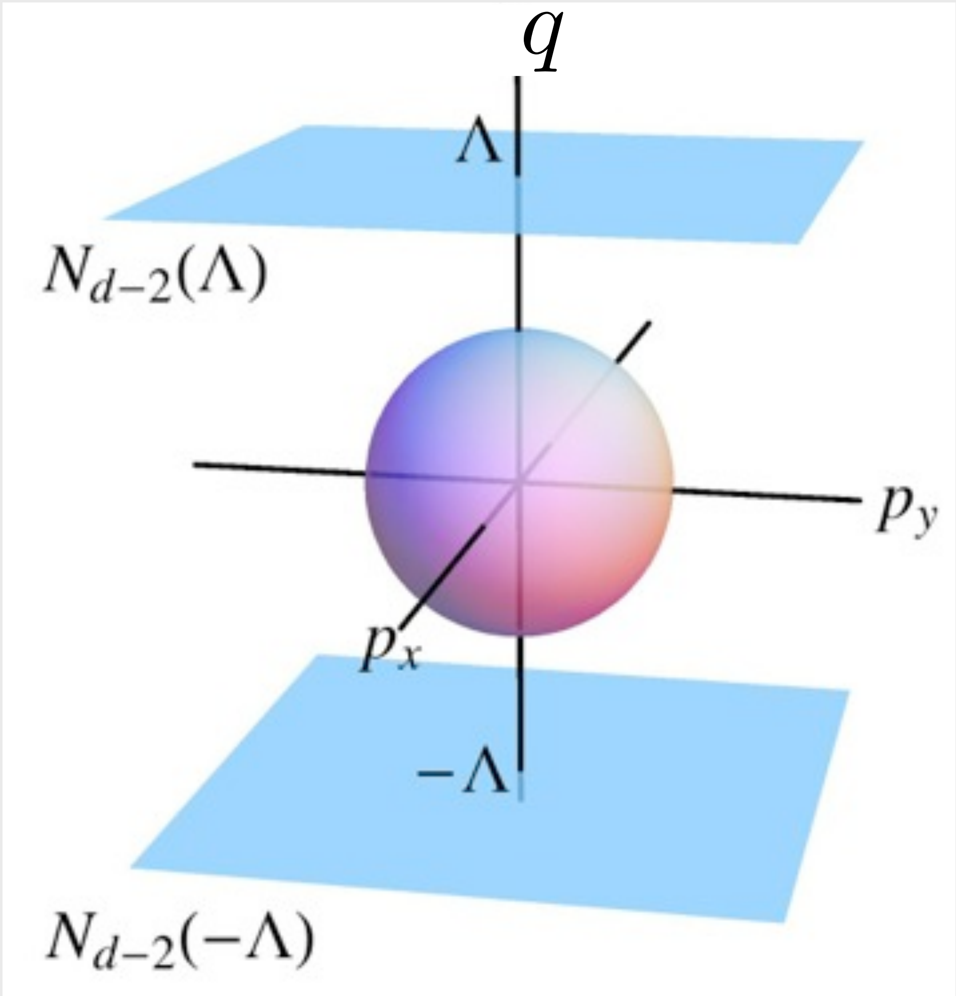

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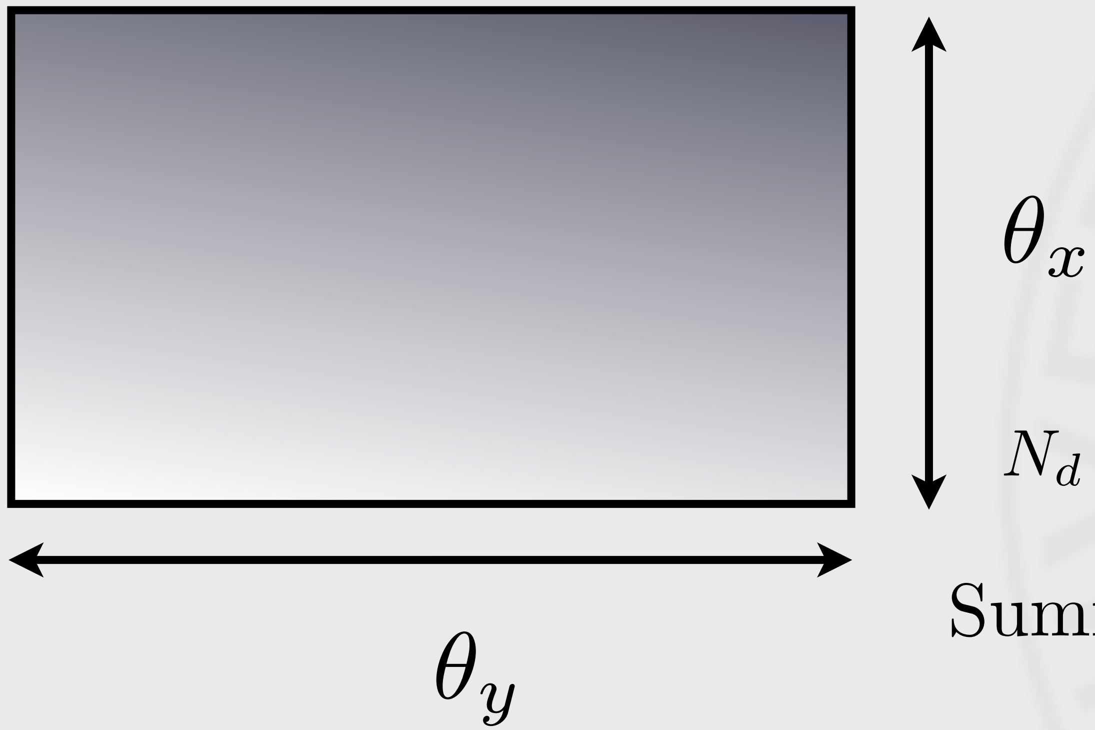
Yes, it does have the right edge invariant.

Disorder

Old idea of Thouless, Wu, Niu: impose phases across the system



$$G_{ij}(\omega, \theta_x, \theta_y \ldots)$$

