Quantum Condensation: Disorder and Instability

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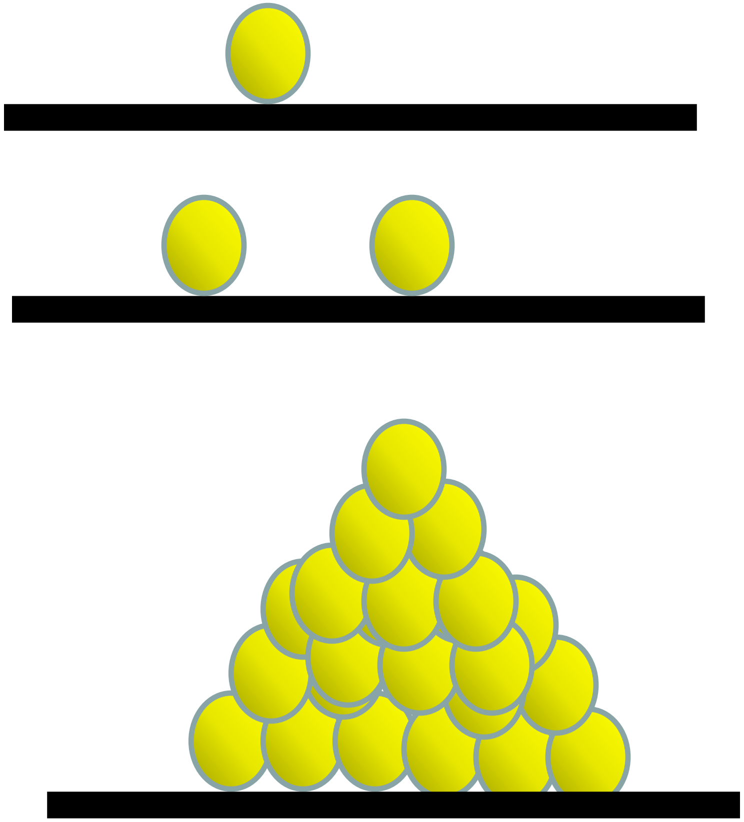
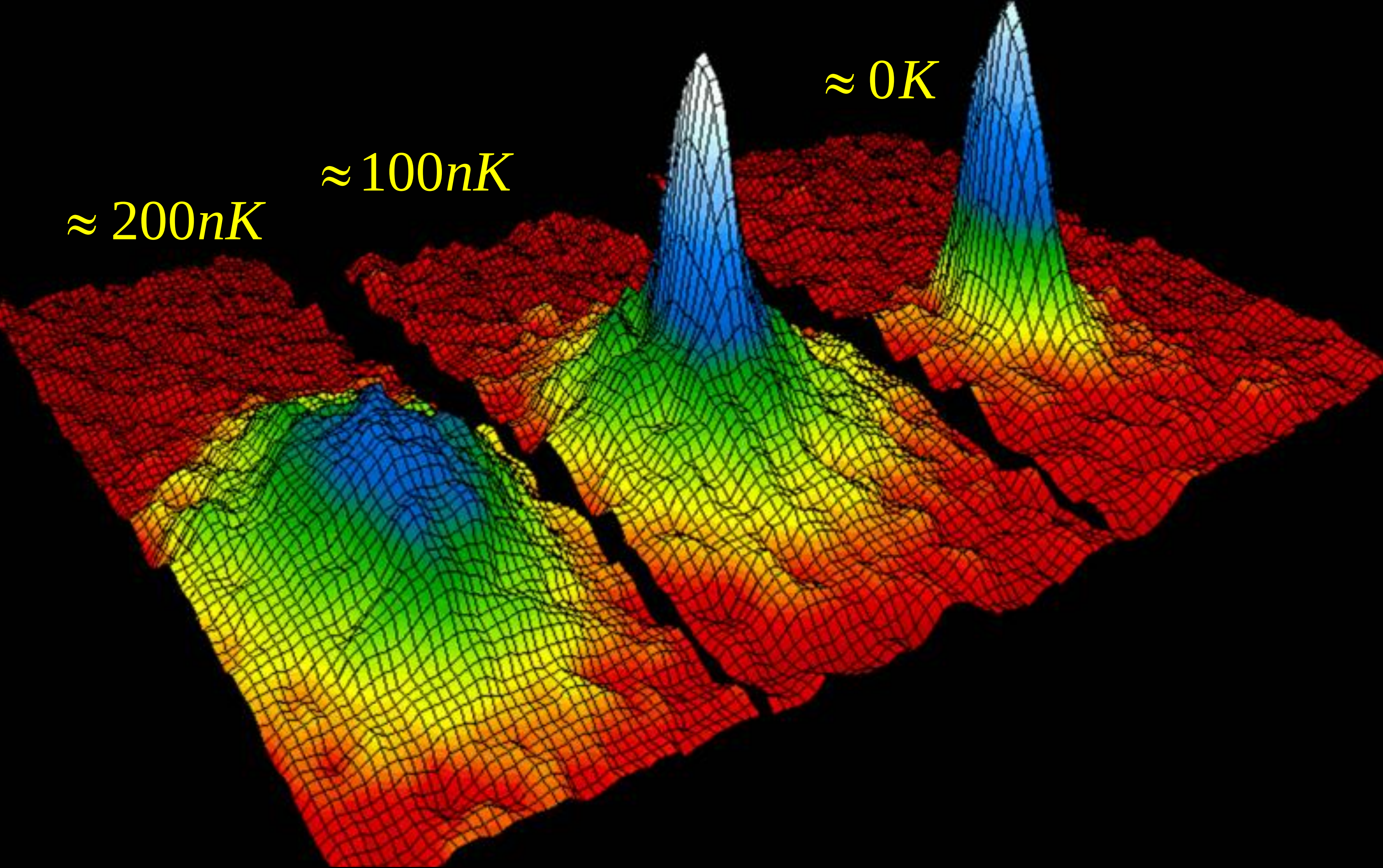
1.Bose Condensation in the presence of disorder

Bose-Einstein Condensation

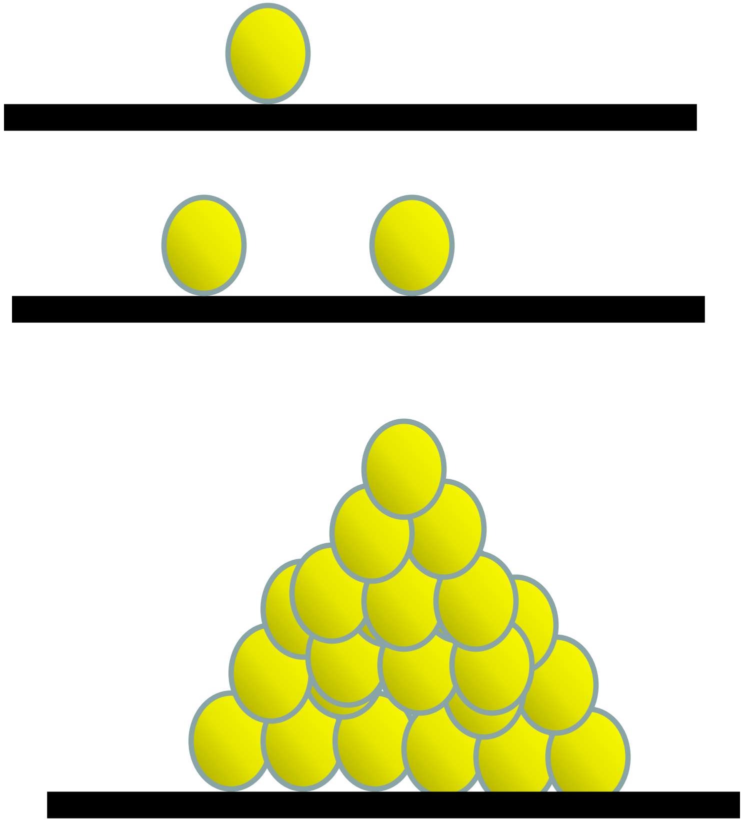


Macroscopic occupation of a single quantum state



Bose–Einstein condensation: 2D velocity distribution of Rb atoms at different temperatures, JILA Science, 1995

Bose-Einstein Condensation

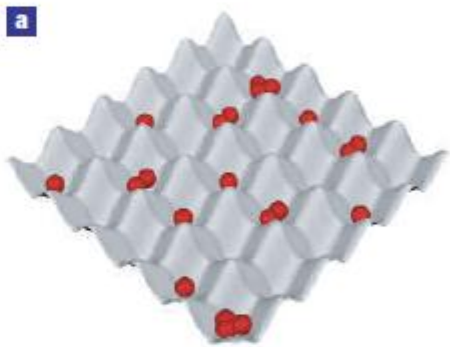


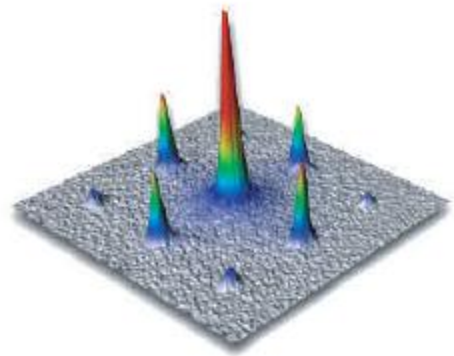
Macroscopic occupation
of a single quantum state

Q: 1. External Potential
2. Interaction between the particles ?

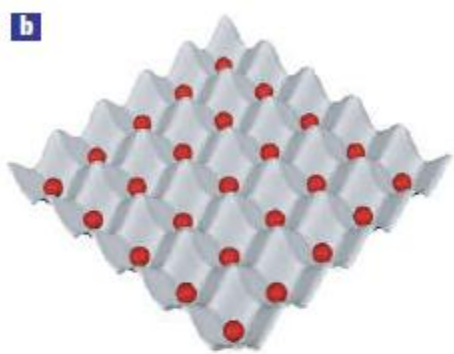
Markus Greiner, Olaf Mandel, Tilman Esslinger, Theodor W. Hänsch & Immanuel Bloch "Quantum phase transition from a superfluid to a Mott insulator in a gas of ultracold atoms" *Nature* **415**, 39-44 (3 January 2002)

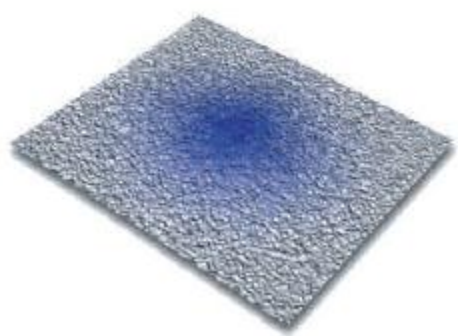
Superfluid





Mott Insulator

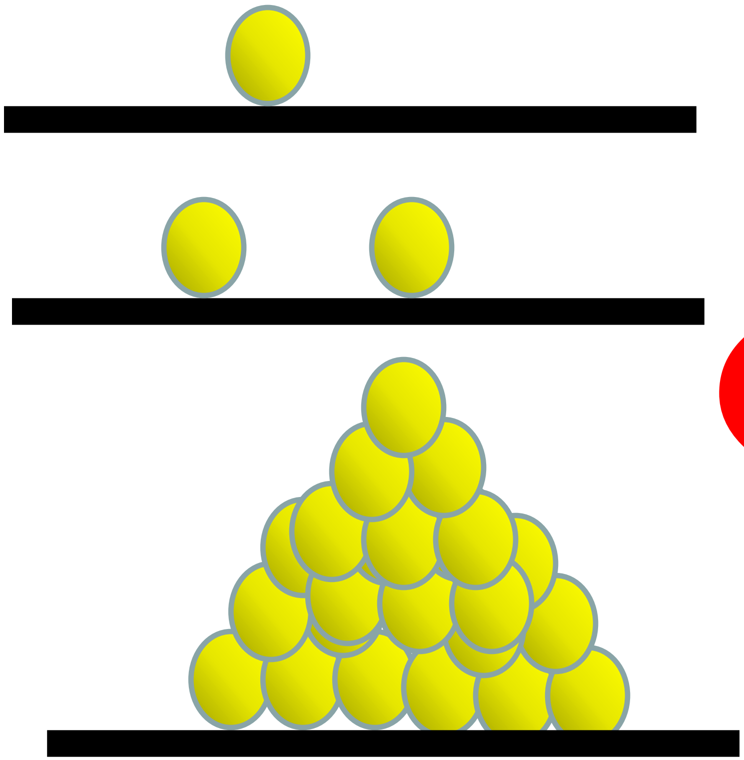




In the lattice

After release of the potential

Bose-Einstein Condensation



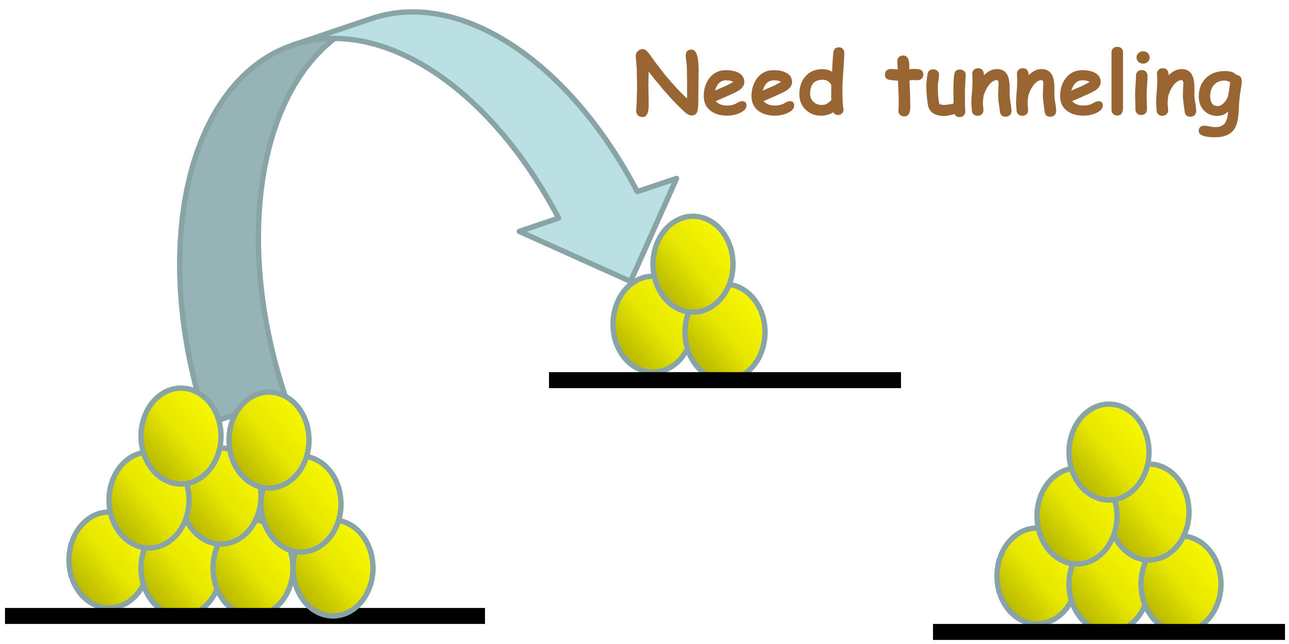
Macroscopic occupation of a single quantum state ?

Q: 1. External Potential ?
2. Interaction between the particles

In general the problem of bosons subject to an external potential is pathological:
at zero temperature all of them will find themselves in the one-particle ground state even if it is a localized one.
Even weak interaction is relevant!

Bose-Einstein Condensation

Disorder – Localized one-particle states
Weak Interaction



~~Macroscopic occupation of a single quantum state~~

Global Phase Coherence – single wave function

Bose-Einstein Condensation

$\Psi\left(\vec{r},t\right)$ Wave function of the condensate

Gross-Pitaevskii equation

$$i\hbar\frac{\partial\Psi\left(\vec{r},t\right)}{\partial t}=\left(-\frac{\hbar^2}{2m}\nabla^2+V\left(\vec{r}\right)+g\left|\Psi\left(\vec{r},t\right)\right|^2\right)\Psi\left(\vec{r},t\right)$$

$V\left(\vec{r}\right)$ External potential - smooth

g Coupling constant Short Range interaction

Localized one-particle states?

