

Part II: Functional RG for Fermi systems

A natural way of dealing with **many energy scales** in interacting electron systems and a powerful source of **new approximations**.

- applicable to **microscopic** models (not only field theory)
- no adjustable parameters
- RG treatment of **infrared singularities** built in

1. Generating functionals
2. Exact flow equations
3. Truncations

Review:

Rev. Mod. Phys. **84**, 299 (2012)

1. Generating functionals

Textbook: Negele & Orland,
Quantum Many-Particle Systems

Interacting Fermi system with bare **action**

$$\mathcal{S}[\psi, \bar{\psi}] = -(\bar{\psi}, G_0^{-1}\psi) + V[\psi, \bar{\psi}]$$

$\psi_K, \bar{\psi}_K$ Grassmann variables, $K =$ quantum numbers + Matsubara frequency

G_0 bare propagator, $V[\psi, \bar{\psi}]$ interaction

$$(\bar{\psi}, G_0^{-1}\psi) = \sum_K \bar{\psi}_K (G_0^{-1}\psi)_K \quad \text{with} \quad (G_0^{-1}\psi)_K = \sum_{K'} (G_0^{-1})_{KK'} \psi_{K'}$$

Spin- $\frac{1}{2}$ fermions with momentum \mathbf{k} and spin orientation σ : $K = (k_0, \mathbf{k}, \sigma)$

Bare propagator in case of translation and spin-rotation invariance:

$$G_0(K) = \frac{1}{ik_0 - \xi_{\mathbf{k}}} \quad (\text{diagonal}), \quad \text{where} \quad \xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu$$

Two-particle interaction:

$$V[\psi, \bar{\psi}] = \frac{1}{4} \sum_{K_1, K_2} \sum_{K'_1, K'_2} V(K'_1, K'_2; K_1, K_2) \bar{\psi}_{K'_1} \psi_{K_1} \bar{\psi}_{K'_2} \psi_{K_2}$$

Generating functional for **connected Green functions**

$$\mathcal{G}[\eta, \bar{\eta}] = -\log \left\{ \int \prod_K d\psi_K d\bar{\psi}_K e^{-S[\psi, \bar{\psi}]} e^{(\bar{\eta}, \psi) + (\bar{\psi}, \eta)} \right\}$$

Connected m -particle Green function

$$G^{(m)}(K_1, \dots, K_m; K'_1, \dots, K'_m) = - \underbrace{\langle \psi_{K_1} \dots \psi_{K_m} \bar{\psi}_{K'_m} \dots \bar{\psi}_{K'_1} \rangle_c}_{\text{connected average}} = \frac{\partial^m}{\partial \eta_{K'_1} \dots \partial \eta_{K'_m}} \frac{\partial^m}{\partial \bar{\eta}_{K_m} \dots \partial \bar{\eta}_{K_1}} \mathcal{G}[\eta, \bar{\eta}] \Big|_{\eta = \bar{\eta} = 0}$$

Legendre transform of $\mathcal{G}[\eta, \bar{\eta}]$: **effective action**

$$\Gamma[\psi, \bar{\psi}] = \mathcal{G}[\eta, \bar{\eta}] + (\bar{\psi}, \eta) + (\bar{\eta}, \psi) \quad \text{with} \quad \psi = -\frac{\partial \mathcal{G}}{\partial \bar{\eta}} \quad \text{and} \quad \bar{\psi} = \frac{\partial \mathcal{G}}{\partial \eta}$$

generates one-particle irreducible (1PI) **vertex functions** $\Gamma^{(m)}$

$$\Gamma^{(1)} = G^{-1} = G_0^{-1} - \Sigma$$

Reciprocity relations at finite source fields:

$$\frac{\partial \Gamma}{\partial \psi} = -\bar{\eta} \quad , \quad \frac{\partial \Gamma}{\partial \bar{\psi}} = \eta$$

$$\mathbf{\Gamma}^{(1)}[\psi, \bar{\psi}] = \left(\mathbf{G}^{(1)}[\eta, \bar{\eta}] \right)^{-1}$$

where $\mathbf{\Gamma}^{(1)}$ and $\mathbf{G}^{(1)}$ are matrices of second derivatives ...

