Part III: Impurities in Luttinger liquids

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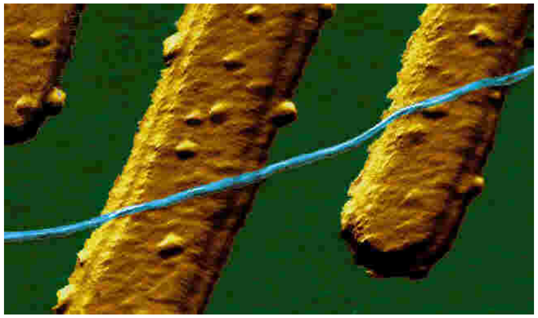
1. Luttinger liquids

One-dimensional interacting Fermi systems without correlation gaps are Luttinger liquids.
(1D counterpart of Fermi liquid in 2D or 3D)

One-dimensional electron systems:

- Complex chemical compounds containing chains
- Quantum wires (in heterostructures)
- Carbon nanotubes
- Edge states



(Dekker's group)

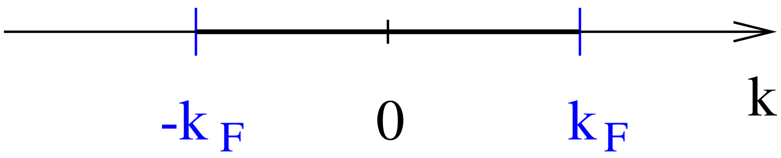
Electronic structure of 1D systems:

Dispersion relations:

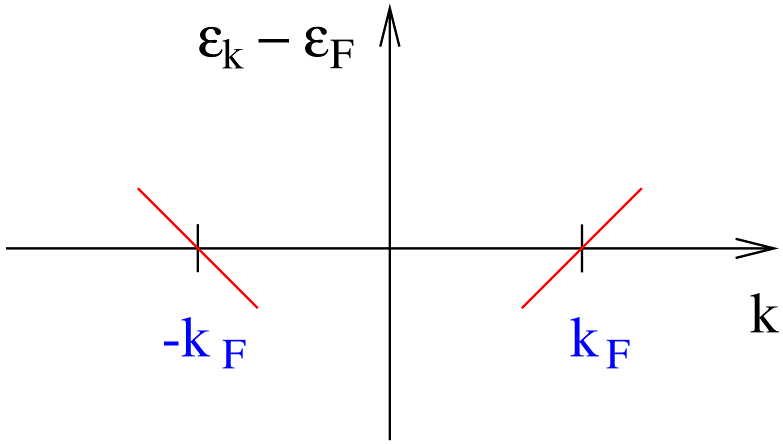
$$\epsilon_k = k^2/2m \quad \text{(low carrier density)}$$

$$\epsilon_k = -2t\cos k \quad \text{(tight binding)}$$

"Fermi surface": 2 points $\pm k_F$



Dispersion relation near Fermi points:



approx. linear:

$$\xi_k = \epsilon_k - \epsilon_F = v_F\,(|k| - k_F)$$

Electron-electron interaction:

has stronger effects than in 2D and 3D systems:

no fermionic quasi-particles, Fermi liquid theory not valid.

Fermi liquid replaced by Luttinger liquid:

- only bosonic low-energy excitations
  (collective charge/spin density oscillations)

- power-laws with non-universal exponents

$\Rightarrow$ Luttinger liquid theory

Textbook: T. Giamarchi: *Quantum physics in one dimension* (2004)

Bulk properties of Luttinger liquids:

- Bosonic low-energy excitations with linear dispersion relation

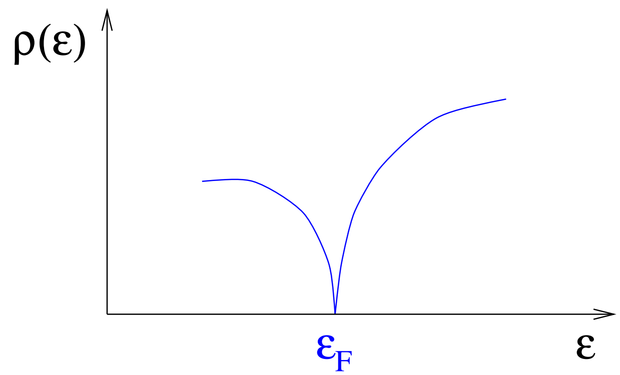
$\xi_q^c = u_c \, q \; , \quad \xi_q^s = u_s \, q$ (charge and spin channel)

$\Rightarrow$ specific heat $c_V \propto T$

- DOS for single-electron excitations:

$$D(\epsilon) \propto |\epsilon - \epsilon_F|^{\alpha}$$

vanishes at Fermi level ($\alpha > 0$)



DOS in principle observable by photoemission or tunneling.

• Density-density correlation function $N(q)$:

finite for $q \rightarrow 0$ (compressibility)

divergent as $|q-2k_F|^{-\alpha_{2k_F}}$ for $q \rightarrow 2k_F$

($\alpha_{2k_F} > 0$ for repulsive interactions)

$\Rightarrow$ enhanced back-scattering ($2k_F$) from impurity.

For spin-rotation invariant (and spinless) systems all exponents can be expressed in terms of one parameter K_ρ.

Asymptotic low energy behavior (power-laws) of Luttinger liquids described by Luttinger model:

$$H_{\mathrm{LM}} = \text{linear } \epsilon_k + \text{forward scattering interactions}$$

It is exactly solvable and scale-invariant (fixed point).

For spinless fermions only one coupling constant, parametrizing interaction between left- and right-movers:

$$H_I = g \int dx \, n_+(x) \, n_-(x)$$

2. Impurity effects

How does a single non-magnetic impurity (potential scatterer) affect properties of a Luttinger liquid?