Condensation centers

Discrete version of the GP equation

Gross-Pitaevskii equation

$$i\hbar \frac{\partial \Psi(\vec{r},t)}{\partial t} = \left(-\frac{\hbar^2}{2m} \nabla^2 + V(\vec{r}) + g\left|\Psi(\vec{r},t)\right|^2 \right) \Psi(\vec{r},t)$$

Discrete version:

$$\Psi(\vec{r},t) \leftarrow \Psi_i(t)$$

"Wave functions" at different condensation centers

$$i\hbar \frac{\partial \Psi_i(t)}{\partial t} = \left(\varepsilon_i(\vec{r}) + g\left|\Psi_i(t)\right|^2 \right) \Psi_i(t) + \sum_{j \neq i} J_{ij} \Psi_i(t)$$

J_{ij} "Josephson coupling" between the condensation centers $\boldsymbol{i}$ and $\boldsymbol{j}$

ε_i One-particle energy at the condensation center $\boldsymbol{i}$

Gross-Pitaevskii equation

Discrete version: $\Psi_i(t)$ "Wave functions" at different condensation centers (CC)

$$i\hbar\frac{\partial\Psi_i(t)}{\partial t}=\left(\varepsilon_i(\vec{r})+g\left|\Psi_i(t)\right|^2\right)\Psi_i(t)+\sum_{j\neq i}J_{ij}\Psi_i(t)$$

J_{ij} "Josephson coupling" between CC $\boldsymbol{i}$ and $\boldsymbol{j}$

ε_i One-particle energies

Large $\left|\Psi_i(t)\right|^2$ ⇨ Small quantum fluctuations of $\left|\Psi_i(t)\right|^2$ ⇨ Only phase matters

$$\Psi_i(t)=\left|\Psi_i(t)\right|e^{i\varphi_i}$$ ⇨ XY spin model

Superfluid – Insulator transition

Large $\left|\Psi_i(t)\right|^2$ ⇨ Small quantum fluctuations of $\left|\Psi_i(t)\right|^2$ ⇨ Only phase matters

$$\Psi_i(t) = \left|\Psi_i(t)\right| e^{i\varphi_i}$$ ⇨ *XY* spin model

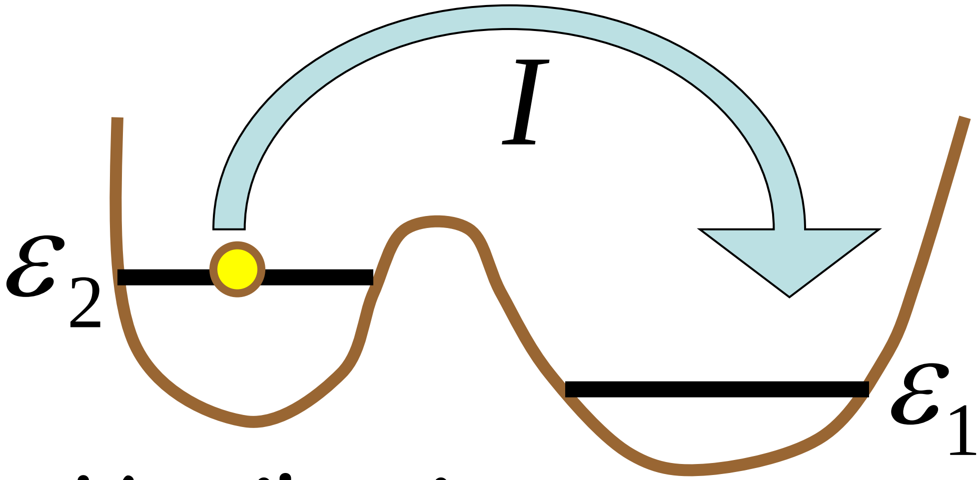
Ordered phase – Superfluid

Disordered phase – Insulator

Reason for the transition quantum fluctuations of the phase due to the onsite interaction

Energy scales in the problem:
- Coupling
- Dispersion in the one-particle energies
- Interaction energy

Two well problem



Hamiltonian

$$\hat{H} = \begin{pmatrix} \varepsilon_1 & I \\ I & \varepsilon_2 \end{pmatrix} \xrightarrow{\text{diagonalize}} \hat{H} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

$$E_2 - E_1 = \sqrt{(\varepsilon_2 - \varepsilon_1)^2 + I^2}$$

$$\hat{H} = \begin{pmatrix} \varepsilon_1 & I \\ I & \varepsilon_2 \end{pmatrix} \xrightarrow{\text{diagonalize}} \hat{H} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

$$E_2 - E_1 = \sqrt{(\varepsilon_2 - \varepsilon_1)^2 + I^2} \approx \begin{cases} \varepsilon_2 - \varepsilon_1 & \varepsilon_2 - \varepsilon_1 >> I \\ I & \varepsilon_2 - \varepsilon_1 << I \end{cases}$$

von Neumann & Wigner "noncrossing rule"

Level repulsion

v. Neumann J. & Wigner E. 1929 Phys. Zeit. v.30, p.467

What about the eigenfunctions ?

$$\hat{H} = \begin{pmatrix} \varepsilon_1 & I \\ I & \varepsilon_2 \end{pmatrix}$$

$$E_2 - E_1 = \sqrt{(\varepsilon_2 - \varepsilon_1)^2 + I^2} \approx \begin{matrix} \varepsilon_2 - \varepsilon_1 & \varepsilon_2 - \varepsilon_1 >> I \\ I & \varepsilon_2 - \varepsilon_1 << I \end{matrix}$$

What about the eigenfunctions ?

$$\phi_1, \varepsilon_1; \phi_2, \varepsilon_2 \quad \Leftarrow \quad \psi_1, E_1; \psi_2, E_2$$

$$\varepsilon_2 - \varepsilon_1 >> I$$

$$\psi_{1,2} = \varphi_{1,2} + O\left(\frac{I}{\varepsilon_2 - \varepsilon_1}\right)\varphi_{2,1}$$

Off-resonance

Eigenfunctions are close to the original on-site wave functions

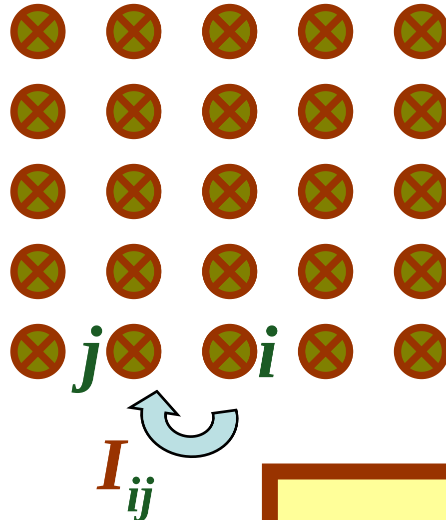
$$\varepsilon_2 - \varepsilon_1 << I$$

$$\psi_{1,2} \approx \varphi_{1,2} \pm \varphi_{2,1}$$

Resonance

In both eigenstates the probability is equally shared between the sites

Anderson Model



- *Lattice - tight binding model*
- *Onsite energies* ε_i *- random*
- *Hopping matrix elements* I_{ij}

$-W < \varepsilon_i < W$

uniformly distributed

$$I_{ij} = \begin{cases} I & i \text{ and } j \text{ are nearest neighbors} \\ 0 & \text{otherwise} \end{cases}$$

Anderson Transition

$$I_c = f(d) * W$$

$I < I_c$

Insulator

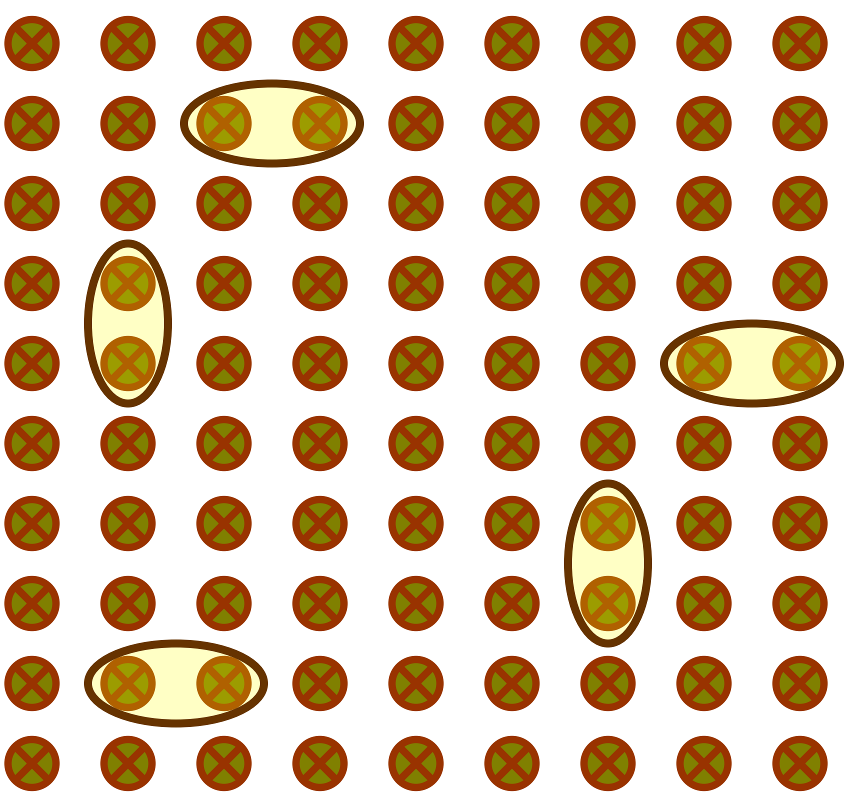
All eigenstates are localized

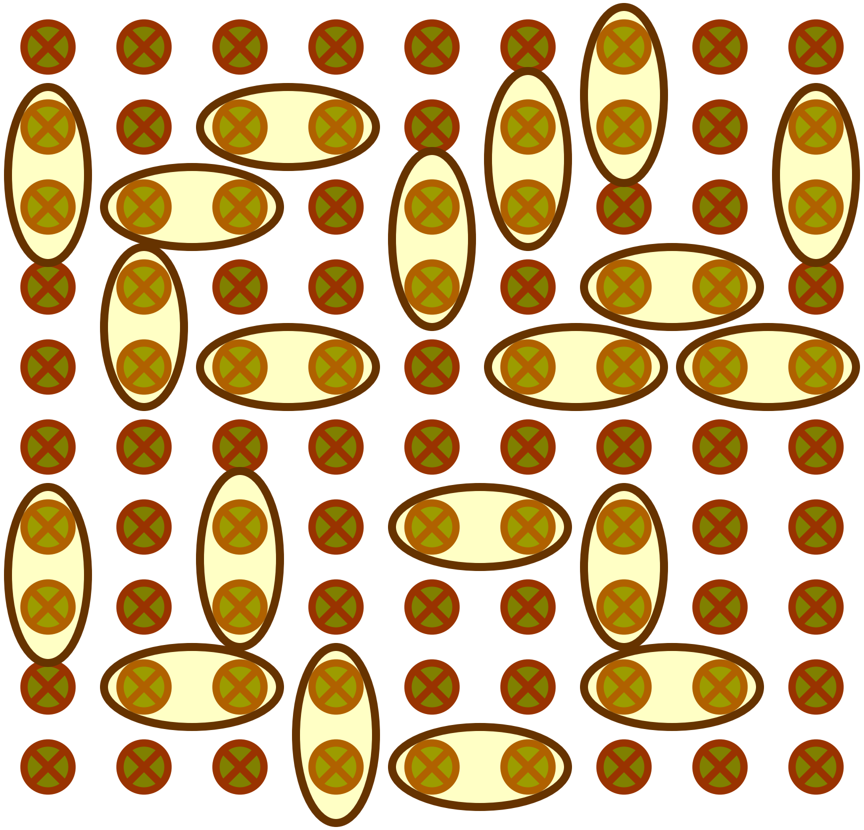
Localization length ξ

$I > I_c$

Metal

There appear states extended all over the whole system





Anderson insulator

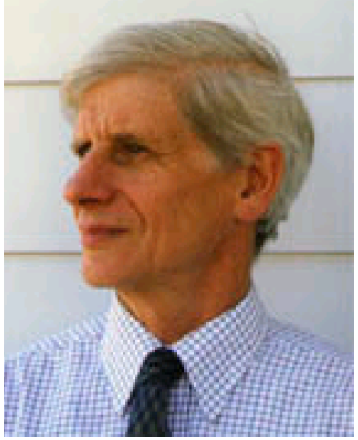
Few isolated resonances

Anderson metal

There are many resonances
and they overlap

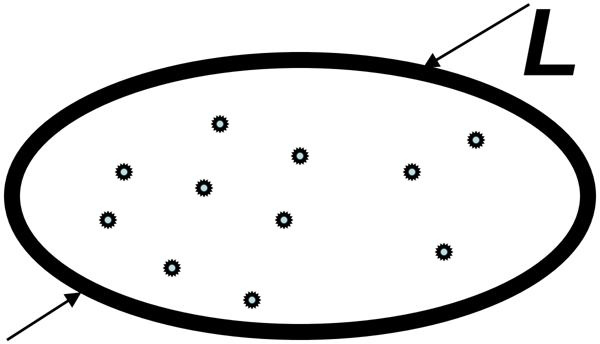
Transition: Typically each site is in the resonance with some other one

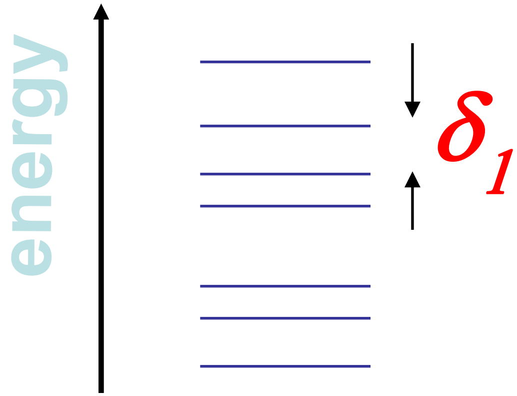
Energy scales (*Thouless, 1972*)



1. Mean level spacing

$$\delta_1 = \left(\nu L^d\right)^{-1}$$





L *system size*

d *# of dimensions*

2. Thouless energy

$$E_T = 2\pi\hbar D L^{-2}$$

D diffusion constant

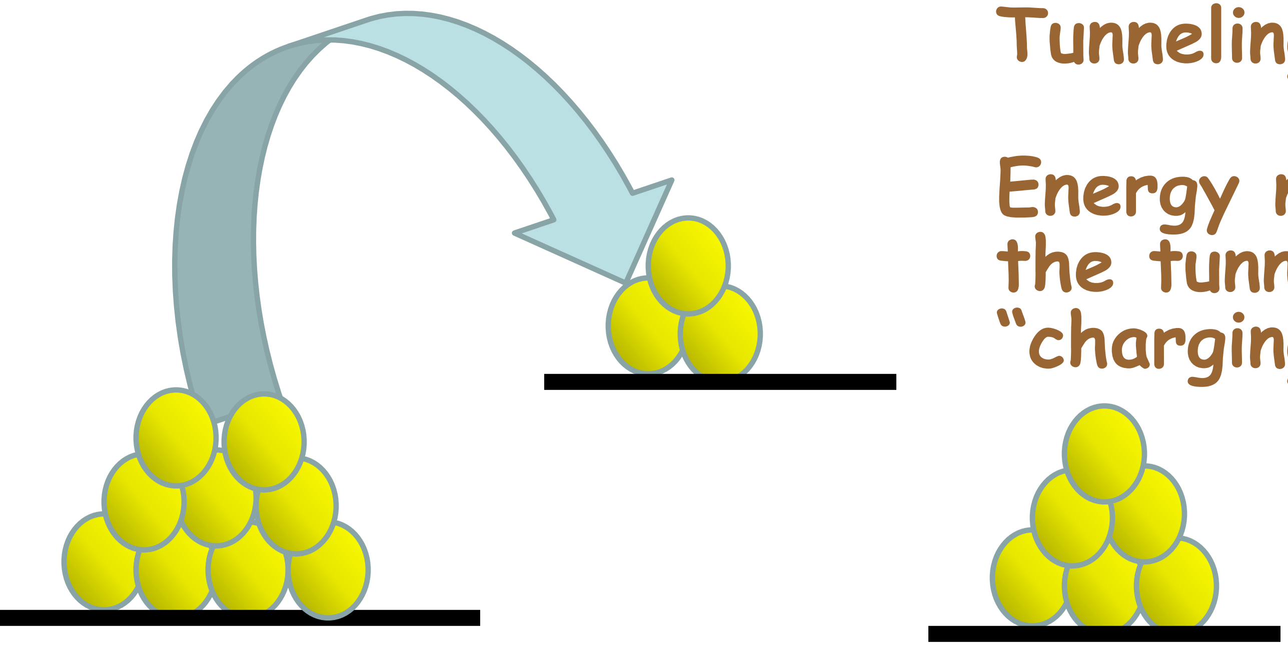
*E_T has a meaning of the **inverse diffusion time** of the traveling through the system or the **escape rate** (for open systems)*

$$\boxed{g \equiv E_T / \delta_1}$$

dimensionless
Thouless
conductance

$$g = \left(2\pi\hbar/e^2\right)G = R_Q G$$

Superfluid – Insulator transition



Tunneling amplitude J

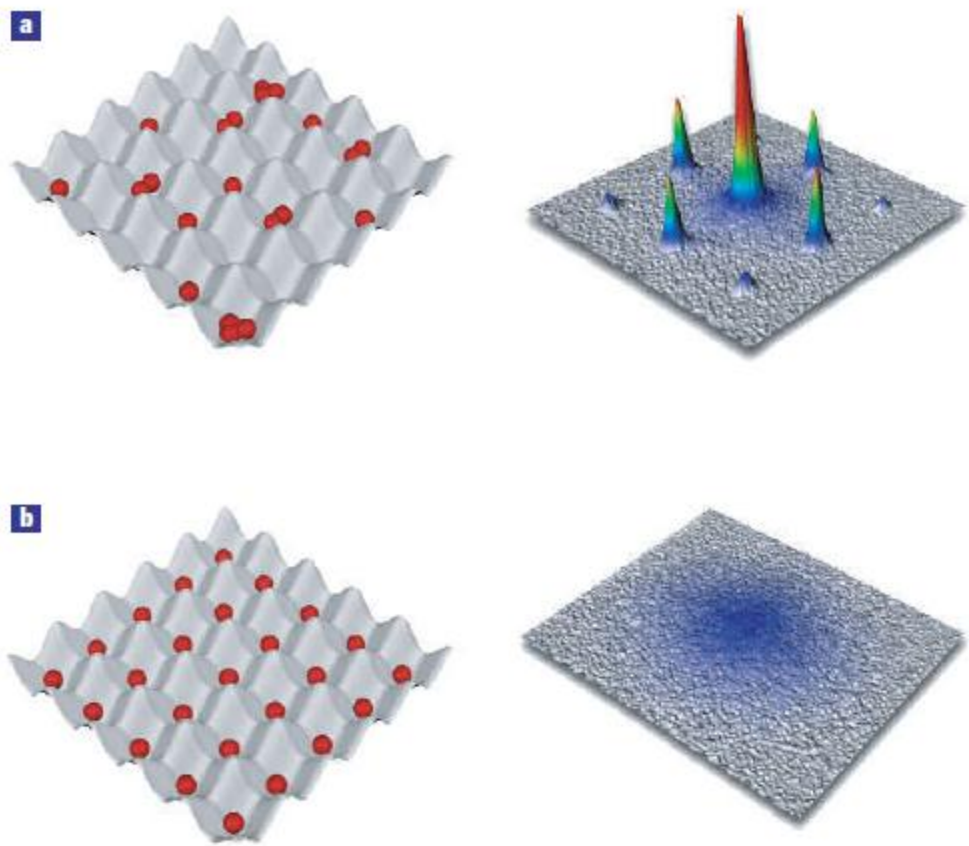
Energy needed for the tunneling "charging energy" E_c