$$N_d = C_d \, \epsilon_{\alpha_0 \ldots \alpha_d} \, \mathrm{tr} \int d\omega d^d\theta \, G^{-1}\partial_{\alpha_0} G \ldots G^{-1}\partial_{\alpha_d} G$$

Summation over each $\alpha = \omega, \theta_1, \ldots, \theta_d$ is implied

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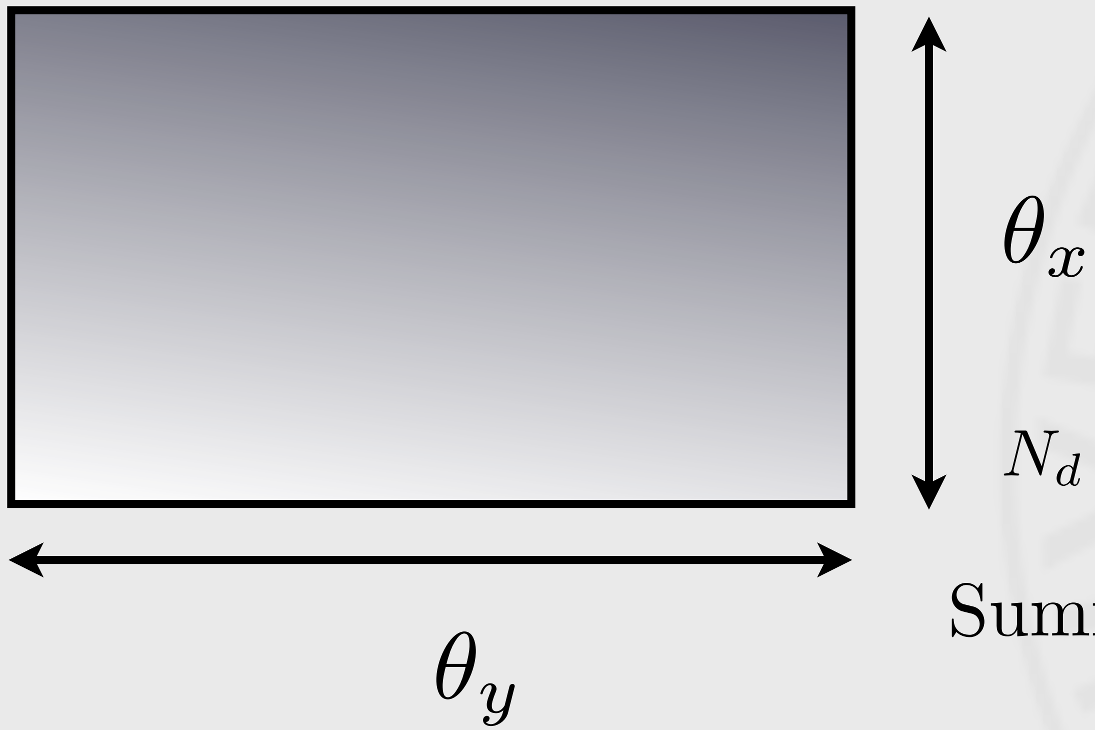