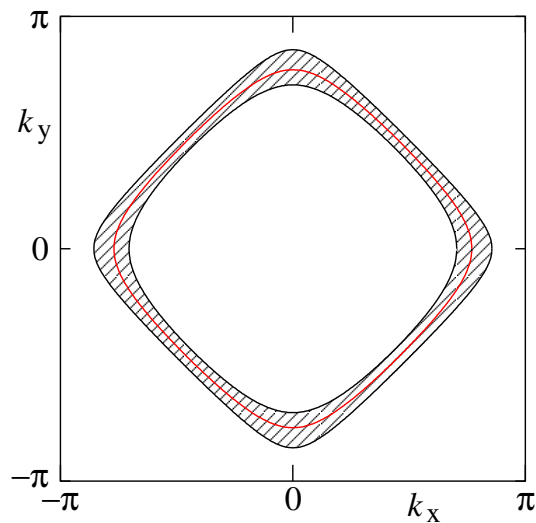
$$\mathbf{G}^{(1)}[\eta, \bar{\eta}] = \begin{pmatrix} \frac{\partial^2 \mathcal{G}}{\partial \bar{\eta}_K \partial \eta_{K'}} & \frac{\partial^2 \mathcal{G}}{\partial \bar{\eta}_K \partial \bar{\eta}_{K'}} \\ \frac{\partial^2 \mathcal{G}}{\partial \eta_K \partial \eta_{K'}} & \frac{\partial^2 \mathcal{G}}{\partial \eta_K \partial \bar{\eta}_{K'}} \end{pmatrix} = - \begin{pmatrix} \langle \psi_K \bar{\psi}_{K'} \rangle & \langle \psi_K \psi_{K'} \rangle \\ \langle \bar{\psi}_K \bar{\psi}_{K'} \rangle & \langle \bar{\psi}_K \psi_{K'} \rangle \end{pmatrix}$$

$$\mathbf{\Gamma}^{(1)}[\psi, \bar{\psi}] = \begin{pmatrix} \frac{\partial^2 \Gamma}{\partial \bar{\psi}_K \partial \psi_{K'}} & \frac{\partial^2 \Gamma}{\partial \bar{\psi}_K \partial \bar{\psi}_{K'}} \\ \frac{\partial^2 \Gamma}{\partial \psi_K \partial \psi_{K'}} & \frac{\partial^2 \Gamma}{\partial \psi_K \partial \bar{\psi}_{K'}} \end{pmatrix}$$

2. Exact flow equations

Impose **infrared cutoff** at energy scale $\Lambda > 0$, e.g. a momentum cutoff

$$G_0^\Lambda(k_0, \mathbf{k}) = \frac{\Theta^\Lambda(\mathbf{k})}{ik_0 - \xi_{\mathbf{k}}} \quad \text{with} \quad \Theta^\Lambda(\mathbf{k}) = \Theta(|\xi_{\mathbf{k}}| - \Lambda)$$



*Momentum space region around the **Fermi surface** excluded by a sharp momentum cutoff in a **2D** lattice model*

Cutoff **regularizes divergence** of $G_0(k_0, \mathbf{k})$ in $k_0 = 0$, $\xi_{\mathbf{k}} = 0$ (Fermi surface)

Other choices: smooth cutoff, frequency cutoff,

mixed momentum-frequency cutoff $\Theta^\Lambda(\sqrt{\xi_{\mathbf{k}}^2 + k_0^2})$

Cutoff excludes "soft modes" below scale Λ from functional integral.

Λ -dependent functionals $\mathcal{G}^\Lambda[\eta, \bar{\eta}]$ and $\Gamma^\Lambda[\psi, \bar{\psi}]$.

Functionals \mathcal{G} and Γ recovered for $\Lambda \rightarrow 0$.

Exact flow equation for Γ^Λ :

$$\frac{d}{d\Lambda} \Gamma^\Lambda[\psi, \bar{\psi}] = -(\bar{\psi}, \dot{Q}_0^\Lambda \psi) - \frac{1}{2} \text{tr} \left[\dot{Q}_0^\Lambda \left(\Gamma^{(1)\Lambda}[\psi, \bar{\psi}] \right)^{-1} \right]$$

$$Q_0^\Lambda = (G_0^\Lambda)^{-1} \quad \dot{Q}_0^\Lambda = \partial_\Lambda Q_0^\Lambda$$