$J > E_c$ Ordered phase – Superfluid

$J < E_c$ Disordered phase – Insulator

Markus Greiner, Olaf Mandel, Tilman Esslinger, Theodor W. Hänsch & Immanuel Bloch "**Quantum phase transition from a superfluid to a Mott insulator in a gas of ultracold atoms**" *Nature* **415**, 39-44 (3 January 2002)

Superfluid



$$J > E_c$$

Mott Insulator

$$J < E_c$$

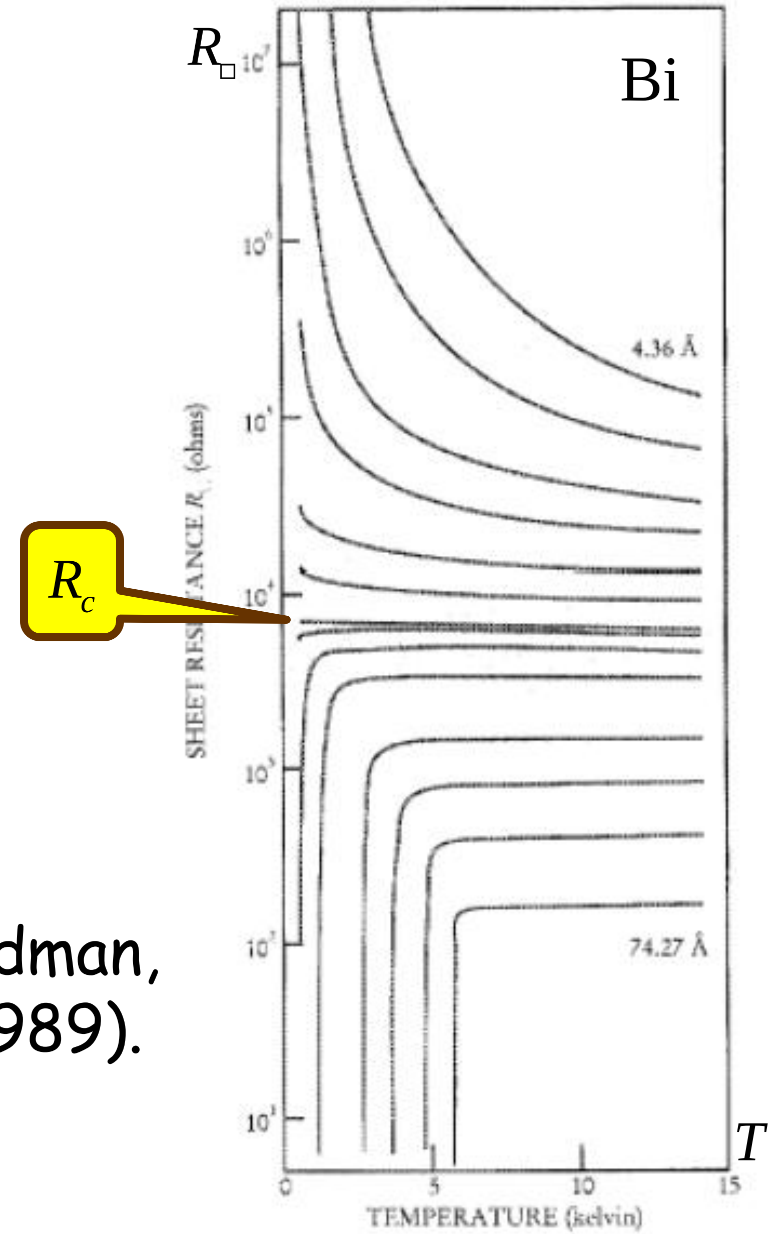
In the lattice

After release of the potential

2. Superconductor-Insulator Transition in two dimensions

Superconductor-Insulator Transition

? $$R_c = \frac{R_Q}{g_c} \qquad R_Q \equiv \frac{h}{4e^2}$$?



D. B. Haviland, Y. Liu and A. M. Goldman, Phys. Rev. Lett., 62, 2180-2183, (1989).

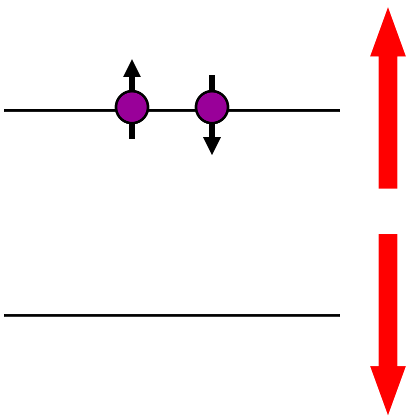
$$\hat{H}_{BCS} = \sum_{\alpha,\sigma=\uparrow,\downarrow} \varepsilon_\alpha a^+_{\alpha\sigma} a_{\alpha\sigma} + \lambda_{BCS} \sum_{\alpha,\beta} a^+_{\alpha\uparrow} a^+_{\alpha\downarrow} a_{\beta\uparrow} a_{\beta\downarrow}$$

Anderson spin chain

P.W. Anderson:
Phys. Rev. **112,**1800, 1958

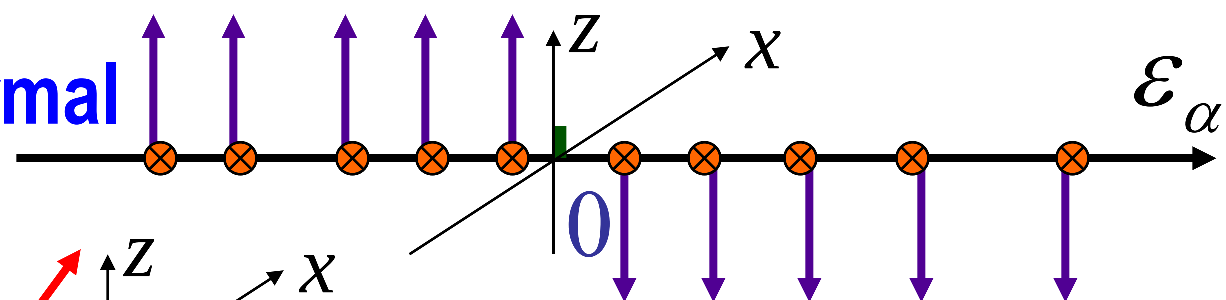
$$\hat{K}^z_\alpha = \frac{1}{2}\left(\sum_{\sigma=\uparrow,\downarrow} \varepsilon_\alpha a^+_{\alpha\sigma} a_{\alpha\sigma} - 1\right) \qquad \hat{K}^+_\alpha = a^+_{\alpha\uparrow} a^+_{\alpha\downarrow} \qquad \hat{K}^-_\alpha = a_{\alpha\uparrow} a_{\alpha\downarrow}$$

SU_2 algebra
spin 1/2

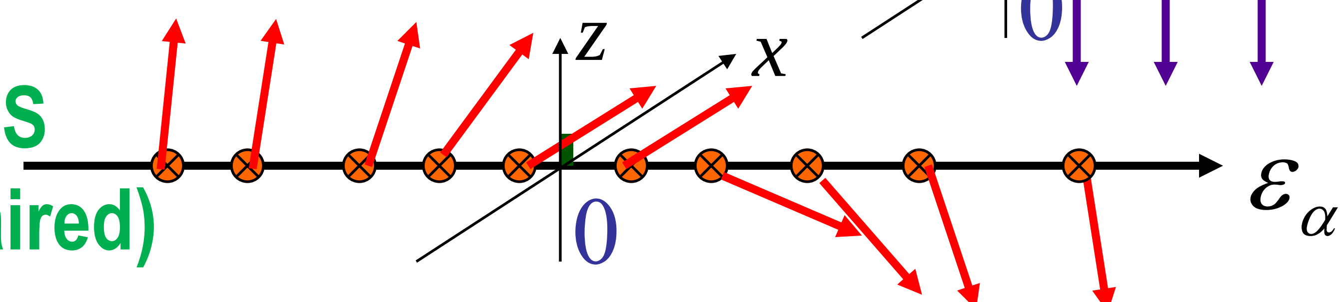


$$\hat{H}_{BCS} = \sum_{\alpha,\sigma=\uparrow,\downarrow} \varepsilon_\alpha \hat{K}^z_\alpha + \lambda_{BCS} \sum_{\alpha,\beta} \hat{K}^+_\alpha \hat{K}^-_\beta$$

normal



BCS (paired)



Disorder-caused corrections to T_c, Δ Suppression of Superconductivity by Coulomb interaction

Ovchinnikov 1973,
Maekawa & Fukuyama 1982
Finkelshtein 1987

1. Quantum corrections in normal metals in 2D

Weak localization, e-e interactions

$$\frac{\delta g(\omega)}{g} \propto \frac{1}{g} \ln\left(\frac{\omega\tau}{\hbar}\right)$$

$$g \equiv \frac{2\pi\hbar}{4e^2}\sigma$$

Dimensionless Thouless conductance

τ mean free time

ω frequency, infrared cutoff

2. Correction to T_c due to the Coulomb repulsion

$$T_c \propto \hbar\theta_D \exp\left(-1/\lambda_{BCS}\right)$$

$$\frac{\delta\lambda_{BCS}}{\lambda_{BCS}} \propto \frac{1}{g} \ln\left(\frac{\omega\tau}{\hbar}\right)$$

$$\frac{\delta T_c}{T_{c0}} \propto \frac{\delta\lambda_{BCS}}{\left(\lambda_{BCS}\right)^2} \propto \frac{1}{g}\left[\ln\left(\frac{T_c\tau}{\hbar}\right)\right]^3$$

$$\theta_D\tau \approx 1$$

Anderson Theorem

Neither superconductor order parameter Δ nor transition temperature T_c depend on disorder

Provided that

Δ is homogenous in space

Corrections to the Anderson theorem – due to inhomogeneity in Δ

$$\frac{\delta T_c}{T_c} \propto \frac{-1}{g}\left(\log\frac{\hbar}{T_c\tau}\right)^3 \qquad T_c\tau < \hbar$$

Interpretation :

$$T_c = \theta_D \exp\left(-\frac{1}{\lambda_{BCS}}\right)$$

SC temperature

Debye temperature

Dimensionless BCS coupling constant

$$\lambda_{BCS} = \left[\ln\left(T_c / \theta_D\right)\right]^{-1}$$

Pure BCS interaction (no Coulomb repulsion):

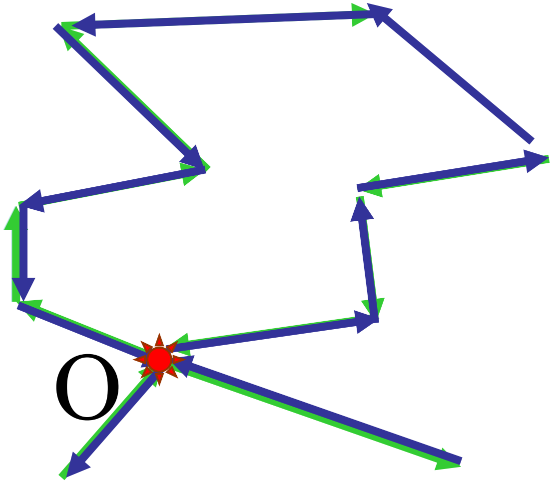
$$\boxed{\lambda_{eff} = \lambda_{BCS}\left[1 + \frac{\#}{g}\ln\left(\frac{1}{T_c\tau}\right)\right]}$$

Interpretation continued: "weak localization" logarithm

$$\lambda_{eff} = \lambda_{BCS}\left[1 + \frac{\#}{g}\ln\left(\frac{1}{T_c \tau}\right)\right]$$

Q: Why logarithm?

A: Return and interference



$$\lambda_{BCS} = \left[\ln\left({T_c}/{\theta_D} \right) \right]^{-1}$$

$$\lambda_{eff} = \lambda_{BCS}\left[1 + \frac{\#}{g} \ln\left(\frac{1}{T_c \tau} \right) \right]$$

• In the universal $(g = \infty)$ limit the effective coupling constant equals to the bare one - **Anderson theorem**

• If there is only **BCS** attraction, then **disorder increases** T_c and Δ by optimizing spatial dependence of Δ.

!

T_c and Δ **reach maxima at the point of Anderson localization**

Problem in conventional superconductors: Coulomb Interaction

Interpretation continued: Coulomb Interaction

Anderson theorem - the gap is homogenous in space.

Without Coulomb interaction adjustment of the gap to the random potential strengthens superconductivity.

Homogenous gap in the presence of disorder violates electroneutrality; Coulomb interaction tries to restore it and thus suppresses superconductivity

$$\lambda_{eff} = \lambda_{BCS} - \frac{\#}{g} \ln\left(\frac{\hbar}{T_c \tau}\right)$$

Perturbation theory:

$$\frac{\delta T_c}{T_c} = \delta\left(\frac{-1}{\lambda}\right) = \frac{\delta\lambda}{\lambda_{BCS}^2} \propto -\frac{1}{g} \ln\left(\frac{\hbar}{T_c \tau}\right)\left[\ln\left(\frac{\theta_D}{T_c}\right)\right]^2$$

Disorder-caused corrections to T_c, Δ Suppression of Superconductivity by Coulomb interaction

Ovchinnikov 1973,
Maekawa & Fukuyama 1982
Finkelshtein 1987

2. Correction to T_c due to the Coulomb repulsion

$$T_c \propto \hbar\theta_D \exp\left(-1/\lambda_{BCS}\right)$$

$$\frac{\delta\lambda_{BCS}}{\lambda_{BCS}} \propto \frac{1}{g}\ln\left(\frac{\omega\tau}{\hbar}\right)$$

$$\Rightarrow \frac{\delta T_c}{T_{c0}} \propto \frac{\delta\lambda_{BCS}}{\left(\lambda_{BCS}\right)^2} \propto \frac{1}{g}\left[\ln\left(\frac{T_c\tau}{\hbar}\right)\right]^3$$

$$\theta_D\tau \approx 1$$

3. One can sum up triple log corrections neglecting single log terms, i.e. neglecting effects of Anderson localization. !

Disorder-caused corrections to T_c, Δ Suppression of Superconductivity by Coulomb interaction

Ovchinnikov 1973,
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Finkelshtein 1987

$$\frac{\delta T_c}{T_{c0}} \propto \frac{\delta\lambda_{BCS}}{\left(\lambda_{BCS}\right)^2} \propto \frac{1}{g}\left[\ln\left(\frac{T_c\tau}{\hbar}\right)\right]^3$$

One can sum up triple log corrections neglecting localization effects.

4.

$$T_c = \frac{\hbar}{\tau}\left[\frac{\sqrt{g}-\ln\left(\hbar/\tau T_{c0}\right)}{\sqrt{g}+\ln\left(\hbar/\tau T_{c0}\right)}\right]^{\frac{\sqrt{g}}{2}}$$

Finkelshtein (1987)
renormalization group

Aleiner (unpublished)
BCS-like mean field

5.

$$g \le \left[\ln\left(\hbar/\tau T_{c0}\right)\right]^2 \approx \frac{1}{\lambda_{BCS}^2} \quad \Rightarrow \quad T_c = 0$$

Disorder-caused corrections to T_c, Δ Suppression of Superconductivity by Coulomb interaction

Ovchinnikov 1973,
Maekawa & Fukuyama 1982
Finkelshtein 1987

4.
$$T_c = \frac{\hbar}{\tau}\left[\frac{\sqrt{g} - \ln\left(\hbar/\tau T_{c0}\right)}{\sqrt{g} + \ln\left(\hbar/\tau T_{c0}\right)}\right]^{\frac{\sqrt{g}}{2}}$$

5.
$$g \le \frac{\#}{\lambda_{BCS}^2} \Rightarrow T_c = 0$$

$$g_c \approx \frac{1}{\lambda_{BCS}^2} \gg 1 \Rightarrow R_c = \frac{R_Q}{g_c} \ll R_Q$$

Conclusion:

critical resistance is **much smaller** than the quantum resistance

$$R_c \ll R_Q$$

!