Non-interacting system:

Impurity induces Friedel oscillations (density oscillations with wave vector $2k_F$)

DOS near impurity finite at Fermi level

Conductance reduced by a finite factor (transmission probability)

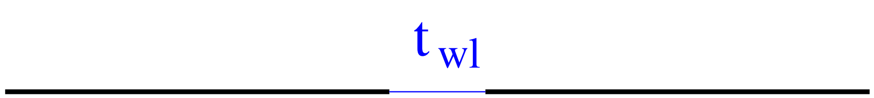
Kane, Fisher '92: impurity in interacting system (spinless Luttinger liquid)

- Weak impurity potential:

Backscattering amplitude V_{2k_F} generated by impurity grows as $\Lambda^{K_\rho - 1}$
for decreasing energy scale Λ.
($K_\rho < 1$ for repulsive interactions; V_{2k_F} is "relevant" perturbation of pure LL)

$\Rightarrow$ Low energy probes see high barrier even if (bare) impurity potential is weak!

- Weak link:



DOS at boundary of LL vanishes as $|\epsilon - \epsilon_F|^{\alpha_B}$ $\Rightarrow$

Tunneling amplitude $t_{\rm wl}$ between two weakly coupled
chains scales to zero as Λ^{α_B} with $\alpha_B = K_\rho^{-1} - 1 > 0$ at low energy scales.
($t_{\rm wl}$ is "irrelevant" perturbation of split chain)

Hypothesis (Kane, Fisher):

Any impurity effectively "cuts the chain" at low energy scales and physical properties obey weak link or boundary scaling. $\Rightarrow$

DOS near impurity:

$$D_i(\epsilon) \propto |\epsilon - \epsilon_F|^{\alpha_B} \quad \text{for } \epsilon \to \epsilon_F \text{ at } T = 0$$

Conductance through impurity:

$$G(T) \propto T^{2\alpha_B} \quad \text{for } T \to 0$$

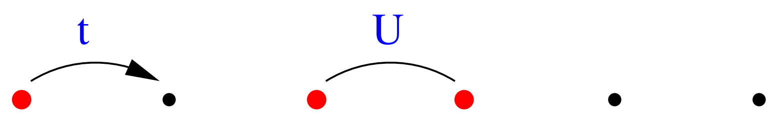
supported within effective bosonic field theory by

- refermionization (Kane, Fisher '92)
- QMC (Moon et al. '93; Egger, Grabert '95)
- Bethe ansatz (Fendley, Ludwig, Saleur '95)

and also within "poor man's" fermionic RG
(Yue, Glazman, Matveev '93)

3. Microscopic model

Spinless fermion model:



nearest neighbor hopping t

nearest neighbor interaction U

$$H_{\rm sf} = -t\sum_j \left(c^\dagger_{j+1}c_j + c^\dagger_j c_{j+1}\right) + U\sum_j n_j n_{j+1}$$

Properties (without impurities):

- exactly solvable by Bethe ansatz
- Luttinger liquid except for $|U| > 2t$ at half-filling
- charge density wave for $U > 2t$ at half-filling

Impurity potential added to bulk hamiltonian $H_{\rm sf}$:

general form: $$H_{\rm imp} = \sum_{j,j'} V_{j'j}\, c^\dagger_{j'} c_j$$

"site impurity":

$$H_{\rm imp} = V n_{j_0} \quad (j_0 \text{ impurity site})$$

"hopping impurity":

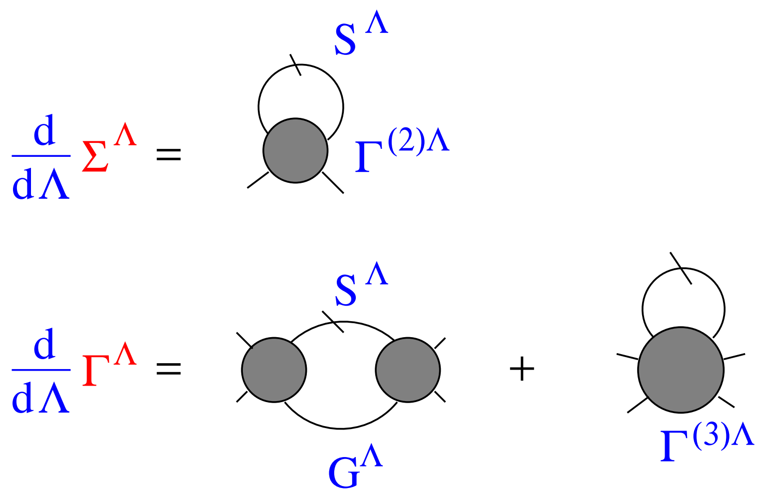
$$H_{\rm imp} = (t - t')\left(c^\dagger_{j_0+1} c_{j_0} + c^\dagger_{j_0} c_{j_0+1}\right)$$

Later also double barrier (two site or hopping impurities)

4. Flow equations

Starting point (for approximations):

Exact hierarchy of differential flow equations for 1-particle irreducible vertex functions with infrared cutoff Λ:



etc. for $\Gamma^{(3)\Lambda}$, $\Gamma^{(4)\Lambda}$, ...

where

$$G^{\Lambda} = \left[(G_0^{\Lambda})^{-1} - \Sigma^{\Lambda}\right]^{-1} \qquad S^{\Lambda} = \left[1 - G_0^{\Lambda}\Sigma^{\Lambda}\right]^{-1} \frac{dG_0^{\Lambda}}{d\Lambda} \left[1 - \Sigma^{\Lambda} G_0^{\Lambda}\right]^{-1}$$

Cutoff:

At $T = 0$ sharp frequency cutoff: $G_0^\Lambda = \Theta(|\omega| - \Lambda)\, G_0$

At finite T (discrete Matsubara frequencies) soft cutoff with width $2\pi T$

G_0 bare propagator without impurities and interaction

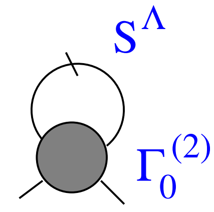
Approximations:

Scheme 1 (first order):

Approximate $\Gamma^{(2)\Lambda} \approx \Gamma_0^{(2)}$ (ignore flow of 2-particle vertex)

$\Rightarrow$ Σ^Λ tridiagonal matrix in real space

$$\frac{\mathrm{d}}{\mathrm{d}\Lambda}\Sigma^\Lambda =$$



Flow equation very simple; at $T = 0$:

$$\frac{d}{d\Lambda}\Sigma^\Lambda_{j,j} = -\frac{U}{2\pi}\sum_{s=\pm 1}\sum_{\omega=\pm\Lambda}\tilde{G}^\Lambda_{j+s,j+s}(i\omega) \qquad \frac{d}{d\Lambda}\Sigma^\Lambda_{j,j\pm 1} = \frac{U}{2\pi}\sum_{\omega=\pm\Lambda}\tilde{G}^\Lambda_{j,j\pm 1}(i\omega)$$

where $\tilde{G}^\Lambda(i\omega) = [G_0^{-1}(i\omega) - \Sigma^\Lambda]^{-1}$.

