Functional renormalization group for interacting electrons

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Part I: Intro: Correlated electrons and RG

Part II: Functional RG for Fermi systems

Part III: Impurities in Luttinger liquids

Review:

W. Metzner, M. Salmhofer, C. Honerkamp, V. Meden, and K. Schönhammer, Rev. Mod. Phys. **84**, 299 (2012)

Part I: Intro: Correlated electrons and RG

1. Energy scales in correlated electron systems
2. Perturbation theory and infrared divergences
3. Renormalization group idea

1. Energy scales in correlated electron systems

Interaction between (valence) electrons in solids $\Rightarrow$

- Spontaneous symmetry breaking (magnetic order, superconductivity)
- Correlation gaps without symmetry-breaking (e.g. Mott metal-insulator transition)
- Kondo effect
- Exotic liquids (*Luttinger liquids*, quantum critical systems)
- ...

The most striking phenomena involve electronic correlations beyond conventional mean-field theories (Hartree-Fock, LDA etc.).

Scale problem:

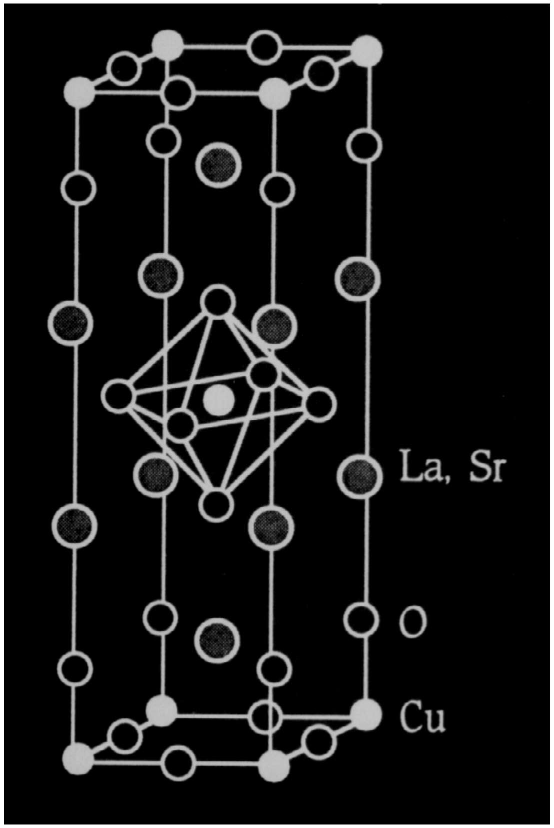
Very different behavior on different energy scales

Collective phenomena, coherence, and composite objects often emerge at scales far below bare energy scales of microscopic Hamiltonian

$\Longrightarrow$ PROBLEM

- for straightforward numerical treatments of microscopic systems
- for conventional many-body methods which treat all scales at once and within the same approximation (e.g. summing subsets of Feynman diagrams)