QUANTUM PHASE TRANSITION
Theory of Dirty Bosons

Fisher, Grinstein and Girvin 1990
Wen and Zee 1990
Fisher 1990

Only phase fluctuations of the order parameter are important near the superconductor - insulator transition

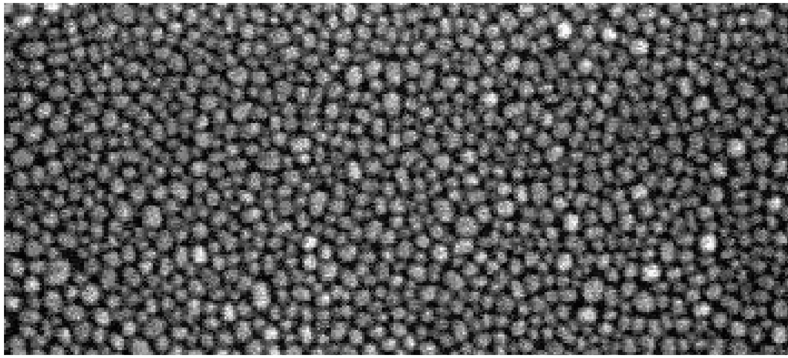
CONCLUSIONS

1. **Exactly at the transition point and at $T \rightarrow 0$ conductance tends to a universal value g_{qc}**

$$g_{qc} = \frac{4e^2}{h} \approx 6K\Omega \quad ??$$

2. **Close to the transition point magnetic field and temperature dependencies demonstrate universal scaling**

Granular films.



Single grain: charging energy $E_c^{(0)}$
one-particle mean level spacing δ_1
SC transition temperature T_{c0}
SC gap Δ

2D array: tunneling conductance g_t

dwell time $\tau_{esc} = \hbar \left(g_t \delta_1 \right)^{-1}$

normal state sheet resistance $R_{\square}^{(N)} = \dfrac{R_Q}{g_t}$

Below T_{c0}: Josephson coupling $E_J = g_t \Delta$ Ambegoakar & Baratoff (1963)

Quantum Phase Transition

Dirty Boson Theory $\Delta \to \infty$

Fisher, Grinstein & Girvin 1990

Wen and Zee 1990

Fisher 1990

Only phase fluctuations

1. Exactly at the transition point and at $T \to 0$ conductance tends to a ~~universal~~ value

$$\boxed{R_c = g_c R_Q \sim R_Q = \frac{h}{4e^2} \approx 6K\Omega \quad ??}$$

Superconductor-insulator transition in granular films.

$\Delta \to \infty$ ⇒ only two energy scales

✓ E_J Josephson energy

✓ E_c charging energy

Efetov 1980

$E_J > E_c$ superconductor

$E_c > E_J$ insulator

?

Problem: E_c is renormalized

$$E_c \neq E_c^{(0)}$$

RPA renormalization

Charging energy of a grain

$$\frac{1}{E_c(-i\omega)} = \frac{1}{E_c^{(0)}} + \frac{g_t}{-i\omega + g_t\delta_1}$$

$$E_c \equiv E_c(\Delta)$$

Conventional dynamical screening

$$\frac{1}{U_{eff}(q,\omega)} = \frac{1}{U_0(q)} + \frac{\nu D q^2}{-i\omega + Dq^2}$$

Ambegoakar, Eckern & Schon (1982, 1984)

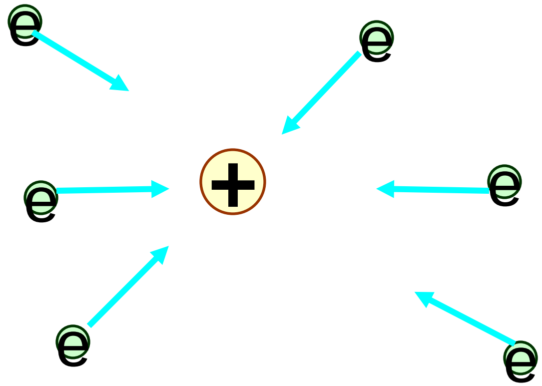
Example: Coulomb interaction

$d = 3$

$$U_0(q) = \frac{4\pi e^2}{q^2} \qquad U_{eff}(q,\omega) = \frac{4\pi e^2}{q^2 + \kappa^2 \dfrac{Dq^2}{-i\omega + Dq^2}} \qquad \kappa^2 = 4\pi e^2 \nu$$

$d = 2$

$$U_0(q) = \frac{2\pi e^2}{q} \qquad U_{eff}(q,\omega) = \frac{2\pi e^2}{q + \kappa \dfrac{Dq^2}{-i\omega + Dq^2}} \qquad \kappa = 2\pi e^2 \nu$$



dynamical
screening

$$U_{eff}(q,\omega)=\frac{U_0(q)}{1+U_0(q)\nu\dfrac{Dq^2}{-i\omega+Dq^2}}$$

? DISSIPATION
or
DYNAMICAL SCREENING ?

RPA renormalization

Charging energy of a grain

$$\frac{1}{E_c(-i\omega)} = \frac{1}{E_c^{(0)}} + \frac{g_t}{-i\omega + g_t\delta_1}$$

$$E_c \equiv E_c(\Delta)$$

Conventional dynamical screening

$$\frac{1}{U_{eff}(q,\omega)} = \frac{1}{U_0(q)} + \frac{\nu D q^2}{-i\omega + Dq^2}$$

Ambegoakar, Eckern & Schon (1982, 1984)

Granular material		Homogenous media		Correspondence
Bare charging energy	$E_c^{(0)}$	Bare potential in the momentum representation	$U_0(q)$	$E_c^{(0)} \Leftrightarrow \nu U_0(q)$
Effective charging energy	$E_c(\omega)$	Effective potential	$U_{eff}(q,\omega)$	$E_c \Leftrightarrow \nu U_{eff}(q,\omega)$
Mean spacing of fermionic levels	δ_1	Fermionic density of states	ν	$\delta_1 \Leftrightarrow \nu^{-1}$
Dimensionless tunneling conductance	g_t	Diffusion constant	D	$g_t \Leftrightarrow Dq^2$

Superconductor-insulator transition in granular films.

$E_J > E_c$ superconductor

$E_c > E_J$ insulator

$$E_J = g_t \Delta$$

$$\frac{1}{E_c} = \frac{1}{E_c^{(0)}} + \frac{g_t}{\Delta + g_t \delta_1} \Rightarrow E_c \approx \begin{cases} \delta_1 & \delta_1 \gg \Delta, E_c^{(0)} \quad I \\ \dfrac{g_t}{\Delta} & E_c^{(0)} \gg \Delta, \delta_1 \quad II \\ E_c^{(0)} & \Delta \gg E_c^{(0)}, \delta_1 \quad III \end{cases}$$

$\delta_1 < \Delta < E_c$

case II

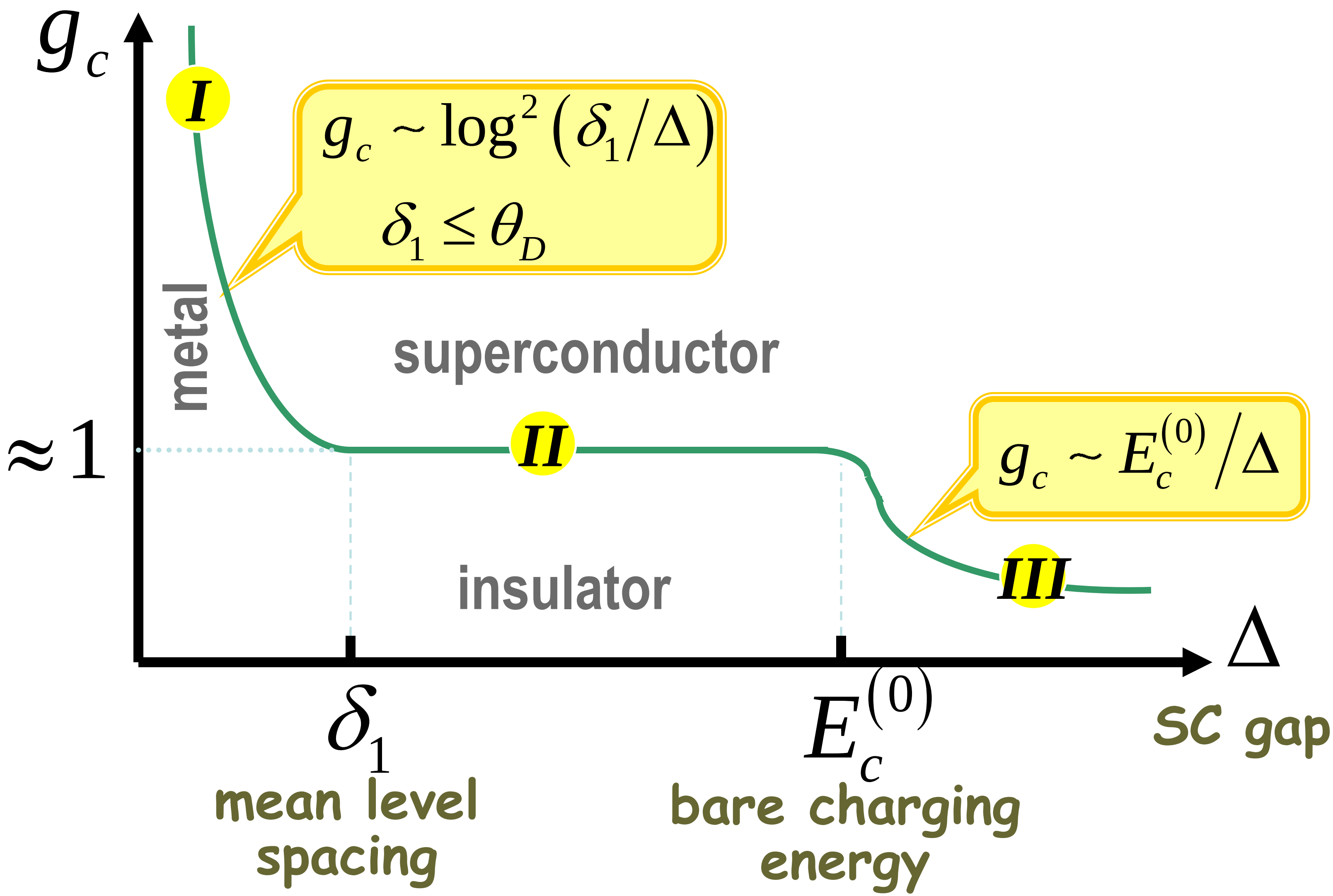
$$E_J = g_t \Delta$$

$$E_c = \frac{\Delta}{g_t}$$

$$\Rightarrow \begin{array}{c} g_c \sim 1 \\ R_c \sim R_Q \end{array}$$

! not $\Delta \to \infty$!

Three Regimes



I. superconductor - metal transition

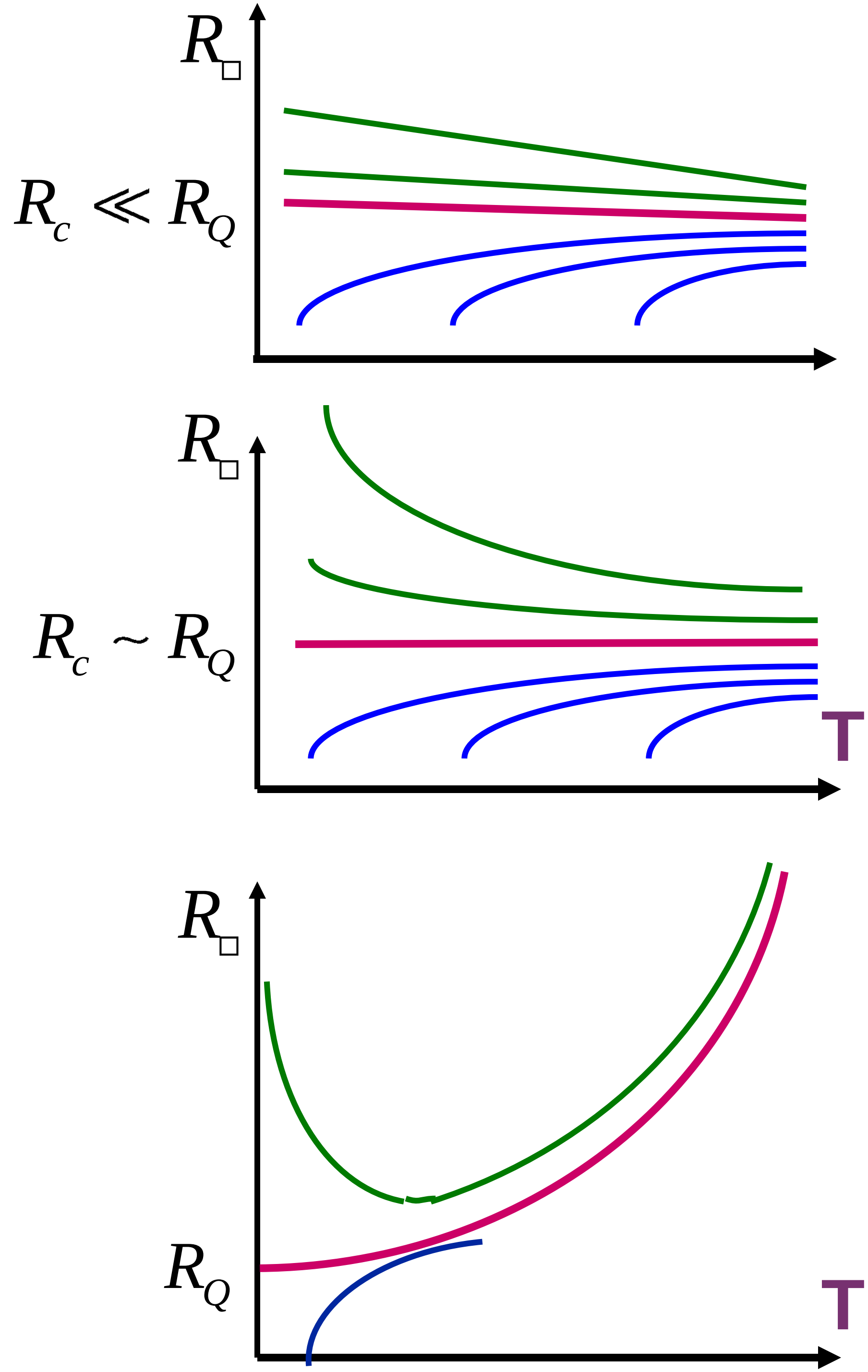
homogenous films

II. superconductor - insulator transition

$$\delta_1 < \Delta < E_c^{(0)}$$

III. superconductor - insulator transition

$$\delta_1 < E_c^{(0)} < \Delta$$



VOLUME 74, NUMBER 15 PHYSICAL REVIEW LETTERS 10 APRIL 1995

Superconducting-Insulating Transition in Two-Dimensional *a*-MoGe Thin Films

Ali Yazdani* and Aharon Kapitulnik
Department of Applied Physics, Stanford University, Stanford, California 94305
(Received 29 November 1994)

Regime *I*
homogenous films

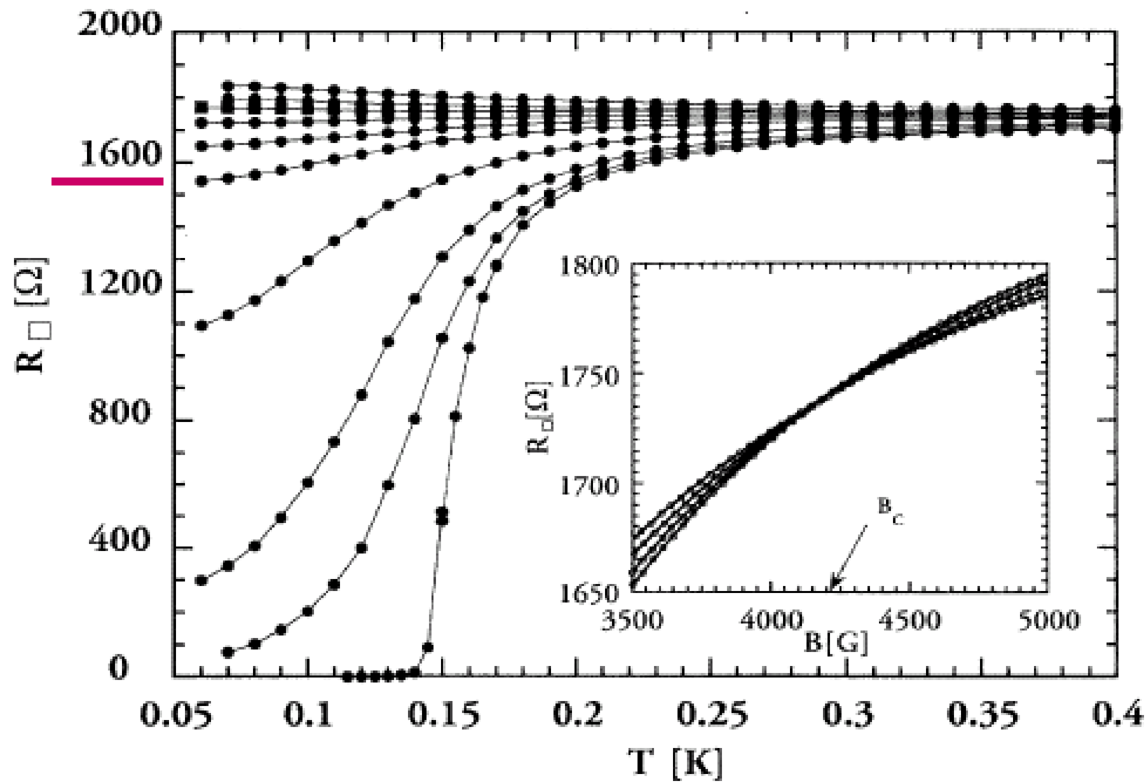
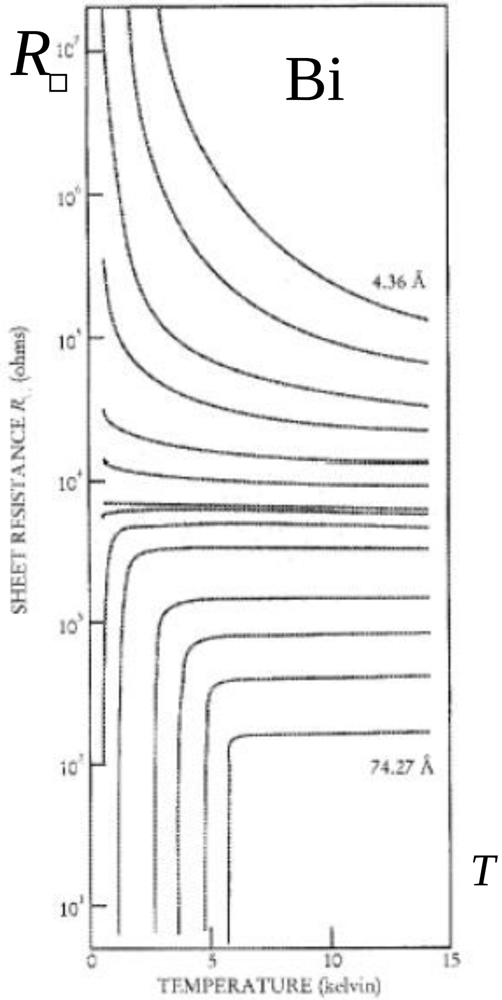


FIG. 1. Zero bias resistance of sample 2 plotted versus temperature at $B = 0, 0.5, 1.0, 2.0, 3.0, 4.0, 4.4, 4.5, 5.5, 6$ kG. In the inset, $R_\square(B,T,E=0)$ for the same sample measured versus field, at $T = 80, 90, 100, 110$ mK.