Kane/Fisher physics already qualitatively captured !

Scheme 2 (second order):

Neglect $\Gamma^{(3)\Lambda}$; approx. $\Gamma^{(2)\Lambda}$ by flowing nearest neighbor interaction U^Λ

$\Rightarrow$ 1-loop flow for U^Λ; flow of Σ^Λ as in scheme 1 with renormalized U^Λ

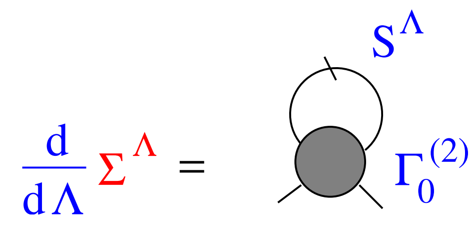
$$\frac{d}{d\Lambda}\Sigma^\Lambda_{j,j} = -\frac{U^\Lambda}{2\pi}\sum_{s=\pm 1}\sum_{\omega=\pm\Lambda}\tilde{G}^\Lambda_{j+s,j+s}(i\omega) \qquad \frac{d}{d\Lambda}\Sigma^\Lambda_{j,j\pm 1} = \frac{U^\Lambda}{2\pi}\sum_{\omega=\pm\Lambda}\tilde{G}^\Lambda_{j,j\pm 1}(i\omega)$$

Works quantitatively even for rather big U

Derivation of flow equation (scheme 1):

Flow equation for self-energy:

$$\frac{d}{d\Lambda}\Sigma^{\Lambda}(1',1) = T\sum_{2,2'} e^{i\omega_2 0^+}\, S^{\Lambda}(2,2')\,\Gamma_0^{(2)}(1',2';1,2)$$



Single-scale propagator

$$S^{\Lambda} = -G^{\Lambda}[\partial_{\Lambda}(G_0^{\Lambda})^{-1}]G^{\Lambda} = \frac{1}{1-G_0^{\Lambda}\Sigma^{\Lambda}}\,\frac{\partial G_0^{\Lambda}}{\partial\Lambda}\,\frac{1}{1-\Sigma^{\Lambda}G_0^{\Lambda}}$$

Self-energy and propagator diagonal in frequency: $\omega_1 = \omega_{1'}$ and $\omega_2 = \omega_{2'}$.

$\Gamma_0^{(2)\Lambda}$ freqency-independent $\Rightarrow$ Σ frequency-independent.

Sharp frequency cutoff ($T = 0$): $G_0^\Lambda(i\omega) = \Theta(|\omega| - \Lambda)\, G_0(i\omega)$ $\Rightarrow$

$$S^\Lambda(i\omega) = -\frac{1}{1 - \Theta(|\omega| - \Lambda) G_0(i\omega) \Sigma^\Lambda} \delta(|\omega| - \Lambda) G_0(i\omega) \frac{1}{1 - \Theta(|\omega| - \Lambda) \Sigma^\Lambda G_0(i\omega)}$$

$\delta(.)$ meets $\Theta(.)$: ill defined!

Consider regularized (smeared) step functions Θ_ϵ with $\delta_\epsilon = \Theta'_\epsilon$, then take limit $\epsilon \to 0$, using

$$\int dx\, \delta_\epsilon(x - \Lambda)\, f[x, \Theta_\epsilon(x - \Lambda)] \xrightarrow{\epsilon \to 0} \int_0^1 dt\, f(\Lambda, t)$$

proof:
substitution $t = \Theta_\epsilon$

Integration can be done analytically, yielding

$$\frac{d}{d\Lambda} \Sigma^\Lambda_{j'_1, j_1} = -\frac{1}{2\pi} \sum_{\omega = \pm\Lambda} \sum_{j_2, j'_2} e^{i\omega 0^+}\, \tilde{G}^\Lambda_{j_2, j'_2}(i\omega)\, \Gamma^{(2)}_{j'_1, j'_2; j_1, j_2}$$

where $\tilde{G}^\Lambda(i\omega) = [G_0^{-1}(i\omega) - \Sigma^\Lambda]^{-1}$

Insert real space structure of bare vertex for spinless fermions with nearest neighbor interaction U:

$$\Gamma^{(2)}_{j_1',j_2';j_1,j_2} = U_{j_1,j_2}\left(\delta_{j_1,j_1'}\delta_{j_2,j_2'} - \delta_{j_1,j_2'}\delta_{j_2,j_1'}\right)$$

$$U_{j_1,j_2} = U\left(\delta_{j_1,j_2-1} + \delta_{j_1,j_2+1}\right)$$

$\Rightarrow$ Flow equations

$$\frac{d}{d\Lambda}\Sigma^\Lambda_{j,j} = -\frac{U}{2\pi}\sum_{s=\pm 1}\sum_{\omega=\pm\Lambda} e^{i\omega 0^+}\,\tilde{G}^\Lambda_{j+s,j+s}(i\omega)$$

$$\frac{d}{d\Lambda}\Sigma^\Lambda_{j,j\pm 1} = \frac{U}{2\pi}\sum_{\omega=\pm\Lambda} e^{i\omega 0^+}\,\tilde{G}^\Lambda_{j,j\pm 1}(i\omega)$$

Convergence factor $e^{i\omega 0^+}$ matters only for $\Lambda \to \infty$

Initial condition at $\Lambda = \Lambda_0 \to \infty$:

$$\Sigma^{\Lambda_0}_{j_1,j_1'} = V_{j_1,j_1'} + \frac{1}{2}\sum_{j_2} \Gamma^{(2)}_{j_1',j_2;j_1,j_2}$$