Example: High temperature superconductors

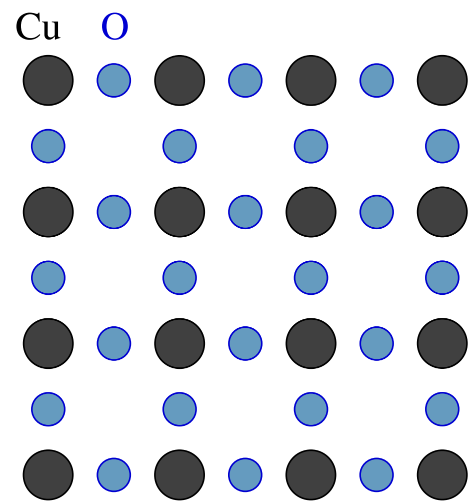


$La_{2-x}Sr_xCuO_4$

Bednorz + Müller 1986

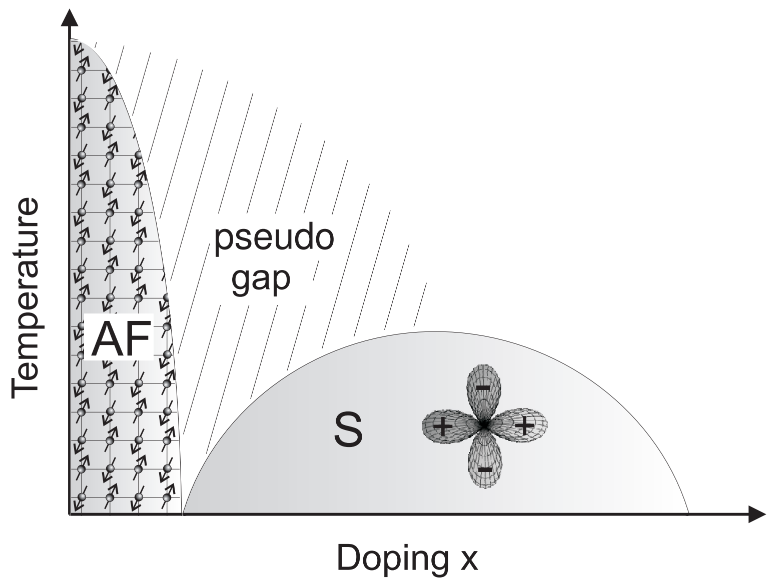
Common structural element:

CuO_2-planes



transverse
coupling
relatively
weak

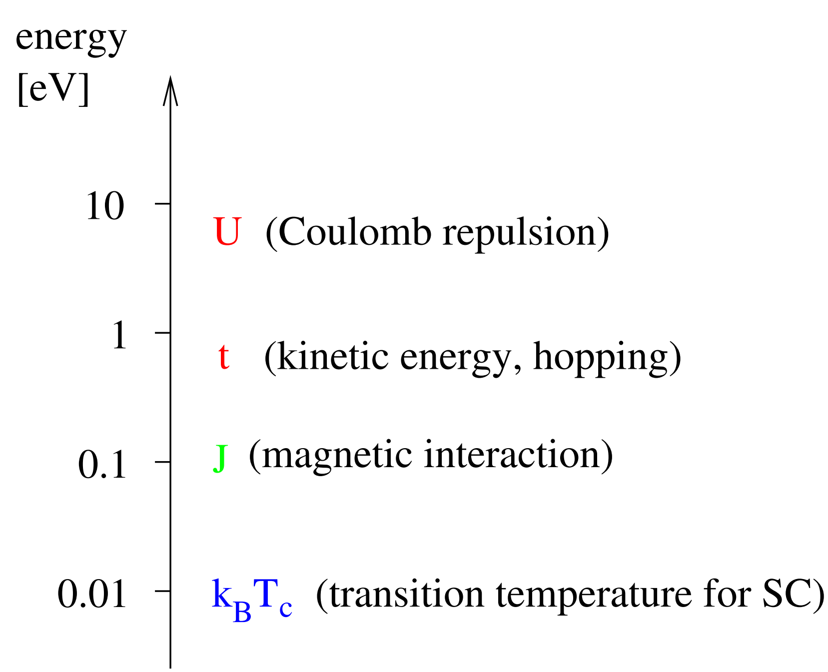
Generic HTSC phase diagram:



- antiferromagnetism
  in undoped compounds
- d-wave superconductivity
  at sufficient doping
- Pseudo gap, non-Fermi liquid
  in "normal" phase at finite T