$$Q_0^\Lambda = \begin{pmatrix} Q_{0, KK'}^\Lambda & 0 \\ 0 & -Q_{0, K'K}^\Lambda \end{pmatrix} \quad \Gamma^{(1)\Lambda}[\psi, \bar{\psi}] = \begin{pmatrix} \frac{\partial^2 \Gamma^\Lambda}{\partial \bar{\psi}_K \partial \psi_{K'}} & \frac{\partial^2 \Gamma^\Lambda}{\partial \bar{\psi}_K \partial \bar{\psi}_{K'}} \\ \frac{\partial^2 \Gamma^\Lambda}{\partial \psi_K \partial \psi_{K'}} & \frac{\partial^2 \Gamma^\Lambda}{\partial \psi_K \partial \bar{\psi}_{K'}} \end{pmatrix}$$

Wetterich '93, Morris '94, Salmhofer + Honerkamp '01

(derivation later)

Expansion in fields:

$$\mathbf{\Gamma}^{(1)\Lambda}[\psi, \bar{\psi}] = (\mathbf{G}^\Lambda)^{-1} - \tilde{\Sigma}^\Lambda[\psi, \bar{\psi}]$$

where $\mathbf{G}^\Lambda = \left(\mathbf{\Gamma}^{(1)\Lambda}[\psi, \bar{\psi}]|_{\psi=\bar{\psi}=0} \right)^{-1} = \begin{pmatrix} G_{KK'}^\Lambda & 0 \\ 0 & -G_{K'K}^\Lambda \end{pmatrix}$

$\tilde{\Sigma}^\Lambda[\psi, \bar{\psi}]$ contains all contributions to $\mathbf{\Gamma}^{(1)\Lambda}[\psi, \bar{\psi}]$ which are at least quadratic in the fields.

$$\left(\mathbf{\Gamma}^{(1)\Lambda}[\psi, \bar{\psi}] \right)^{-1} = (1 - \mathbf{G}^\Lambda \tilde{\Sigma}^\Lambda)^{-1} \mathbf{G}^\Lambda = [1 + \mathbf{G}^\Lambda \tilde{\Sigma}^\Lambda + (\mathbf{G}^\Lambda \tilde{\Sigma}^\Lambda)^2 + \dots] \mathbf{G}^\Lambda \Rightarrow$$

$$\frac{d}{d\Lambda} \mathbf{\Gamma}^\Lambda = -\text{tr}[\dot{Q}_0^\Lambda G^\Lambda] - (\bar{\psi}, \dot{Q}_0^\Lambda \psi) + \frac{1}{2} \text{tr}[\mathbf{S}^\Lambda (\tilde{\Sigma}^\Lambda + \tilde{\Sigma}^\Lambda \mathbf{G}^\Lambda \tilde{\Sigma}^\Lambda + \dots)]$$

where $\mathbf{S}^\Lambda = -\mathbf{G}^\Lambda \dot{Q}_0^\Lambda \mathbf{G}^\Lambda = \frac{d}{d\Lambda} \mathbf{G}^\Lambda|_{\Sigma^\Lambda \text{ fixed}}$ "single scale propagator"

Expand $\mathbf{\Gamma}^\Lambda[\psi, \bar{\psi}]$ and $\tilde{\Sigma}^\Lambda[\psi, \bar{\psi}]$ in powers of ψ and $\bar{\psi}$, compare coefficients \Rightarrow

Flow equations for self-energy $\Sigma^\Lambda = Q_0^\Lambda - \Gamma^{(1)\Lambda}$, two-particle vertex $\Gamma^{(2)\Lambda}$, and many-particle vertices $\Gamma^{(3)\Lambda}$, $\Gamma^{(4)\Lambda}$, etc.

$$\frac{d}{d\Lambda} \Sigma^\Lambda = \text{Diagram: a circle with a shaded center and a loop on top labeled } S^\Lambda$$

$$S^\Lambda = \frac{d}{d\Lambda} G^\Lambda \Big|_{\Sigma^\Lambda \text{ fixed}}$$

$$\frac{d}{d\Lambda} \Gamma^{(2)\Lambda} = \text{Diagram: two shaded circles connected by two arcs, top arc labeled } S^\Lambda, \text{ bottom arc labeled } G^\Lambda, \text{ each circle labeled } \Gamma^{(2)\Lambda} + \text{Diagram: a shaded circle with a loop on top labeled } S^\Lambda \text{ and labeled } \Gamma^{(3)\Lambda}$$