Haviland, Liu & Goldman, PRL, (1989)



McGreer, Nease, Haviland, Martinez, Halley & Goldman, PRL, (1991).

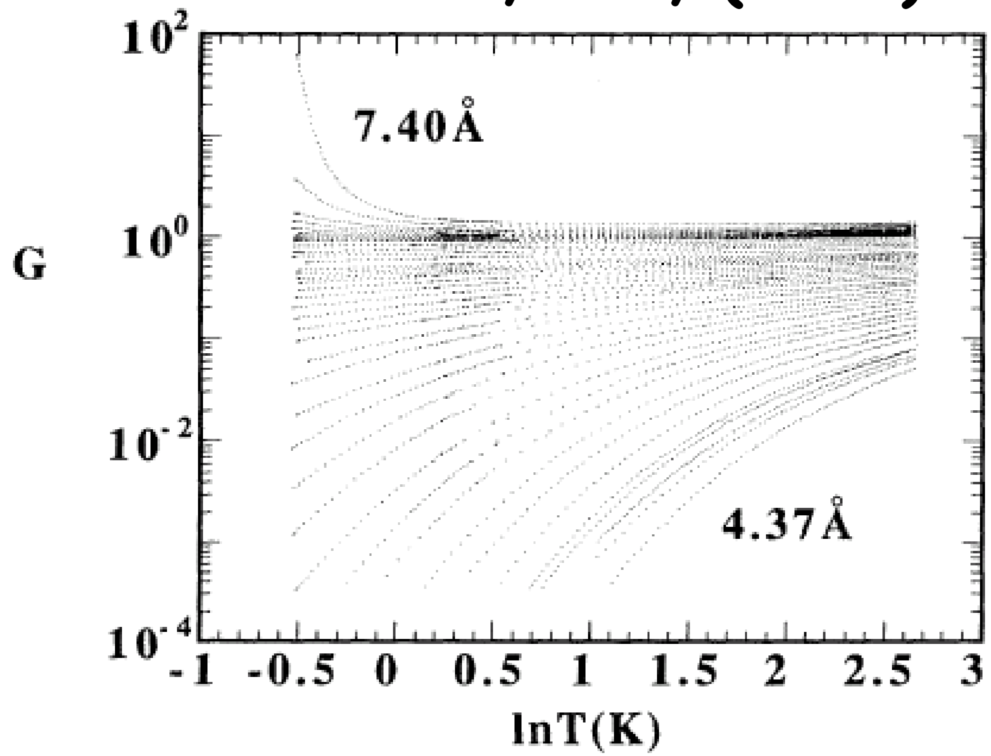


FIG. 1. Logarithm of the conductance G, in units of $4e^2/h$, vs $\ln(T)$ for a number of different Bi films. The thickness of the first (thinnest) and last (thickest) films are indicated.

Regime *II* Inhomogeneous films

$$\Delta < E_c^{(0)}$$

Regime *I*. Homogenous films

Quantum phase transitions in two dimensions: Experiments in Josephson-junction arrays

H. S. J. van der Zant, W. J. Elion, L. J. Geerligs, and J. E. Mooij
Department of Applied Physics and Delft Institute of Microelectronics and Submicron-technology (DIMES), Delft University of Technology Lorentzweg 1, 2628 CJ Delft, The Netherlands

(Received 21 May 1996)

Regime *III*. ?

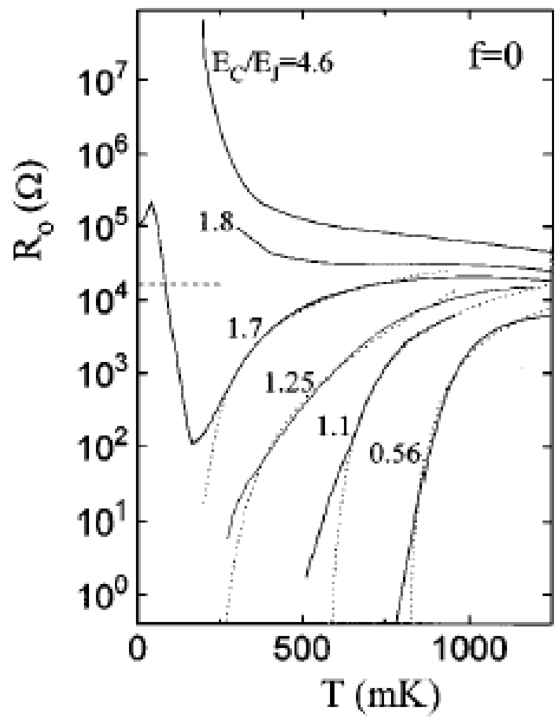
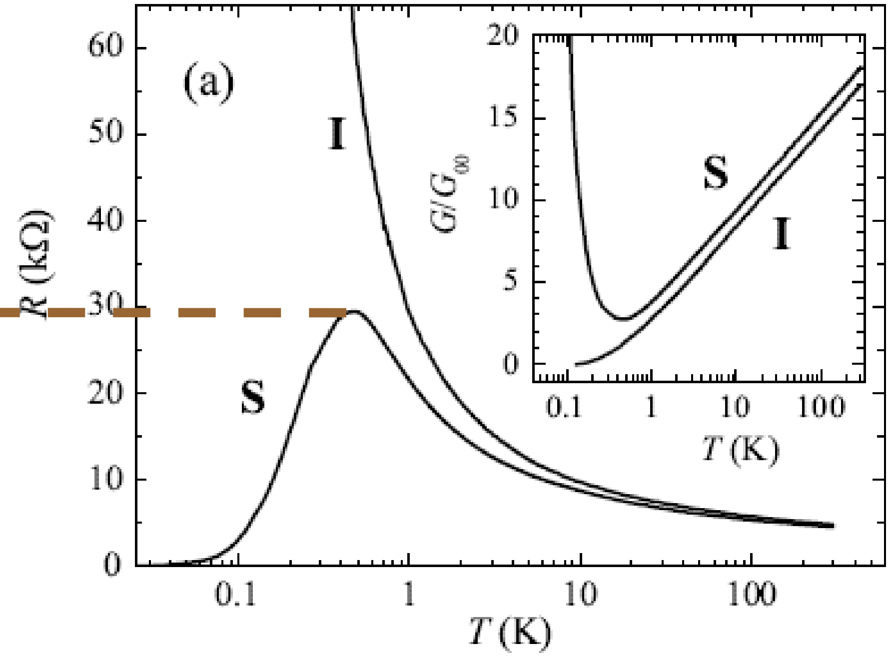


FIG. 5. The zero-field linear resistance per junction measured as a function of temperature for six different arrays. Dotted lines are fits to the vortex-KTB square-root cusp formula. The dashed horizontal line shows the zero-temperature universal resistance ($8R_q/\pi$=16.4 kΩ) of the S–I transition at f=0.

Baturina, Mironov, Vinokur, Baklanov, & Strunk
JETP Lett. 88, 752 (2008)

Regime *III*. ?

$\times 2$

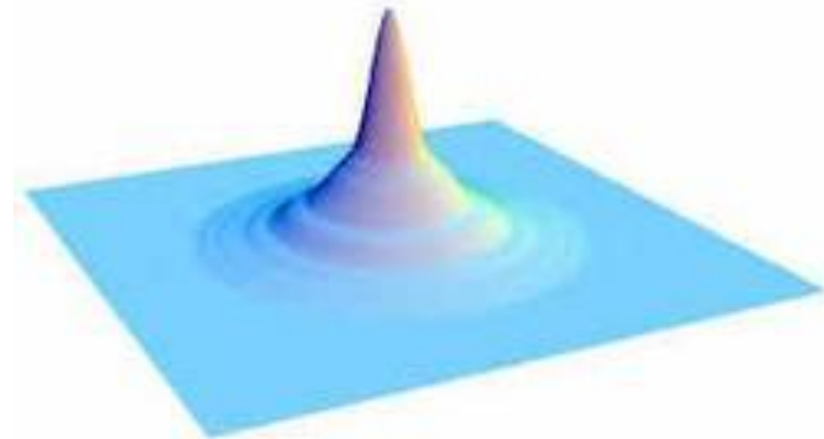


TiN films $50\mu \times 100\mu$

3.Driven Bose Condensation

Q: What is the difference between a pendulum clock and a Bose condensate ?

Absolutely Classical system

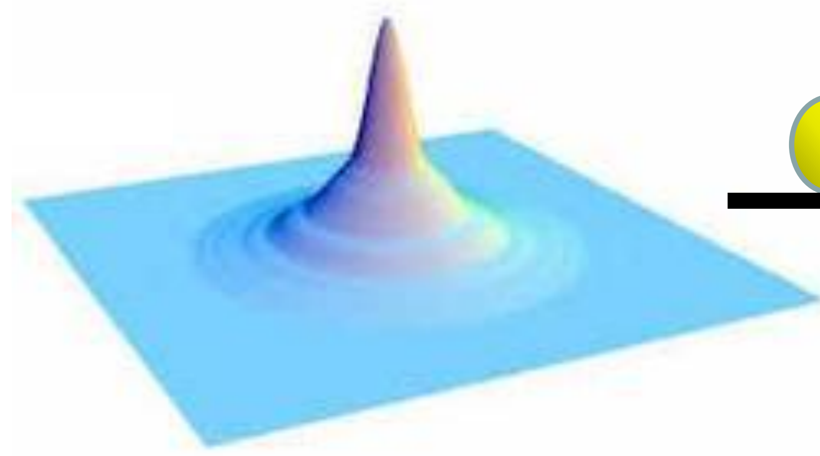


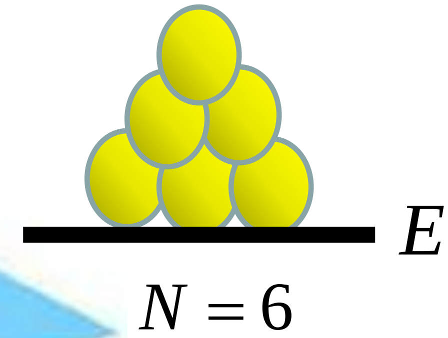
Macroscopic Quantum system

? ? ? ?

Q: What is in common between a pendulum and a Bose condensate ?

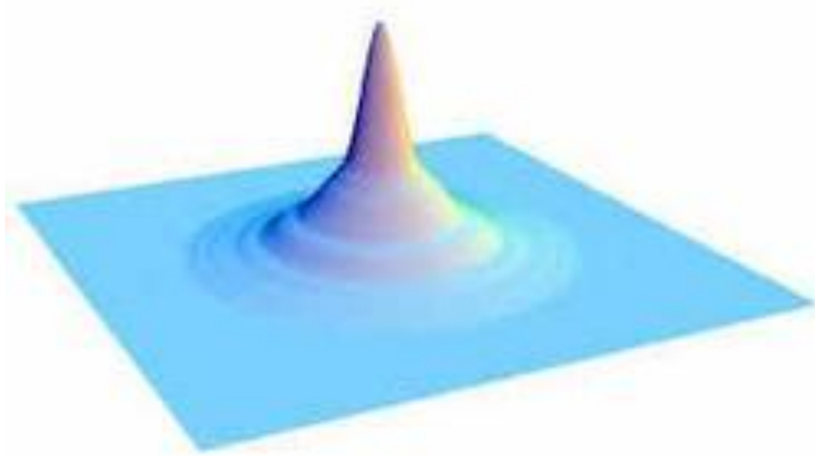
Bose condensate has a lot in common with a pendulum !?





Oscillator	Single bosonic state
Harmonic, N-th state	N bosons without interaction
Nonlinear, N-th state	N interacting bosons
Classical: $N \to \infty$	Macroscopic quantum state

Q: What is the difference between a pendulum clock and a Bose condensate**?**



Q: Can we tell that the pendulum is in a "**macroscopic quantum state**" **?**

A: Of course not**!** But why**?**

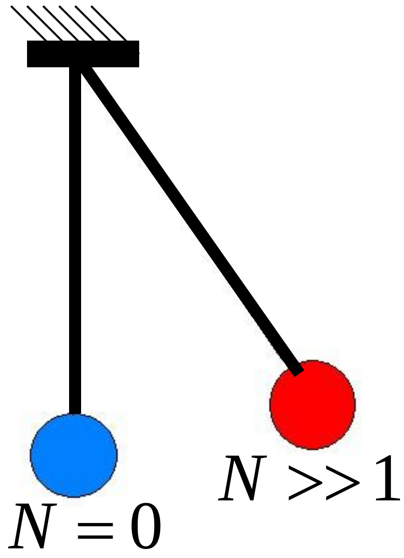
Q: Why the pendulum is not a system in "macroscopic quantum state" **?**

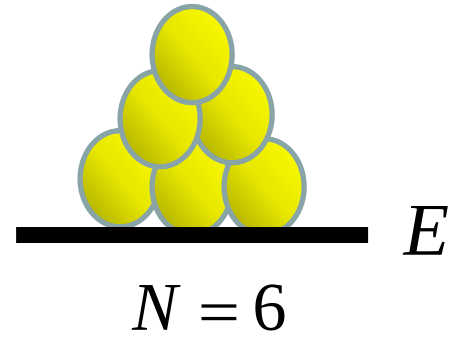
Answer #1 (correct but incomplete):

Bose condensate can be in the thermodynamic equilibrium

The pendulum clock needs energy pumping from e.g. gravity in order to compensate the dissipation.
Driven system

Q: What if the bosons have finite life-time and their number is not conserved ? **?**





Bose Condensate: $N \to \infty$

Q: Why we can not call the pendulum is in a "macroscopic quantum state"?

Another possible answer:

Long range phase coherence.

Can it be used as a signature of the macroscopic quantum state ?

Driven Bose Condensate ?

- stationary state
- pumping + dissipation
- number of bosons is not conserved

Can we call the laser beam a Bose-condensate of phonons ?