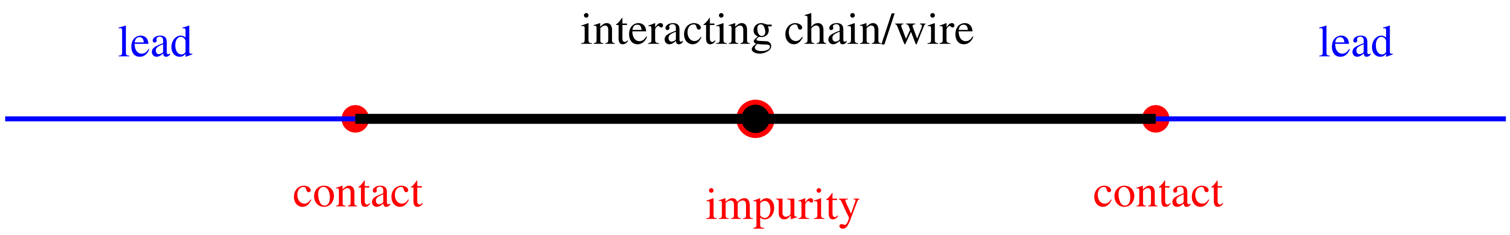
where $V_{j_1,j_1'}$ is the bare impurity potential and
the second term is due to the flow from ∞ to Λ_0 (!)

Flow equations at finite temperatures $T > 0$:

Replace $\omega = \pm\Lambda$ by $\omega = \pm\,\omega_n^{\Lambda}$ in flow equations,
where ω_n^{Λ} is the Matsubara frequency most close to Λ.

Calculation of conductance:

Interacting chain connected to semi-infinite non-interacting leads via smooth or abrupt contacts



Conductance $G(T) = -\frac{e^2}{h} \int d\epsilon \, f'(\epsilon) \, |t(\epsilon)|^2$ with $|t(\epsilon)|^2 \propto |G_{1,N}(\epsilon)|^2$

Propagator $G_{1,N}(\epsilon)$ calculated in presence of leads, which affect the interacting region only via boundary contributions $\Sigma_{1,1}(\epsilon)$ and $\Sigma_{N,N}(\epsilon)$ to the self-energy

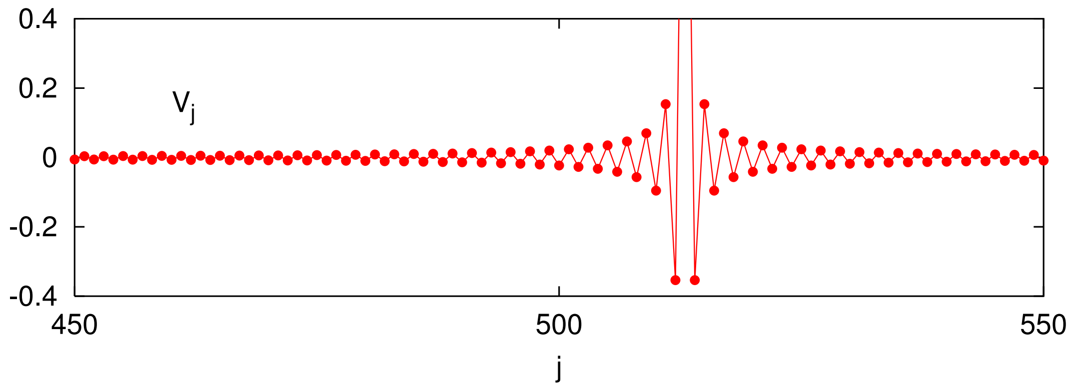
Vertex corrections vanish within our approximation (no inelastic scattering) (see Oguri '01)

fRG features:

- perturbative in U (weak coupling)
- non-perturbative in impurity strength
- arbitrary bare impurity potential (any shape)
- full effective impurity potential
  (cf. "poor man's" RG: only V_{2k_F})
- cheap numerics up to 10^5 sites for $T > 0$ and 10^7 sites at $T = 0$.
- captures all scales, not just asymptotics.

5. Results

Renormalized impurity potential (from self-energy Σ_{jj} at $\Lambda = 0$):

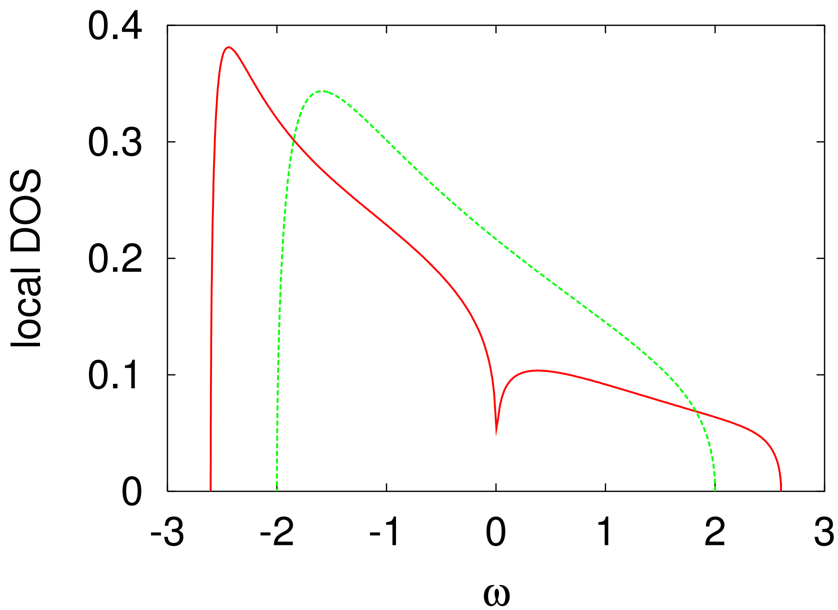


long-range $2k_F$-oscillations ! (associated with Friedel oscillations of density)

$2k_F$-oscillations also in renormalized hopping amplitude around impurity

Results for local DOS near impurity site:

(half-filling, ground state, $U = 1$, $V = 1.5$, 1000 sites)



red: Lutt. liquid

green: Fermi gas

Strong suppression of DOS near Fermi level

Power law with boundary exponent α_B for $\omega \to 0$, $N \to \infty$

Spectral weight at $\omega = 0$ in good agreement with DMRG for $U < 2$.