$$\frac{d}{d\Lambda} \Gamma^{(3)\Lambda} = \text{Diagram: three shaded circles in a triangle, top circle labeled } \Gamma^{(2)\Lambda}, \text{ bottom-left labeled } \Gamma^{(2)\Lambda}, \text{ bottom-right labeled } \Gamma^{(2)\Lambda}, \text{ top arc labeled } S^\Lambda, \text{ bottom arc labeled } G^\Lambda + \text{Diagram: two shaded circles connected by two arcs, top arc labeled } S^\Lambda, \text{ bottom arc labeled } G^\Lambda, \text{ left circle labeled } \Gamma^{(2)\Lambda}, \text{ right circle labeled } \Gamma^{(3)\Lambda} + \text{Diagram: a shaded circle with a loop on top labeled } S^\Lambda \text{ and labeled } \Gamma^{(4)\Lambda}$$

Hierarchy of **1-loop** diagrams; all **one-particle irreducible**

Initial conditions:

$\Sigma^{\Lambda_0} =$ bare single-particle potential (if any)

$\Gamma^{(2)\Lambda_0} =$ antisymmetrized bare two-particle interaction

$\Gamma^{(m)\Lambda_0} = 0$ for $m \geq 3$

$\Gamma^\Lambda[\psi, \bar{\psi}]$ interpolates between regularized bare action $\mathcal{S}^{\Lambda_0}[\psi, \bar{\psi}]$ for $\Lambda = \Lambda_0$ and generating functional for vertex functions $\Gamma[\psi, \bar{\psi}]$ for $\Lambda = 0$.

Derivation of flow equation:

$$e^{-\mathcal{G}^\Lambda[\eta, \bar{\eta}]} = \int \prod_K d\psi_K d\bar{\psi}_K e^{(\bar{\psi}, \dot{Q}_0^\Lambda \psi)} e^{-V[\psi, \bar{\psi}]} e^{(\bar{\eta}, \psi) + (\bar{\psi}, \eta)}$$

Take Λ -derivative on both sides \Rightarrow

$$\begin{aligned} -(\partial_\Lambda \mathcal{G}^\Lambda) e^{-\mathcal{G}^\Lambda} &= \int \prod_K d\psi_K d\bar{\psi}_K (\bar{\psi}, \dot{Q}_0^\Lambda \psi) e^{(\bar{\psi}, \dot{Q}_0^\Lambda \psi)} e^{-V[\psi, \bar{\psi}]} e^{(\bar{\eta}, \psi) + (\bar{\psi}, \eta)} \\ &= -(\partial_\eta, \dot{Q}_0^\Lambda \partial_{\bar{\eta}}) e^{-\mathcal{G}^\Lambda[\eta, \bar{\eta}]} \end{aligned}$$

\Rightarrow Flow equation for \mathcal{G}^Λ

$$\frac{d}{d\Lambda} \mathcal{G}^\Lambda[\eta, \bar{\eta}] = \left(\frac{\partial \mathcal{G}^\Lambda}{\partial \eta}, \dot{Q}_0^\Lambda \frac{\partial \mathcal{G}^\Lambda}{\partial \bar{\eta}} \right) + \text{tr} \left(\dot{Q}_0^\Lambda \frac{\partial^2 \mathcal{G}^\Lambda}{\partial \bar{\eta} \partial \eta} \right)$$

Legendre transform

$$\Gamma^\Lambda[\psi, \bar{\psi}] = \mathcal{G}^\Lambda[\eta^\Lambda, \bar{\eta}^\Lambda] + (\bar{\psi}, \eta^\Lambda) + (\bar{\eta}^\Lambda, \psi)$$

Note that η^Λ and $\bar{\eta}^\Lambda$ are Λ -dependent functions of ψ and $\bar{\psi}$.

$$\frac{d}{d\Lambda} \Gamma^\Lambda[\psi, \bar{\psi}] = \frac{d}{d\Lambda} \mathcal{G}^\Lambda[\eta^\Lambda, \bar{\eta}^\Lambda] + (\bar{\psi}, \partial_\Lambda \eta^\Lambda) + (\partial_\Lambda \bar{\eta}^\Lambda, \psi)$$

The total derivative acts also on the Λ -dependence of η^Λ and $\bar{\eta}^\Lambda$.