Bose-condensation out of equilibrium

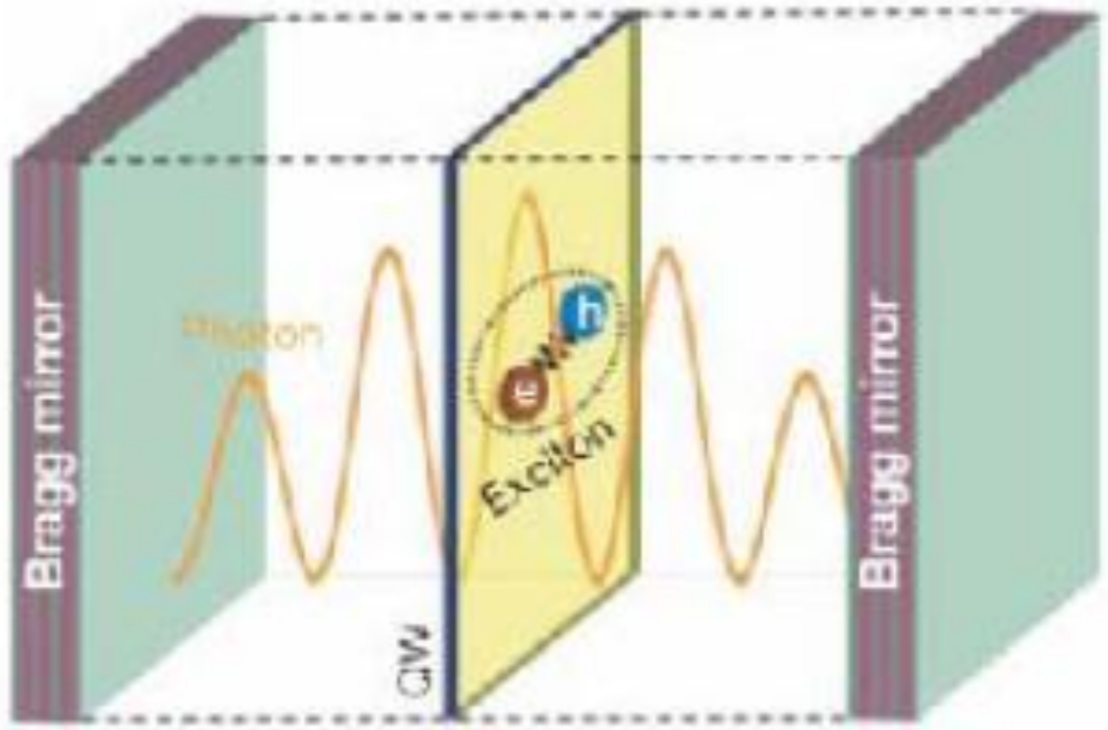
Hui Deng, Gregor Weihs, Charles Santori, Jacqueline Bloch, and Yoshihisa Yamamoto, "**Condensation of semiconductor microcavity exciton polaritons**" *Science* **298**, 199–202 (2002).

J. Kasprzak, M. Richard, S. Kundermann, A. Baas, P. Jeambrun, J. M. J.Keeling, F. M. Marchetti, M. H. Szymanska, R. Andre, J. L. Staehli, V. Savona, P. B. Littlewood, B. Deveaud and Le Si Dang,"**Bose–Einstein condensation of exciton polaritons** "*Nature* 443, **409** (2006).

S. O. Demokritov, V. E. Demidov, O. Dzyapko, G. A. Melkov, A. A. Serga, B. Hillebrands & A. N. Slavin "**Bose–Einstein condensation of quasi-equilibrium magnons at room temperature under pumping**", *Nature* **443**, pp.430-433 (2006)

J.Klaers, J.Schmitt, F.Vewinger & M.Weitz "**Bose–Einstein condensation of photons in an optical microcavity**", *Nature* **468**, pp.545–548 (2010)

Polaritons $\approx$ Photons with weak interaction



2D light
+
excitons

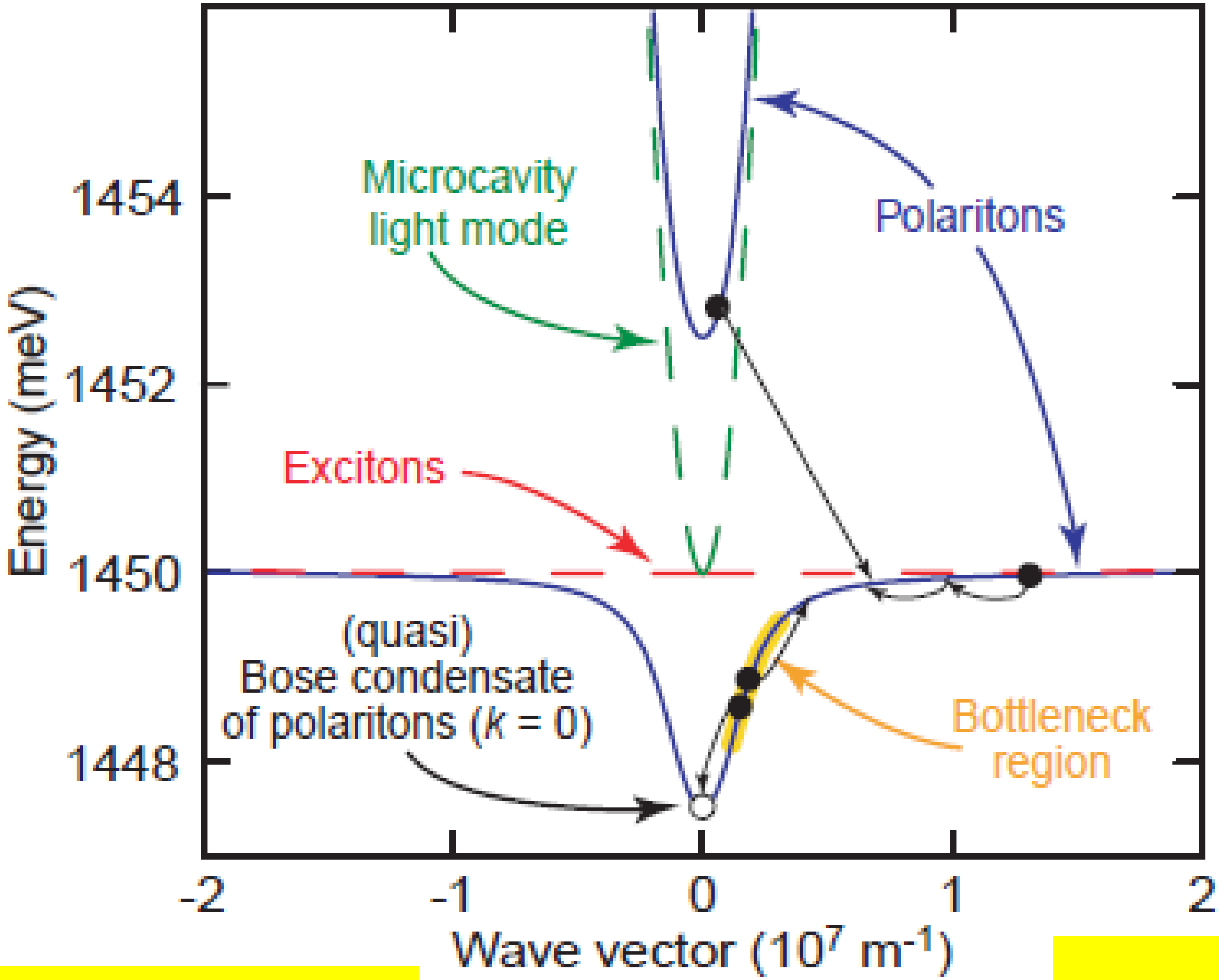
$$\omega = c\sqrt{k_\perp^2 + k_{||}^2}\ ; \quad \omega_0 \equiv ck_{\perp\min}\ ; \quad \omega - \omega_0 << \omega_0$$

$$\omega \approx \omega_0 + \frac{\left(ck_{||}\right)^2}{2\omega_0}$$

constant

2D light

Polaritons $\approx$ **Photons with weak interaction**



$$m^* \sim 10^{-4} m_0$$

Small effective mass

$$T_c \propto (m^*)^{-1}$$

Critical temp. – up to the room temp.

#of incoming particles in unit time n_{in}

# of outgoing particles in unit time n_{out}

Total number of particles N

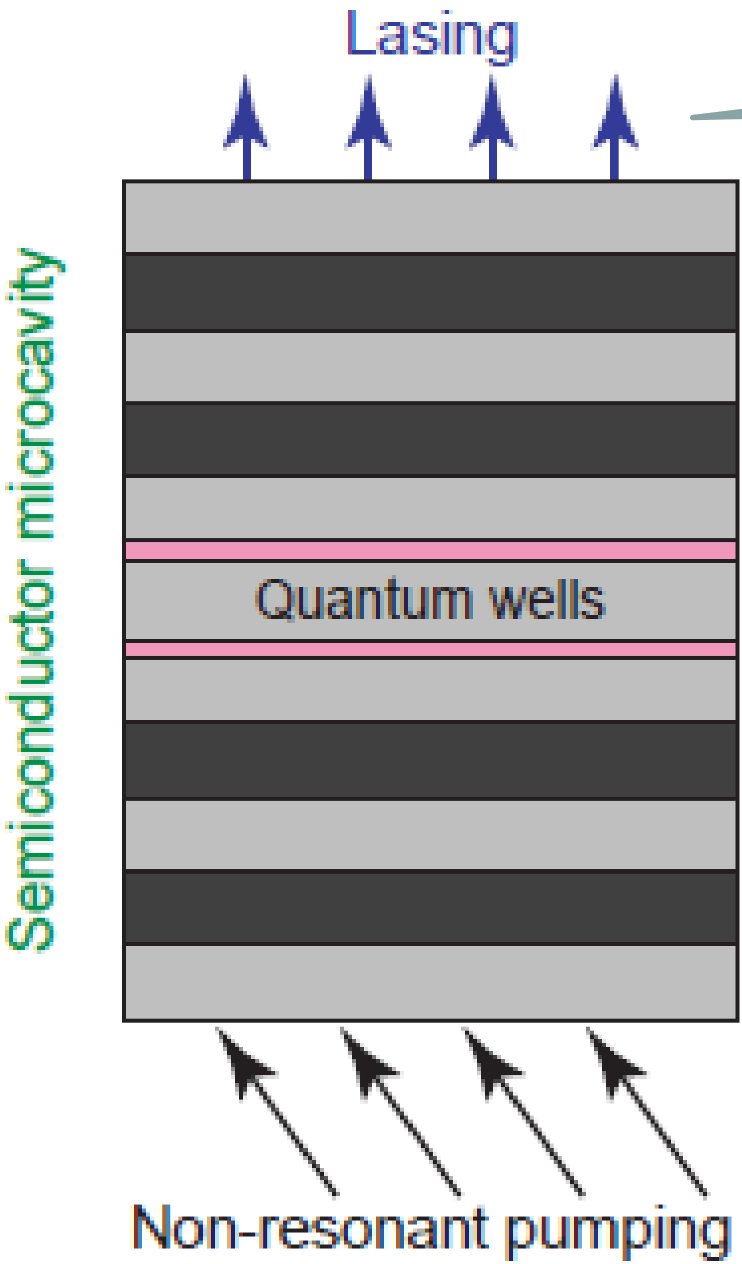
Stationary state: $n_{in} = n_{out}$

Classical particles with finite lifetime Γ^{-1}: $n_{in} = const$ $n_{out} = N\Gamma$

Bosons: $n_{in} = WN$ $n_{out} = N\Gamma$

Threshold

Polaritons ≈ Photons with weak interaction



Dissipation.

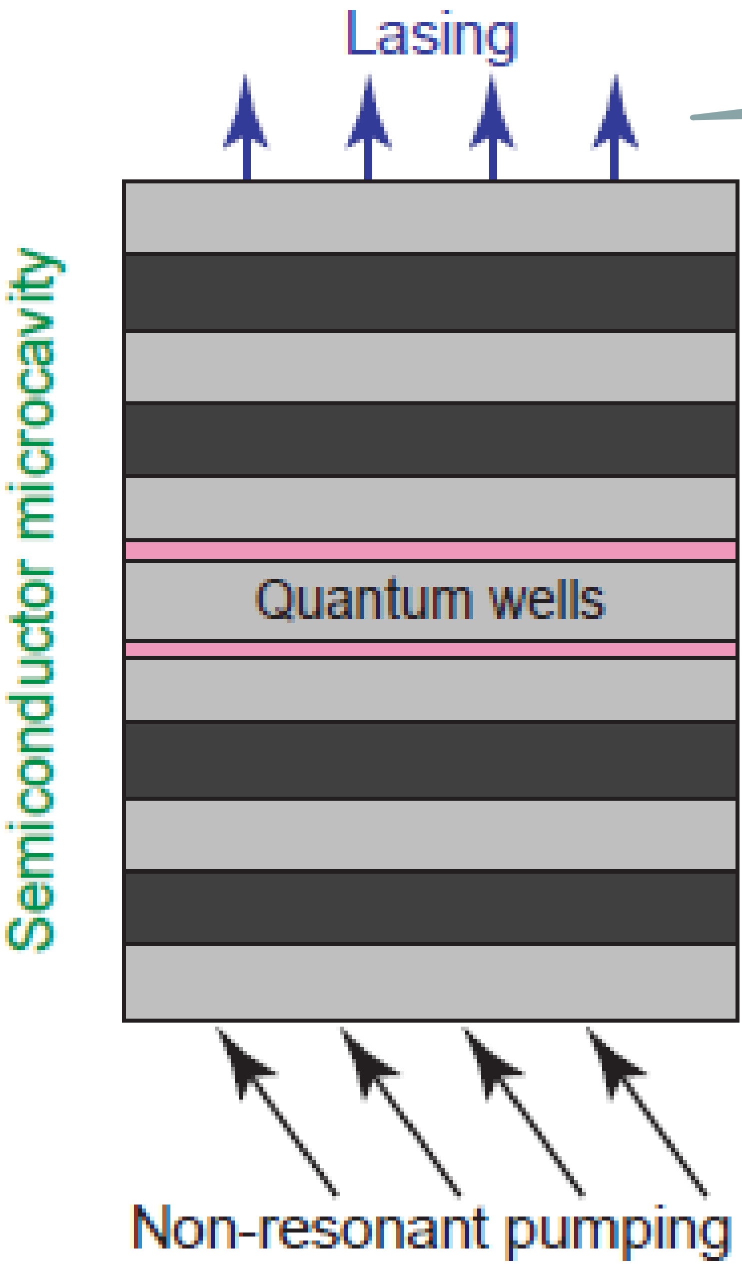
Pumping

Measurements:

1. Far field.
   Angular distribution (small deviations from the normal to the plane) of the emitted light ⟺ momentum distribution of the polaritons.

2. Near field.
   2D density of the polaritons.

Polaritons ≈ Photons with weak interaction



Dissipation.

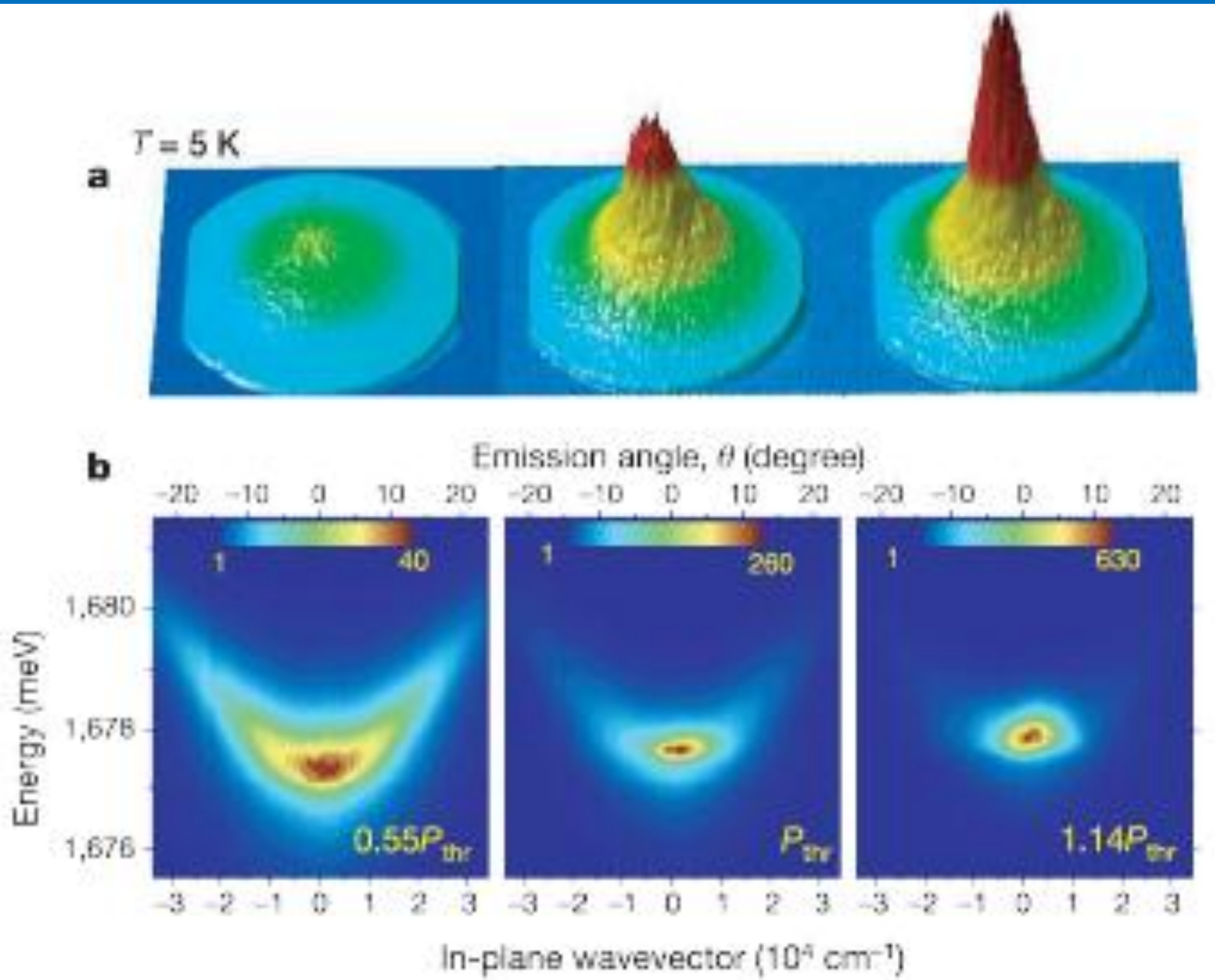
Stationary state:
Pumping = Dissipation

For bosons
both the pumping and the dissipation are proportional to the number of the particles