Log. derivative of spectral weight
at Fermi level as fct. of system size:

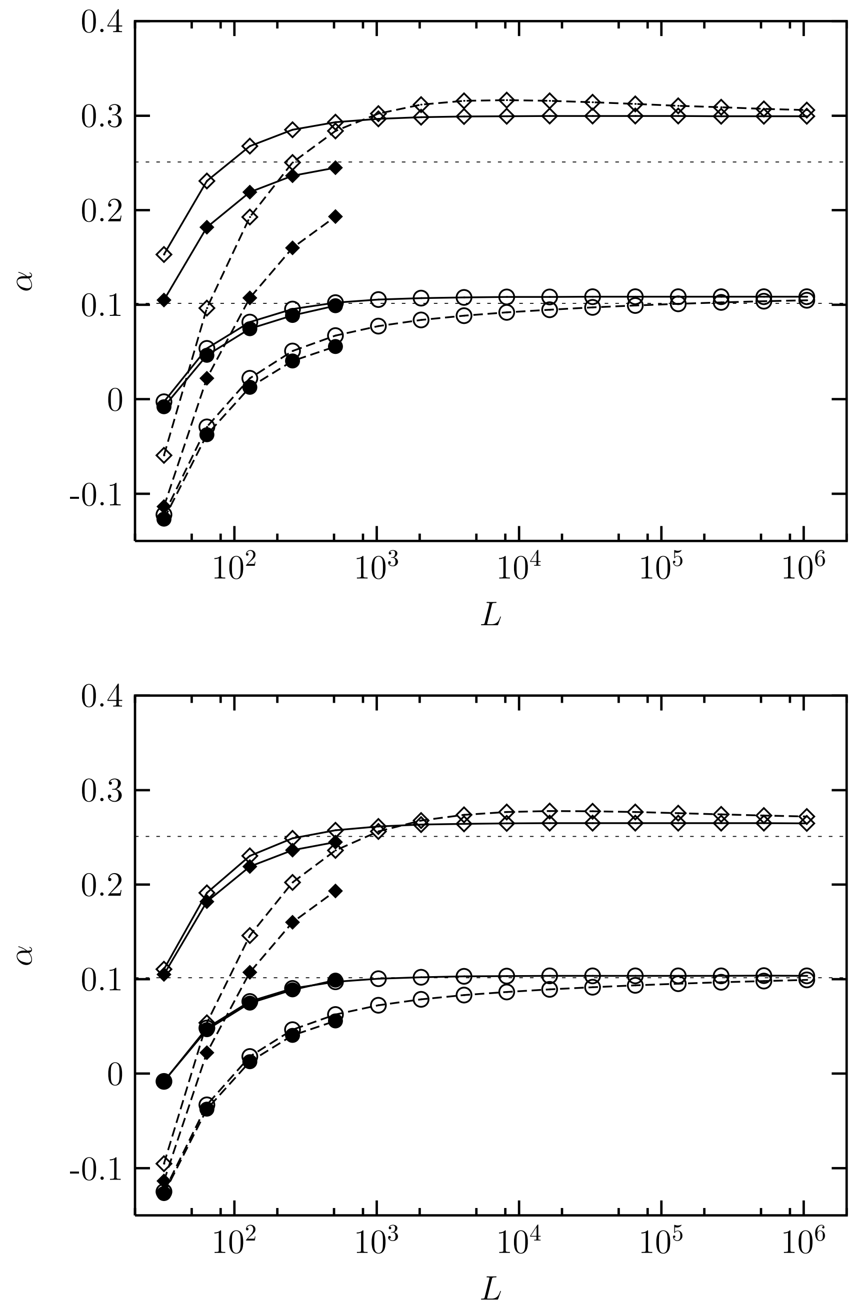
- near boundary (*solid lines*)
- near hopping impurity (*dashed lines*)

circles: quarter-filling, $U = 0.5$
squares: quarter-filling, $U = 1.5$

open symbols: fRG
filled symbols: DMRG

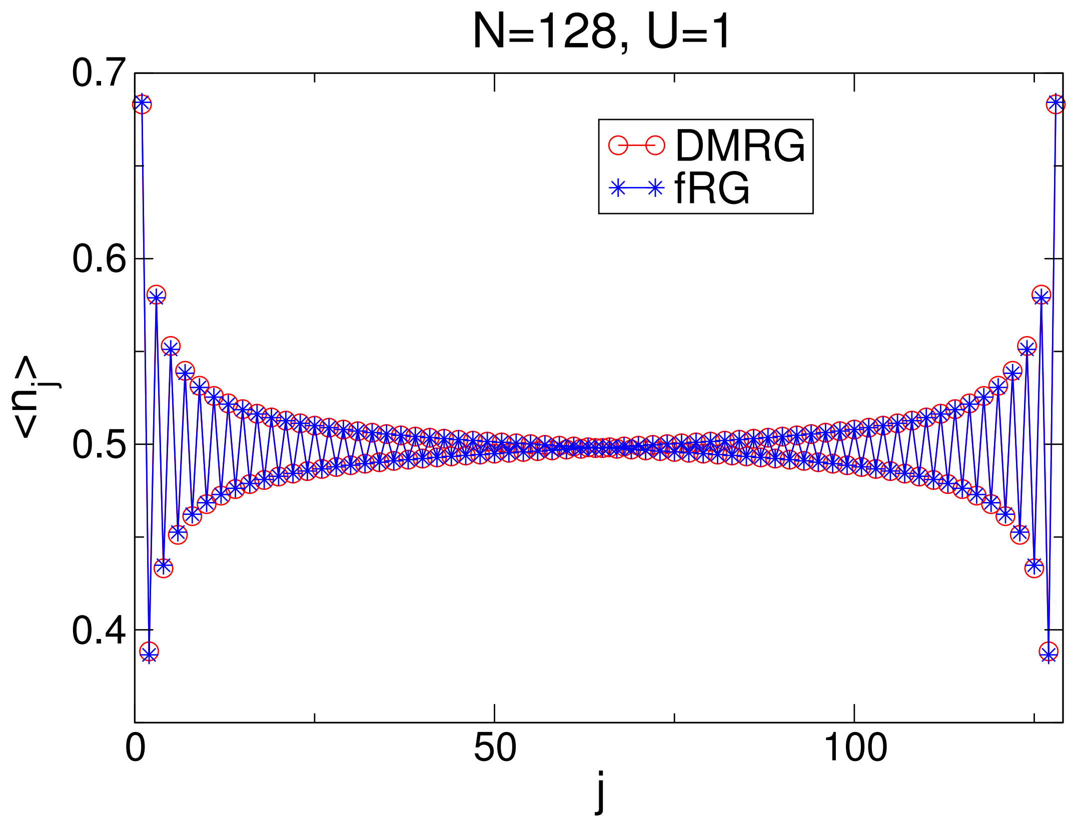
top panel: without vertex renorm.
bottom panel: with vertex renorm.