$$\frac{\partial \mathcal{G}^\Lambda}{\partial \bar{\eta}} = -\psi, \quad \frac{\partial \mathcal{G}^\Lambda}{\partial \eta} = \bar{\psi} \quad \Rightarrow \quad \frac{d}{d\Lambda} \Gamma^\Lambda[\psi, \bar{\psi}] = \frac{d}{d\Lambda} \mathcal{G}^\Lambda[\eta^\Lambda, \bar{\eta}^\Lambda] \Big|_{\eta^\Lambda, \bar{\eta}^\Lambda \text{ fixed}}$$

Insert flow equation for \mathcal{G}^Λ and use reciprocity relations between derivatives of \mathcal{G}^Λ and Γ^Λ

\Rightarrow Flow equation for Γ^Λ

$$\frac{d}{d\Lambda} \Gamma^\Lambda[\psi, \bar{\psi}] = -(\bar{\psi}, \dot{Q}_0^\Lambda \psi) - \frac{1}{2} \text{tr} \left[\dot{Q}_0^\Lambda \left(\Gamma^{(1)\Lambda}[\psi, \bar{\psi}] \right)^{-1} \right]$$

Alternative functional RG versions:

- Polchinski flow equations
- Wick ordered flow equations

3. Truncations

Infinite hierarchy of flow equations usually unsolvable.

Two types of **approximation**:

- Truncation of hierarchy at finite order
- Simplified parametrization of effective interactions

Truncations can be justified for **weak coupling** or **small phase space**.

For **bosonic** fields (e.g. order parameter fluctuations) **non-perturbative** truncations, including all powers in fields, are possible.

See, for example, **Jakubczyk et al., PRL 103, 220602 (2009)**

Simple truncations in **one-particle irreducible** fRG:

- Set $\Gamma^{(3)\Lambda} = 0$, neglect self-energy feedback in flow of $\Gamma^{(2)\Lambda}$:

$$\frac{d}{d\Lambda} \Gamma^{(2)\Lambda} = \text{Diagram}$$

Unbiased **stability analysis**
at weak coupling;
d-wave superconductivity
in 2D Hubbard model

- Compute flow of self-energy with bare interaction (neglecting flow of $\Gamma^{(2)\Lambda}$):

$$\frac{d}{d\Lambda} \Sigma^\Lambda = \text{Diagram}$$

Captures properties
of isolated **impurities**
in 1D **Luttinger liquid**

Power counting:

Which interaction terms are important at **low energy**?

Conventional power counting procedure:

rescale momenta, **frequencies** and **fields** after mode elimination such that quadratic part of action remains invariant; see how interaction terms scale.

Consider **1D chiral** Fermi system with linear dispersion $\xi_k = v k$ at $T = 0$

Effective action

$$\mathcal{S}^\Lambda = \int dk_0 \int_{-\Lambda}^{\Lambda} dk (ik_0 - vk) \bar{\psi}_{k_0,k} \psi_{k_0,k} - V^\Lambda[\psi, \bar{\psi}]$$

Mode elimination reduces Λ : $\Lambda' = \Lambda/s$, $s > 1$

Rescale momentum and frequency: $k = k'/s$, $k_0 = k'_0/s \Rightarrow |k'| \leq \Lambda$

$$dk_0 dk (ik_0 - vk) = [dk'_0 dk' (ik'_0 - vk')]/s^3$$

Compensate by **rescaling** fields $\psi = s^{3/2}\psi'$, $\bar{\psi} = s^{3/2}\bar{\psi}'$

Now see scaling of **interaction** terms:

2-particle interaction: $g \int \prod_{j=1}^3 \underbrace{dk_{j0} dk_j}_{s^{-2}} \underbrace{\bar{\psi} \bar{\psi} \psi \psi}_{s^6}$ invariant, "**marginal**"

k-dependence of g : $g(k) = g(0) + \underbrace{\sum_j \gamma_j k_j}_{\text{extra } s^{-1}} + \dots$ "**irrelevant**"

3-particle interaction: $(s^{-2})^5 (s^{3/2})^6 = s^{-1}$ irrelevant **if** $g_3(0)$ finite

Usually $g_3(0)$ of order Λ^{-1} ! **Not irrelevant** !

Power counting in $d > 1$ cannot be done (easily) by scaling, since quadratic term *cannot be restored by homogeneous scaling of momenta!*

Better look directly at behavior of Feynman diagrams.

Interactions generally "less relevant" in $d > 1$ due to stronger phase space restrictions.