Instability:
lasing threshold

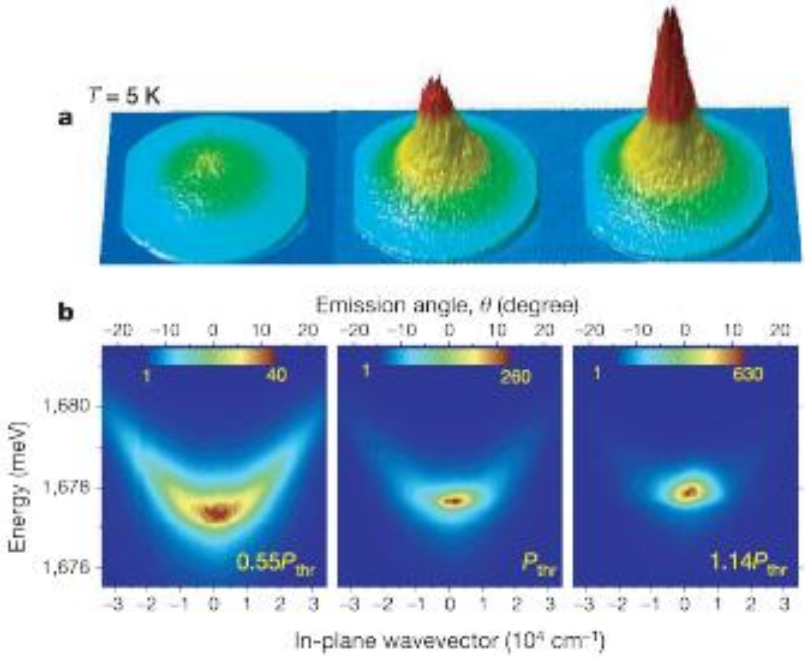
Pumping

First observation of the condensation

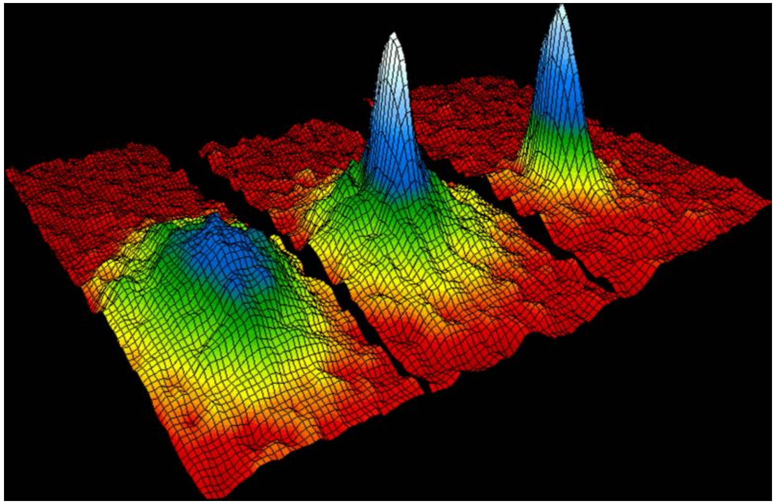


J. Kasprzak, M. Richard, S. Kundermann, A. Baas, P. Jeambrun, J. M. J. Keeling, F. M. Marchetti, M. H. Szymanska, R. Andre, J. L. Staehli, V. Savona, P. B. Littlewood, B. Deveaud and Le Si Dang, Nature 443, 409 (2006).

Compare:

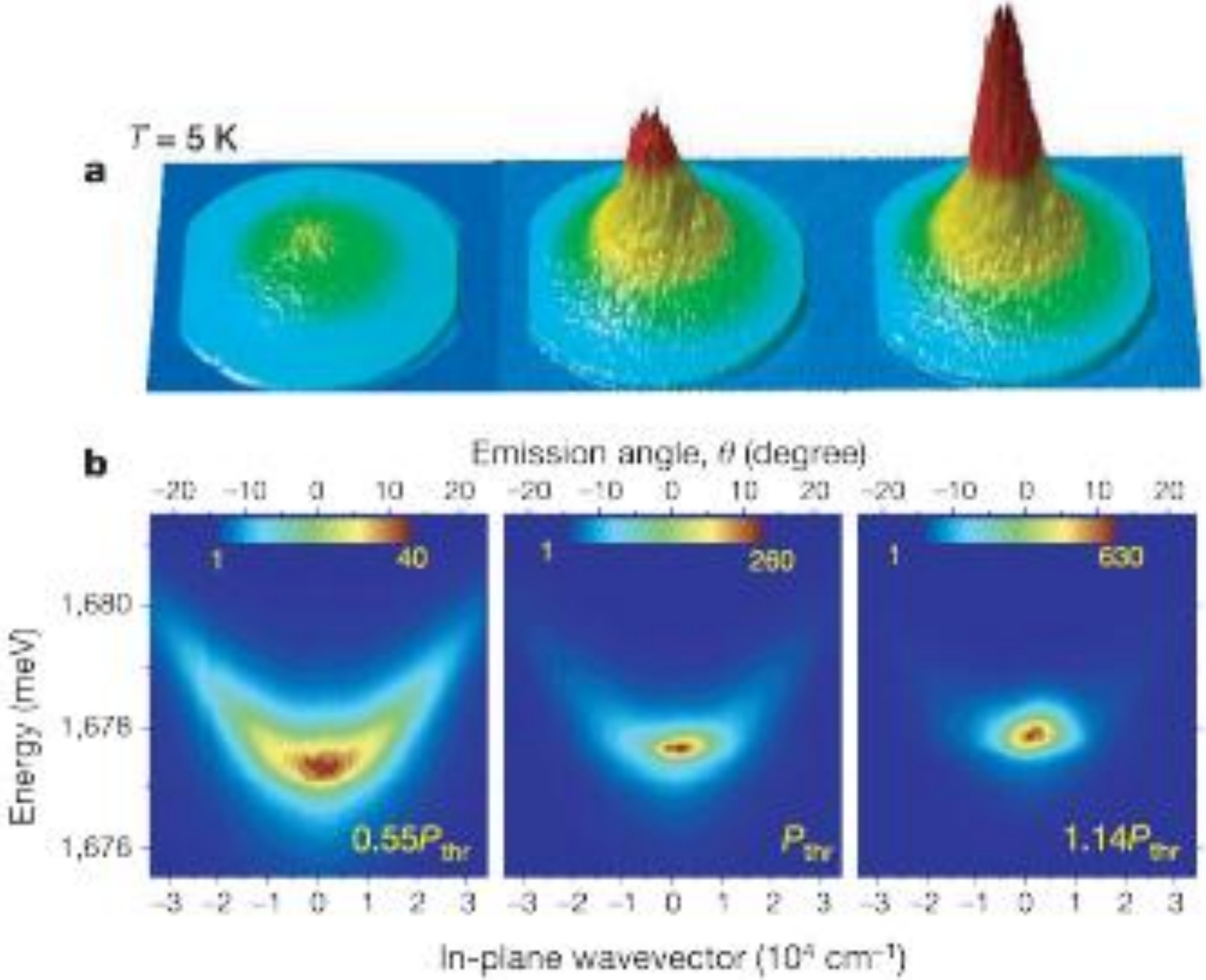


polaritons



atoms

First observation of the condensation



Polaritons condense at zero in-plane momentum

Why ?

J. Kasprzak, M. Richard, S. Kundermann, A. Baas, P. Jeambrun, J. M. J. Keeling, F. M. Marchetti, M. H. Szymanska, R. Andre, J. L. Staehli, V. Savona, P. B. Littlewood, B. Deveaud and Le Si Dang, Nature 443, 409 (2006).

NATURE PHOTONICS | VOL 6 | JANUARY 2012

L. V. Butov and A. V. Kavokin

To the Editor — Several experimental groups have demonstrated that exciton–polaritons quasiparticles made from a mix of light and matter can form a coherent state (that is, a condensate in momentum space) in a semiconductor microcavity. However, **there is little agreement in the community** regarding the nature and associated terminology of this condensate: **is it a Bose–Einstein condensate (BEC), a laser, or something else?** Polaritons are also sometimes described as exhibiting superfluidity. Here we wish to point out that **describing polaritons and their condensate in terms of a BEC and superfluidity may be misleading.**

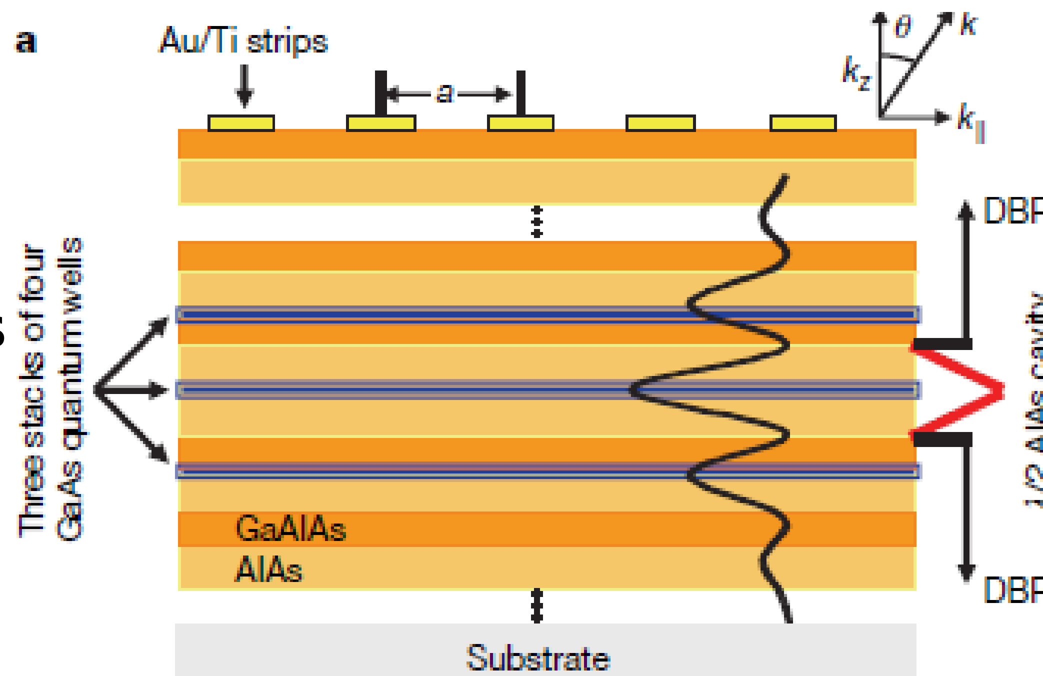
Vol 450 | 22 November 2007 | doi:10.1038/nature06334 nature

Coherent zero-state and π-state in an exciton–polariton condensate array

C. W. Lai[1,2,3], N. Y. Kim[1,2], S. Utsunomiya[3,4], G. Roumpos[1], H. Deng[1], M. D. Fraser[1], T. Byrnes[2,3], P. Recher[1,2], N. Kumada[4], T. Fujisawa[4] & Y. Yamamoto[1,3]

Polariton array: 1.4-mm-wide Au/Ti strips are equally spaced ***a* =2.8 µm**. A $\lambda/2$ AlAs cavity (red lines) is sandwiched by two distributed Bragg reflectors with alternating GaAlAs/AlAs $\lambda/4$ layers (two short dotted vertical lines.). The cavity resonance wavelength λ varies around the quantum well exciton resonance - **776 nm** with tapering.

a



Three stacks of four GaAs quantum wells are positioned at the central three antinodes of the photon field (black oscillatory curve).

Vol 450 | 22 November 2007 | doi:10.1038/nature06334

nature

Coherent zero-state and π-state in an exciton–polariton condensate array

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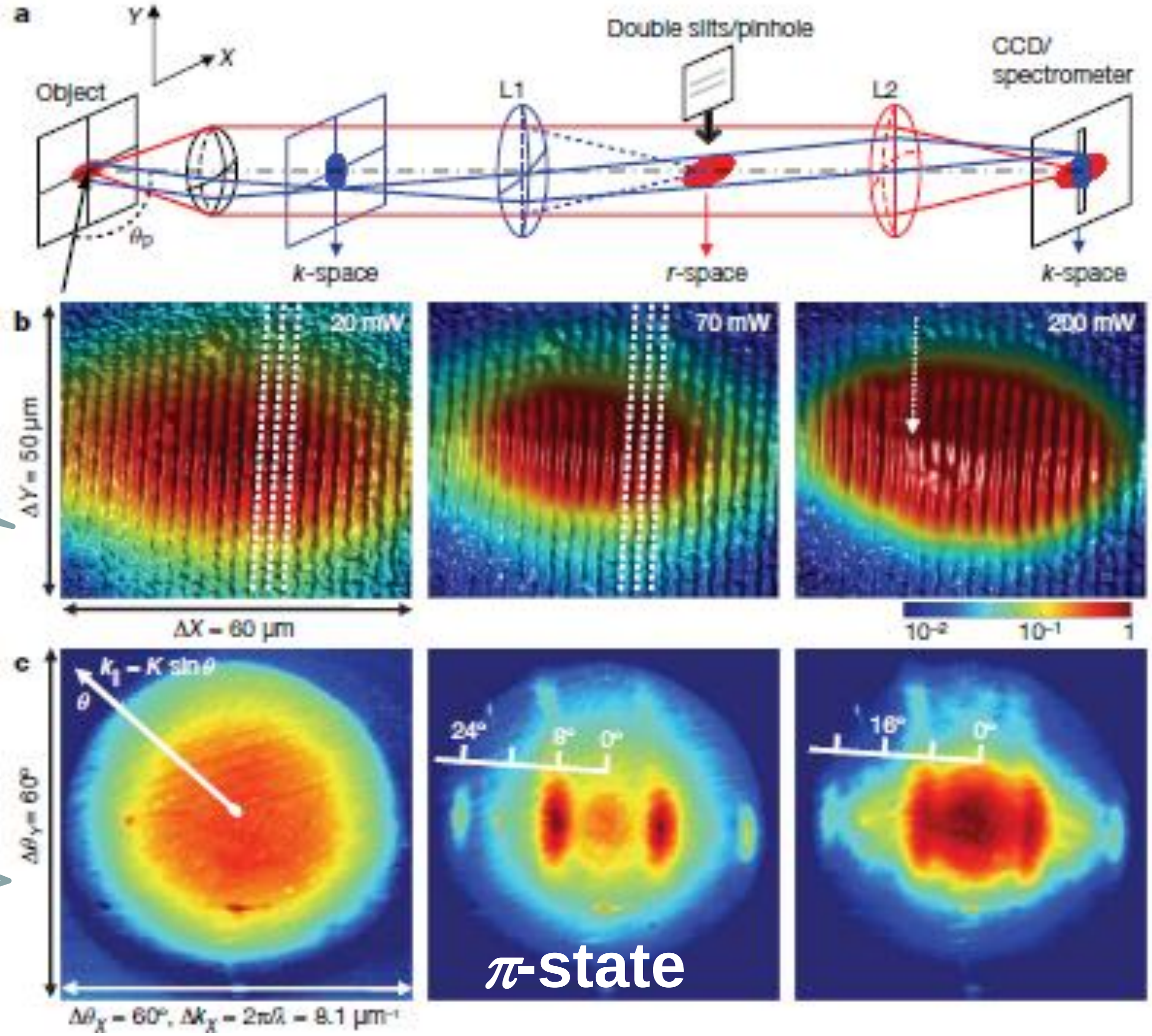
Spatial distribution (near field)

Angular (momentum) distribution (far field)