horizontal lines:
exact boundary exponents



Friedel oscillations from open boundaries:

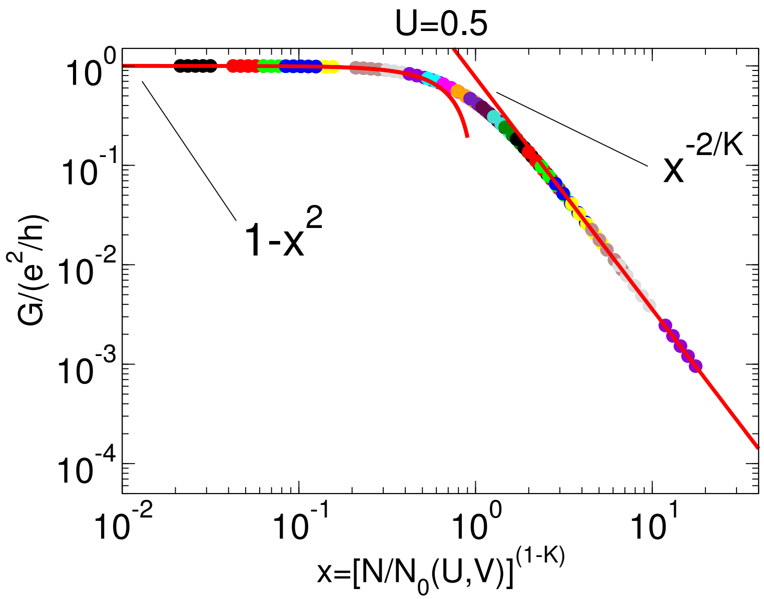
(half-filling, ground state)

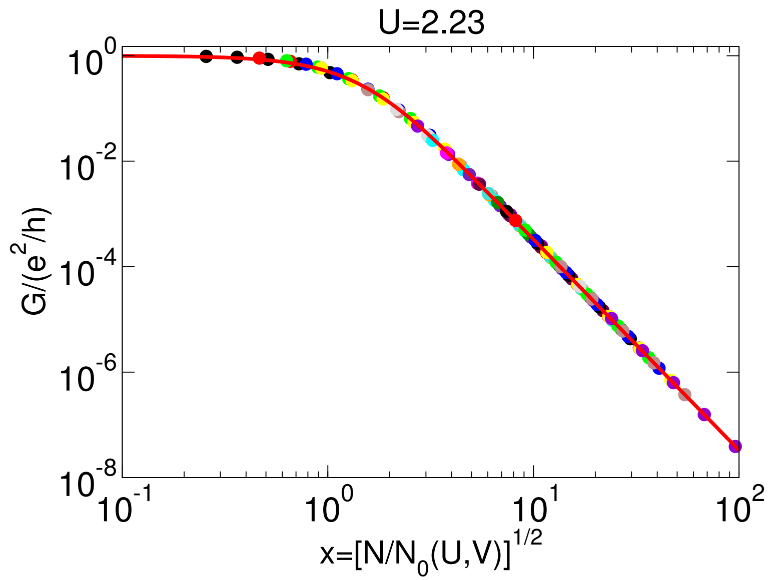


Excellent agreement between fRG and DMRG

One parameter scaling of conductance ($T = 0$):

Single impurity, smooth contacts: $G(N) = \frac{e^2}{h} \tilde{G}_K(x)$, $x = [N/N_0(U,V)]^{1-K}$