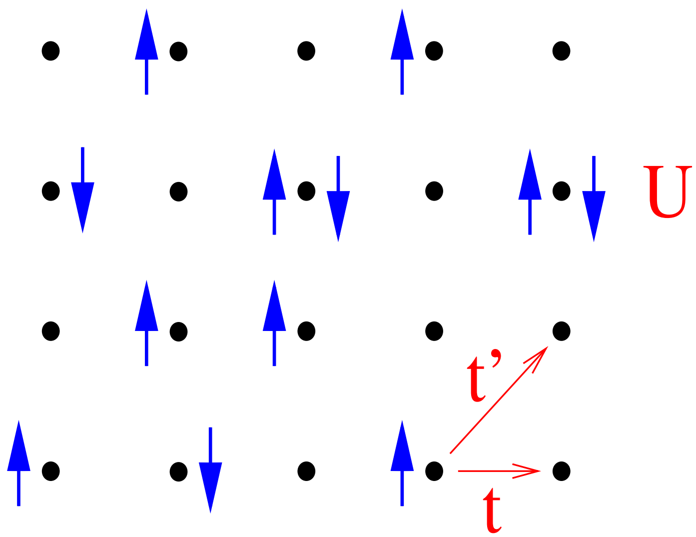
Vast hierarchy of energy scales:

Magnetic interaction and
superconductivity generated
from kinetic energy and
Coulomb interaction



Effective single-band model for $\mathrm{CuO_2}$-planes in HTSC:

2D Hubbard model (Anderson '87, Zhang & Rice '88)



Hamiltonian $H = H_{kin} + H_I$

$$H_{kin} = \sum_{\mathbf{i},\mathbf{j}} \sum_{\sigma} t_{\mathbf{ij}}\, c^{\dagger}_{\mathbf{i}\sigma} c_{\mathbf{j}\sigma} = \sum_{\mathbf{k},\sigma} \epsilon_{\mathbf{k}}\, n_{\mathbf{k}\sigma}$$

$$H_I = U \sum_{\mathbf{j}} n_{\mathbf{j}\uparrow} n_{\mathbf{j}\downarrow}$$

Antiferromagnet at half-filling for sufficiently large U (easy to understand)

Superconductivity?

Phase diagram and other properties extremely hard to compute !

2. Perturbation theory and infrared divergences

Physical properties of interacting electron (and other) systems follow from Green functions

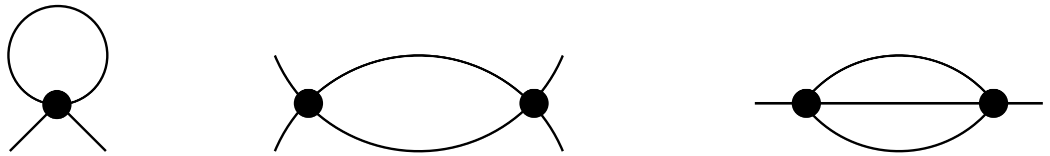
$$G^{(m)}(K_1, \dots, K_m; K'_1, \dots, K'_m) = -\langle \psi_{K_1} \dots \psi_{K_m} \bar{\psi}_{K'_m} \dots \bar{\psi}_{K'_1} \rangle_c$$

with multi-index K containing single-particle quantum numbers and (Matsubara) frequency variable, e.g. $K = (k_0, \mathbf{k}, \sigma)$;

$G^{(m)}$ yields expectation values of m-body operators, m-particle excitation spectra, response functions, $G = G^{(1)}$ yields also thermodynamics.

Expansion of $G^{(m)}$ (or one-particle irreducible vertex functions $\Gamma^{(m)}$) in powers of coupling constant $\quad \Rightarrow$

Perturbative contributions described by Feynman diagrams



etc.

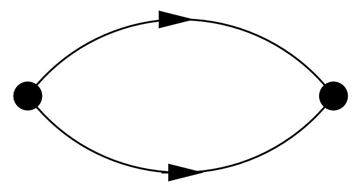
lines $\longleftrightarrow$ bare propagator $G_0(k_0, \mathbf{k}) = \dfrac{1}{ik_0 + \mu - \epsilon_{\mathbf{k}}}$

vertices $\longleftrightarrow$ interaction

Propagator singular for $k_0 = 0$, $\epsilon_{\mathbf{k}} = \mu$ (non-interacting Fermi surface)

$\Rightarrow$ infrared divergences

Infrared divergence in particle-particle bubble:



For vanishing total momentum (Cooper channel)
at $T = 0$

$$\text{pp-bubble} \propto \int dk_0 \int d^d k \, \frac{1}{ik_0 - \xi_{\mathbf{k}}} \, \frac{1}{-ik_0 - \xi_{-\mathbf{k}}} \overset{\xi_{-\mathbf{k}} = \xi_{\mathbf{k}}}{=}$$

$$\int dk_0 \int d^d k \, \frac{1}{k_0^2 + \xi_{\mathbf{k}}^2} = \int dk_0 \int d\xi \, \frac{N(\xi)}{k_0^2 + \xi^2}$$

logarithmically divergent in *any* dimension if $N(0) \neq 0$

$\Rightarrow$ Cooper instability, superconductivity

Note: Propagator divergent on $(d-1)$-dimensional manifold, embedded in $(d+1)$-dimensional space (spanned by k_0 and $\mathbf{k}$)

3. Renormalization group idea

Strategy to deal with hierarchy of energy scales and infrared divergences ?

Main idea (Wilson):
Treat degrees of freedom with different energy scales successively, descending step by step from the highest scale.

In practice, using functional integral representation:
Integrate degrees of freedom (bosonic or fermionic fields) successively, following a suitable hierarchy of energy scales.

$\Rightarrow$ One-parameter family of effective actions $\mathcal{S}^{\Lambda}$, interpolating smoothly between bare action and final effective action (for $\Lambda \to 0$) from which all physical properties can be extracted.

Renormalization group map: $\mathcal{S}^{\Lambda} \mapsto \mathcal{S}^{\Lambda'}$ with $\Lambda' < \Lambda$

Discrete version: $\Lambda' = \Lambda/b$ with $b > 1$

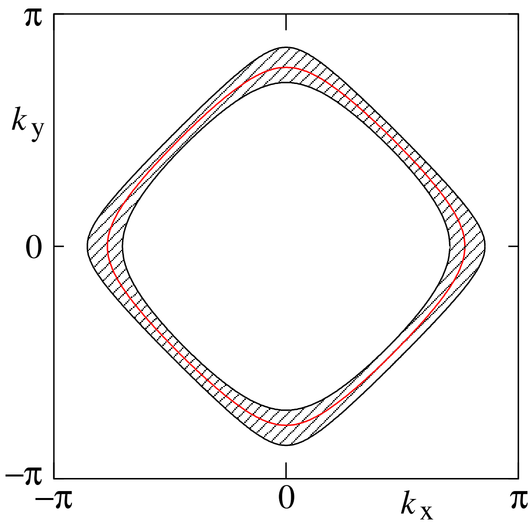
Continuous version: $\Lambda' = \Lambda - d\Lambda$

The final effective action is obtained by iterating the RG map, which amounts to solving a differential flow equation $\partial_{\Lambda}\mathcal{S}^{\Lambda} = \beta^{\Lambda}[\mathcal{S}^{\Lambda}]$ in the continuous version.

Advantage:
Small steps from Λ to Λ' easier to control than going from highest scale Λ_0 to $\Lambda = 0$ in one shot. Easier for:

- rigorous estimates
- controlled approximations (regular perturbative expansions et al.)

Effective actions $\mathcal{S}^{\Lambda}$ can be defined for example by integrating only fields with momenta satisfying $|\xi_{\mathbf{k}}| > \Lambda$, which excludes a momentum shell around the Fermi surface.



Momentum space region around the Fermi surface excluded by a sharp momentum cutoff in a 2D lattice model

History of RG for Fermi systems:

Long tradition in 1D systems, starting in 1970s (Solyom, ...); mostly field-theoretical RG with few couplings.

RG work for 2D or 3D Fermi systems with renormalization of interaction functions started in 1990s and can be classified as

- rigorous:
  Feldman, Trubowitz, Knörrer, Magnen, Rivasseau, Salmhofer;
  Benfatto, Gallavotti; ...
- pedagogical:
  Shankar; Polchinski; ...
- computational (using “functional RG”):
  Zanchi, Schulz; Halboth, Metzner; Honerkamp, Salmhofer, Rice; ...