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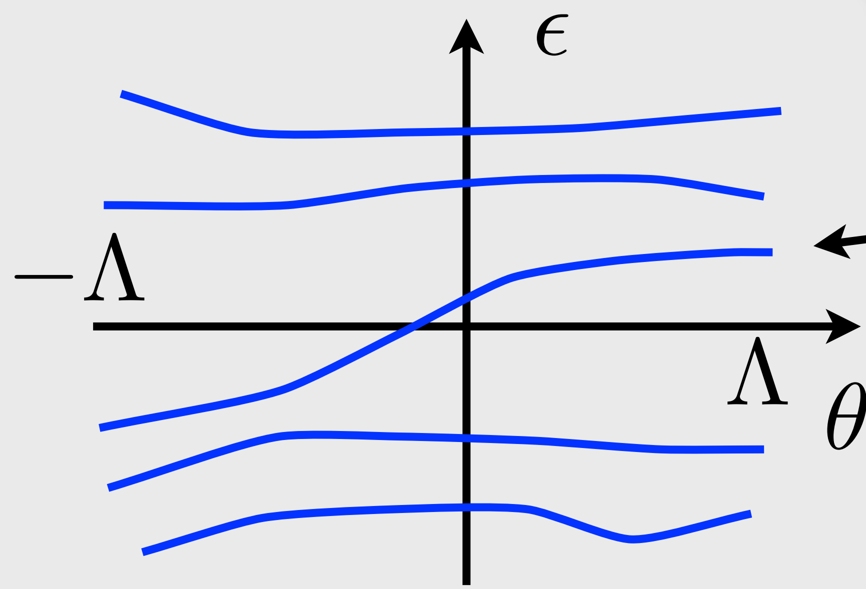
$d = 2:$



This edge level must be delocalized

$$N_0(\Lambda) - N_0(-\Lambda) = 1$$

It follows that an edge of a topological insulator does not localize in the presence of disorder

Sigma models with "topological" terms

Describe lack of localization at the boundary of an insulator

$$S \sim \sigma \int d^{\bar{d}} x \, (\partial_\mu Q)^2 + \text{topological term}$$

Topological term = either WZW term or "Z_2" term.

T - target space

Can be added if either $\pi_{\bar{d}}(T) = \mathbb{Z}_2$ or $\pi_{\bar{d}+1}(T) = \mathbb{Z}$ $\bar{d} = d - 1$

It is believed that these sigma models with topological terms result in the absence of localization

Cartan label	Time evolution operator $\exp\{it\mathcal{H}\}$	Fermionic replica NLσM target space
A	$U(N) \times U(N)/U(N)$	$U(2n)/U(n) \times U(n)$
AIII	$U(N+M)/U(N) \times U(M)$	$U(n) \times U(n)/U(n)$
AI	$U(N)/O(N)$	$Sp(2n)/Sp(n) \times Sp(n)$
BDI	$O(N+M)/O(N) \times O(M)$	$U(2n)/Sp(2n)$
D	$O(N) \times O(N)/O(N)$	$O(2n)/U(n)$
DIII	$SO(2N)/U(N)$	$O(n) \times O(n)/O(n)$
AII	$U(2N)/Sp(2N)$	$O(2n)/O(n) \times O(n)$
CII	$Sp(N+M)/Sp(N) \times Sp(M)$	$U(n)/O(n)$
C	$Sp(2N) \times Sp(2N)/Sp(2N)$	$Sp(2n)/U(n)$
CI	$Sp(2N)/U(N)$	$Sp(2n) \times Sp(2n)/Sp(2n)$

From: Ryu, Schnyder, Furusaki, Ludwig, 2010

10 classes of sigma models first derived by Altland & Zirnbauer to describe localization properties

Justifies 10 symmetry classes of topological insulators