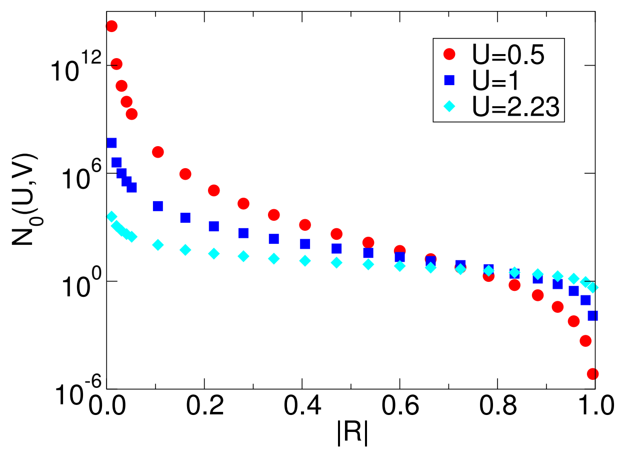




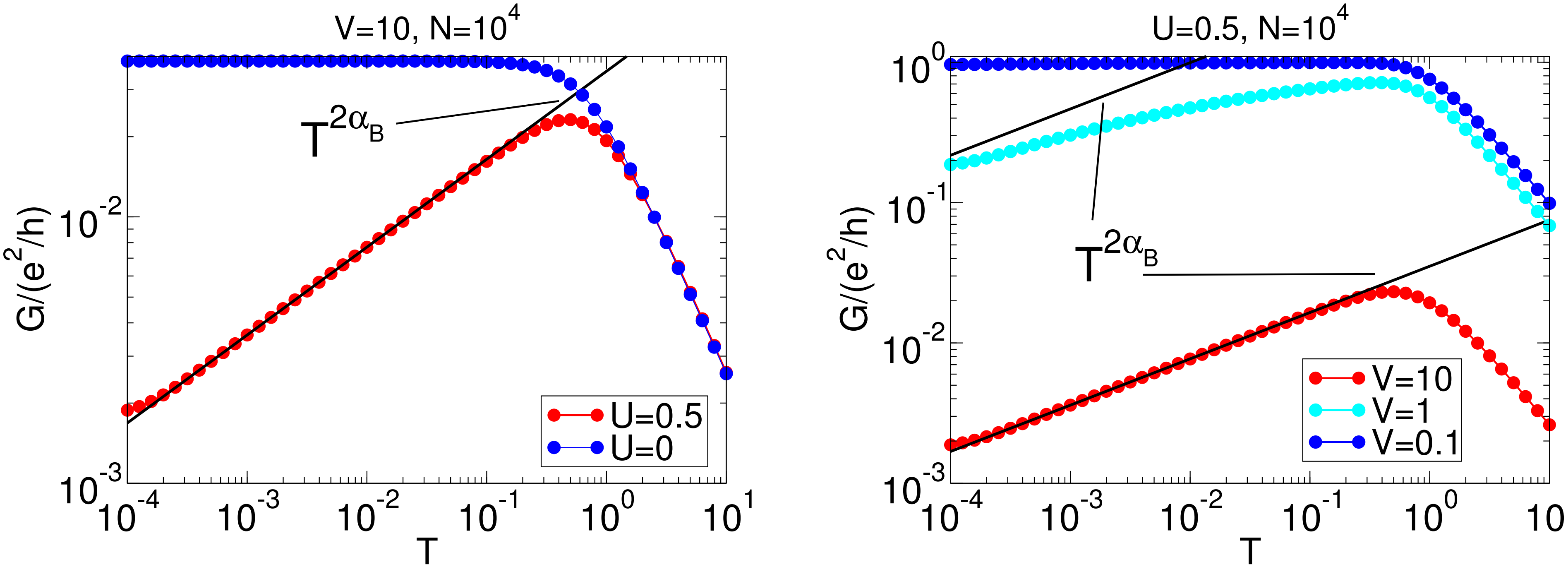
Crossover size

as function of bare

reflection amplitude

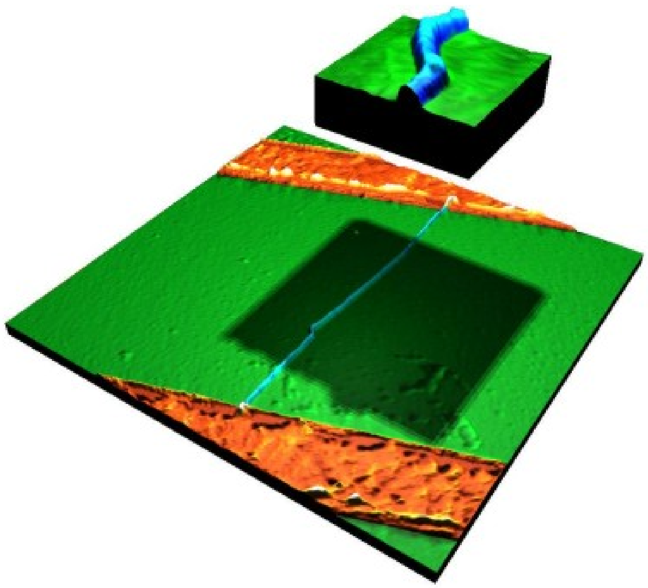


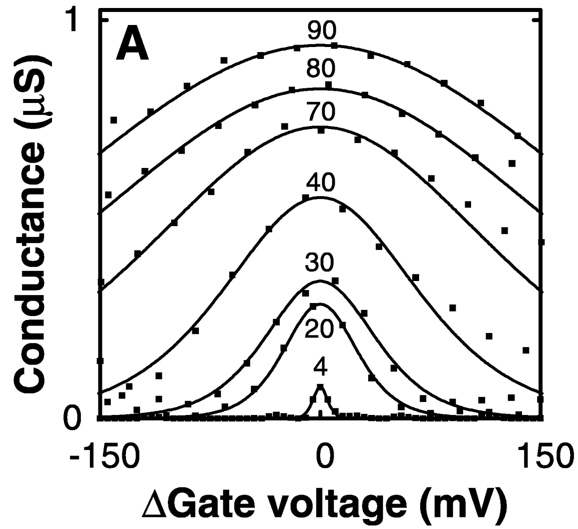
Conductance at $T > 0$ — smooth contacts



Asymptotic power law $G(T) \propto T^{2\alpha}$ reached on accessible scales only for sufficiently strong impurities

Resonant tunneling through double barrier:

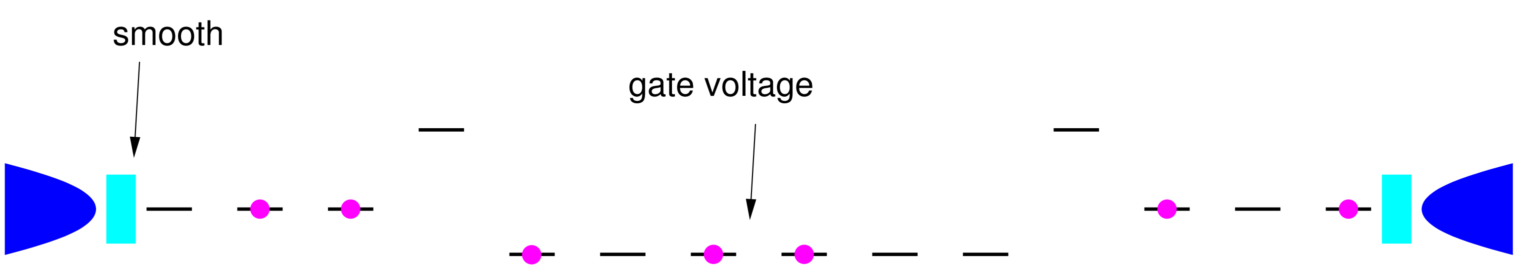




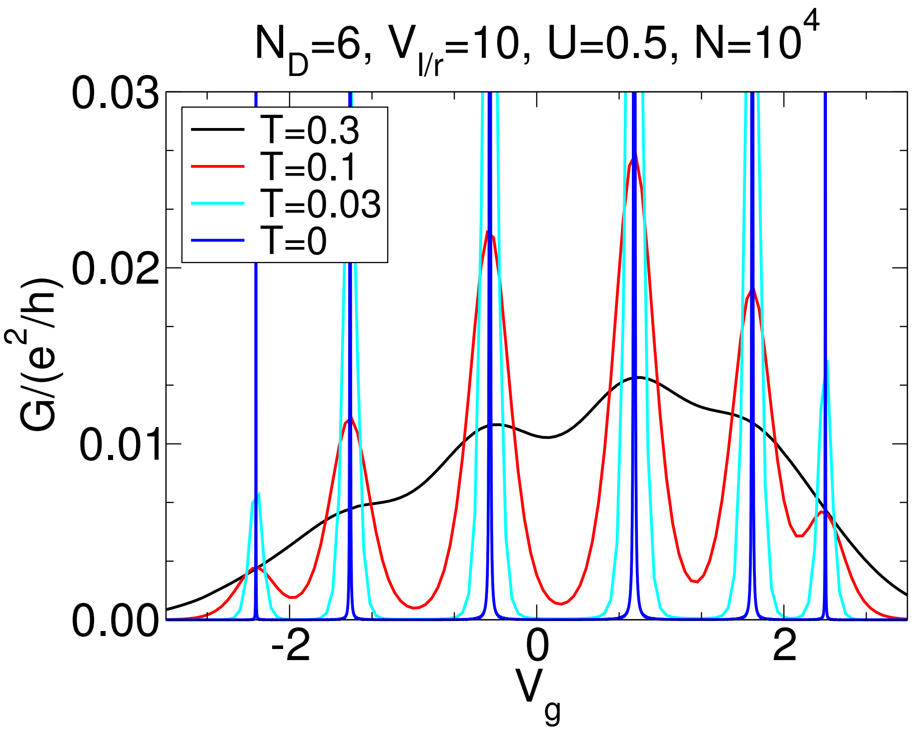
(Dekker's group '01)

Treated theoretically by many groups; controversial results !

Model setup:

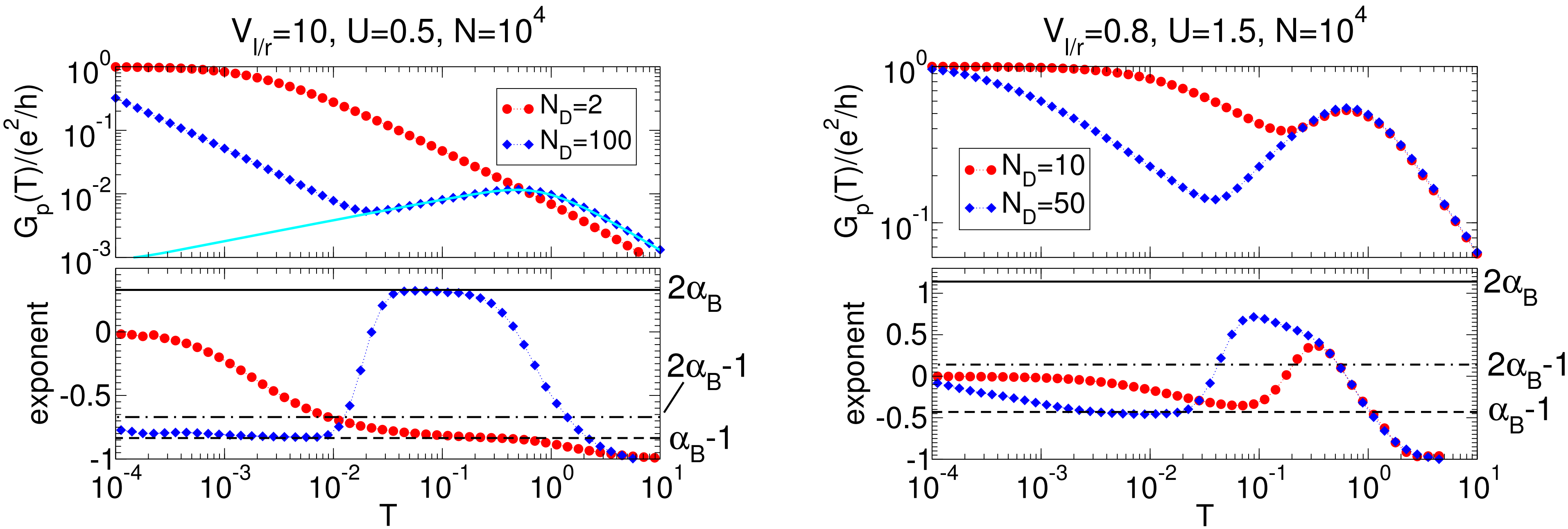


Resonance peaks in conductance as a function of gate voltage:



At $T = 0$, width $w \sim N^{K-1}$

T-dependence of $|t(\epsilon)|^2$ important fRG results for $G_p(T)$ (symmetric double barrier):



Various distinctive power laws, in particular (Furusaki, Nagaosa '93,'98):

- exponent $2\alpha_B$ (looks like independent impurities in series)
- exponent $\alpha_B - 1$ (*"uncorrelated sequential tunneling"*)

No indications of exponent $2\alpha_B - 1$ (*"correlated sequential tunneling"*)

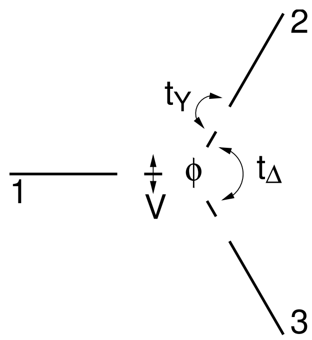
Summary

- fRG is reliable and flexible tool to study Luttinger liquids with impurities
- can be applied to microscopic models, restricted to "weak" coupling
- provides simple physical picture
- interplay of contacts, impurities, and correlations
- method covers all energy scales
- resonant tunneling: universal behavior and crossover captured

Extensions

- include spin:
  e.g. extended Hubbard model: (Andergassen et al. '06)

- more complex geometries:
  e.g. Y-junctions (Barnabé-Thériault et al. '05)



- non-equilibrium transport:
  e.g. through interacting wire (Jakobs et al. '07-'10)