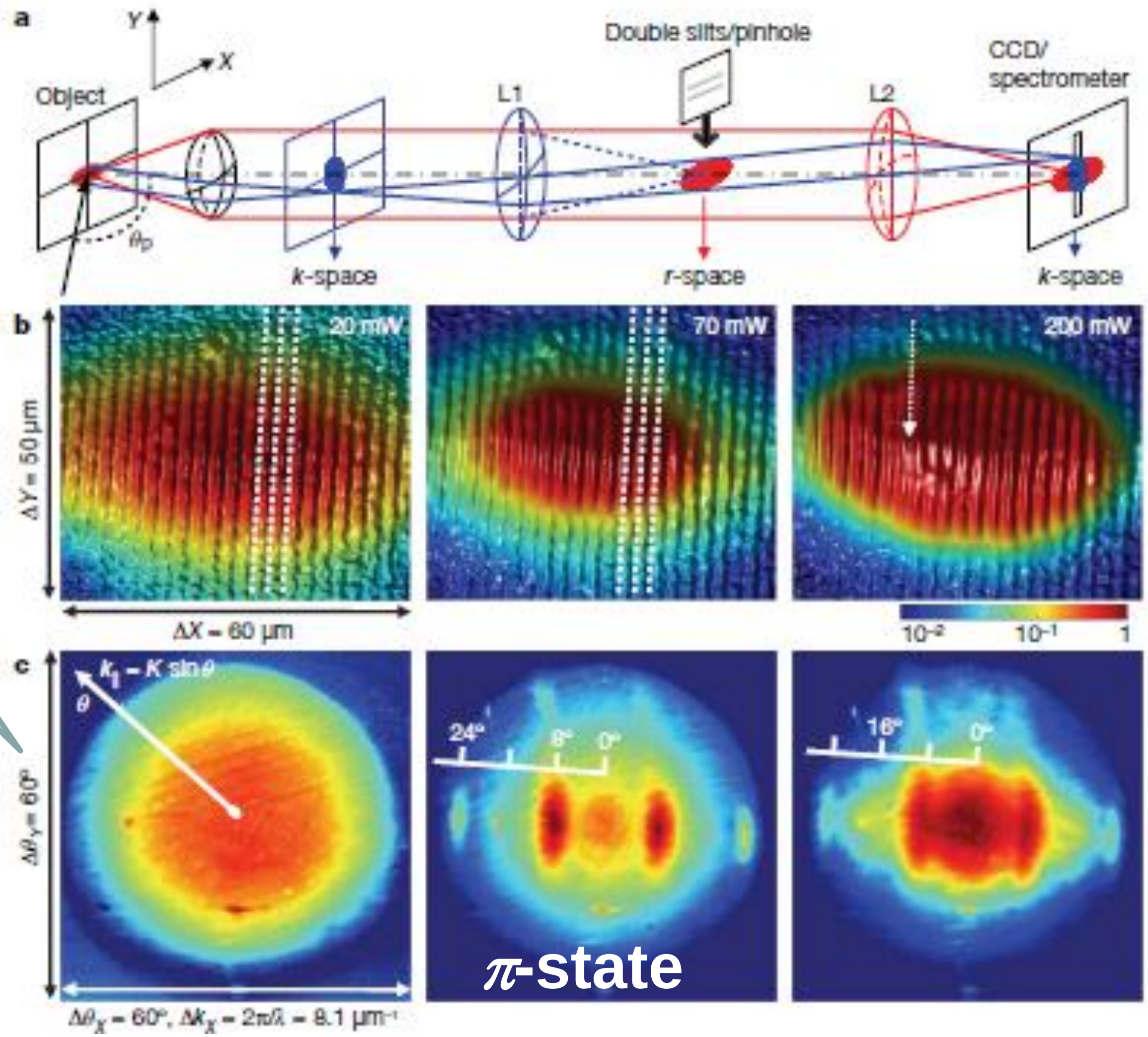


Vol 450 | 22 November 2007 | doi:10.1038/nature06334 nature

Coherent zero-state and π-state in an exciton–polariton condensate array

C. W. Lai[1,2,3], N. Y. Kim[1,2], S. Utsunomiya[3,4], G. Roumpos[1], H. Deng[1], M. D. Fraser[1], T. Byrnes[2,3], P. Recher[1,2], N. Kumada[4], T. Fujisawa[4] & Y. Yamamoto[1,3]

Angular (momentum) distribution (far field)

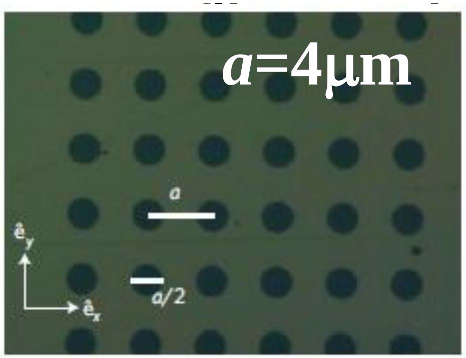


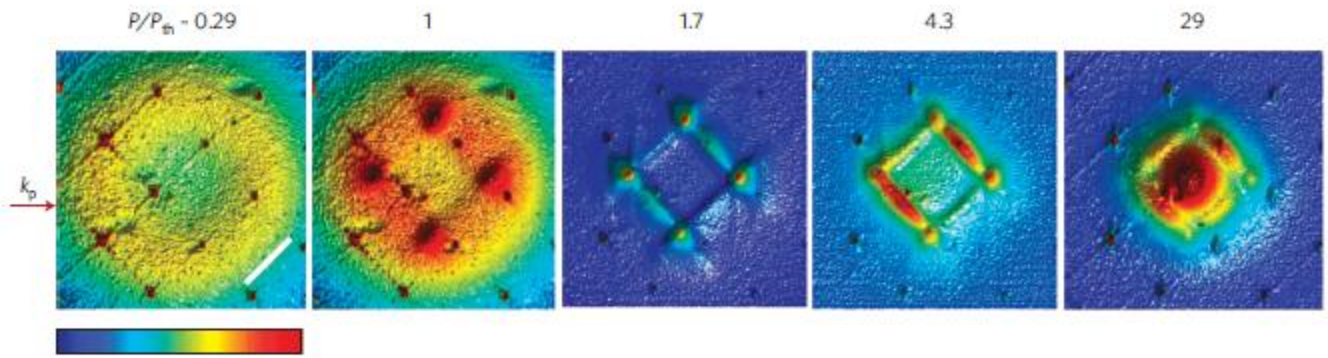
Phases of the neighboring strips differ by π. Condensation to the state with the maximal momentum!

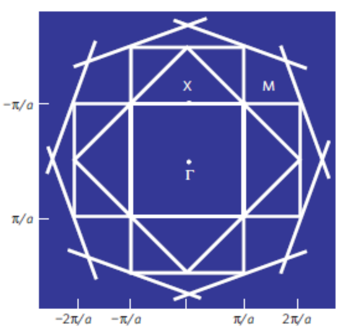
LETTERS
PUBLISHED ONLINE: 19 JUNE 2011 | DOI:10.1038/NPHYS2012

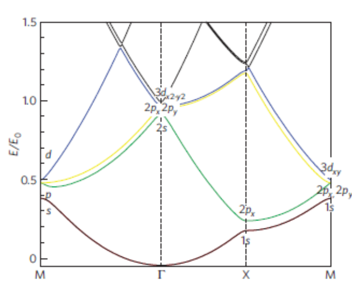
Dynamical *d*-wave condensation of exciton-polaritons in a two-dimensional square-lattice potential

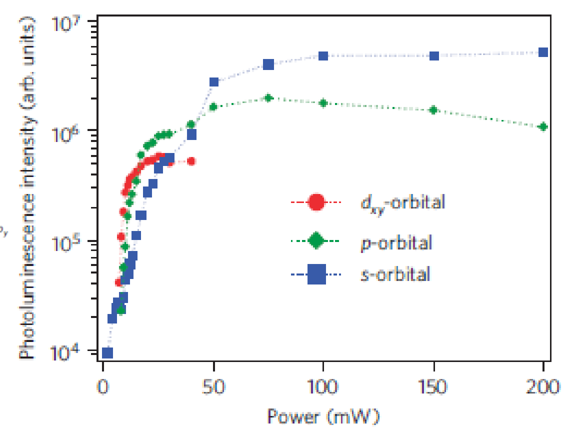
Na Young Kim, Kenichiro Kusudo, CongjunWu, Naoyuki Masumoto, Andreas Löffler,





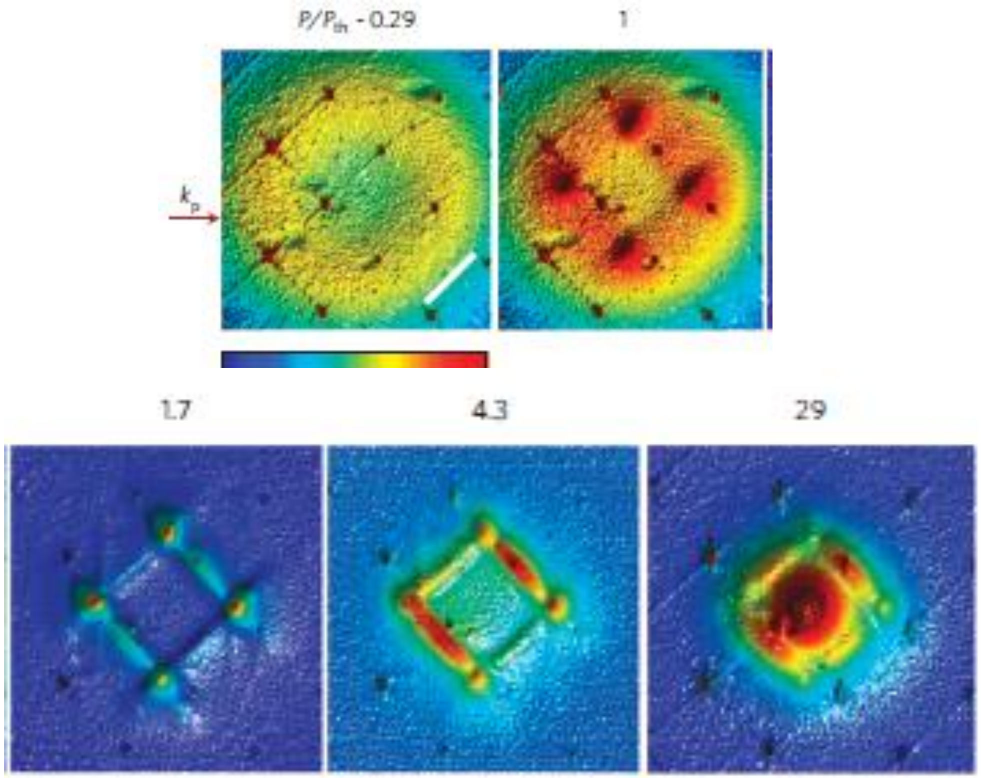




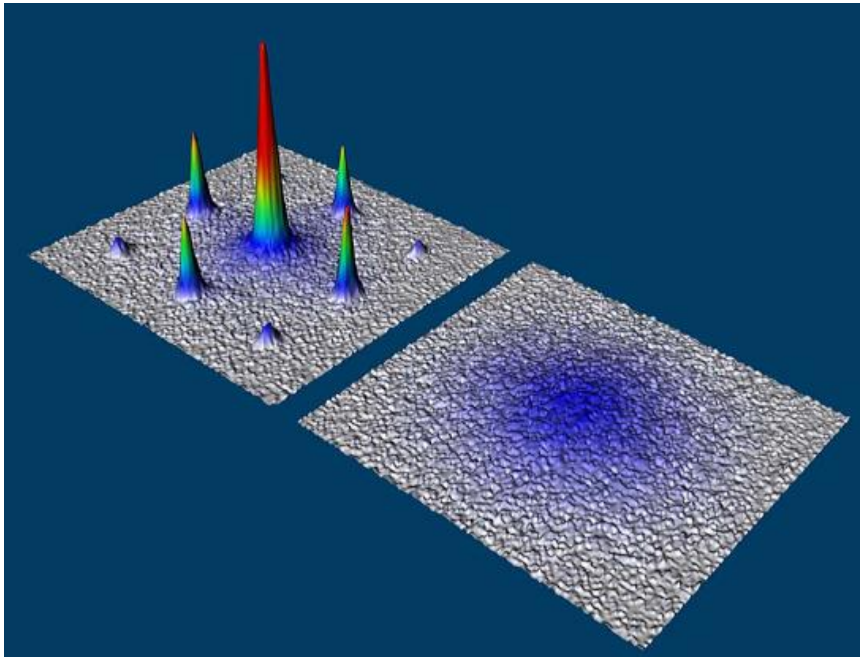


Three condensates simultaneously ?

Condensation in periodic potential. Compare:



polaritons



atoms

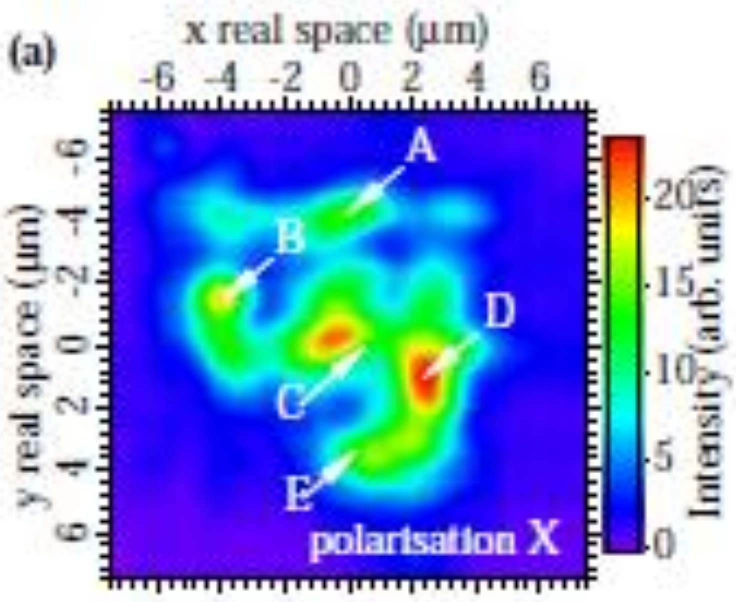
Exciton-polaritons localized by disorder

Coexisting Non-Equilibrium Condensates with Long-Range Spatial Coherence in Semiconductor Microcavities

D. N. Krizhanovskii,[1] K. G. Lagoudakis,[2] M. Wouters,[2] B. Pietka,[2] R. A. Bradley,[1] K. Guda,[1] D. M. Whittaker,[1] M. S. Skolnick,[1] B. Deveaud-Plédran,[2] M. Richard,[3] R. André,[3] and Le Si Dang[3]

[1] *Department of Physics & Astronomy, University of Sheffield, Sheffield S3 7RH, United Kingdom*
[2] *Ecole Polytechnique Fédérale de Lausanne (EPFL), Station 3, CH-1015 Lausanne, Switzerland*
[3] *Institut Néel, CNRS and Université J. Fourier, 38042 Grenoble, France*

Phys. Rev. B **80**, 045317 (2009).



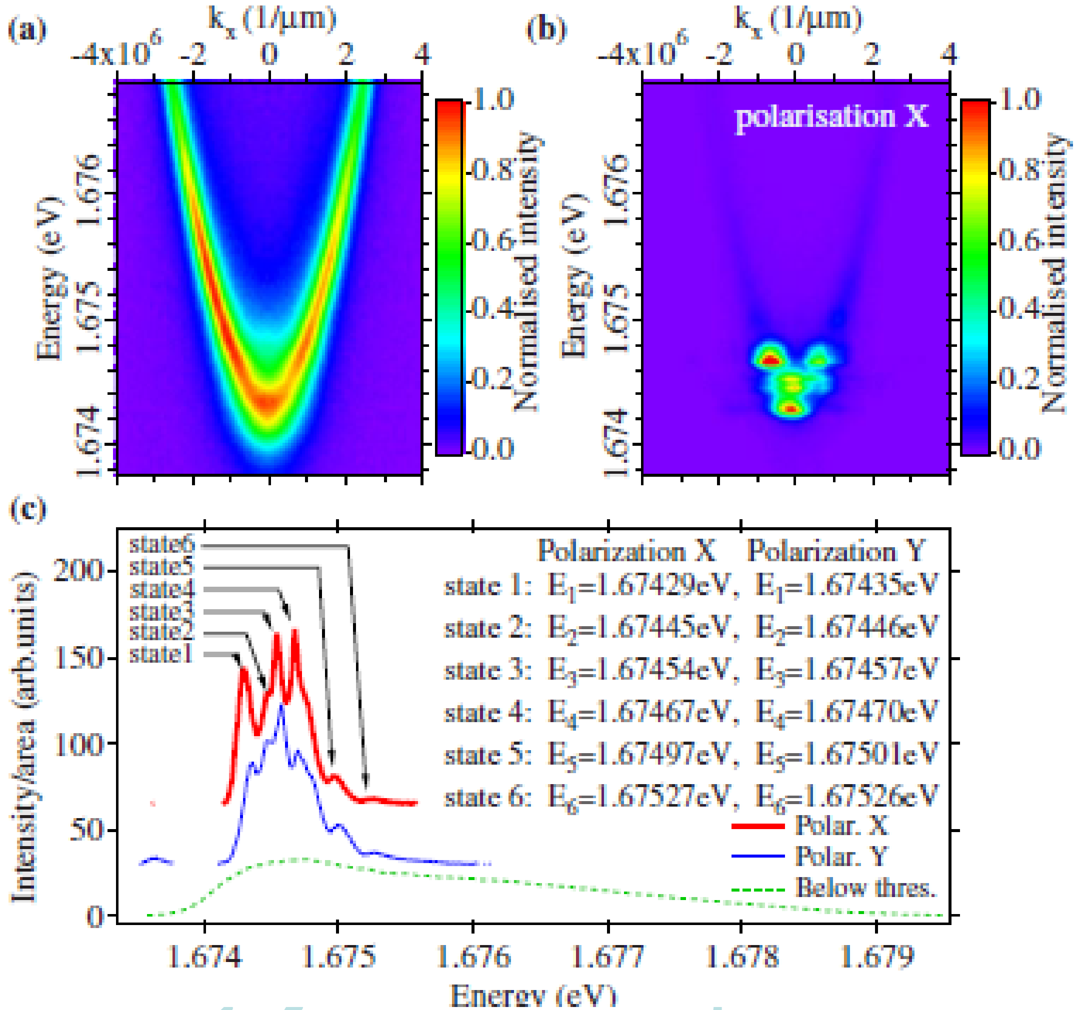
5 bright spots

Natural assumption: each spot is the location of a droplet of the Bose-condensate.

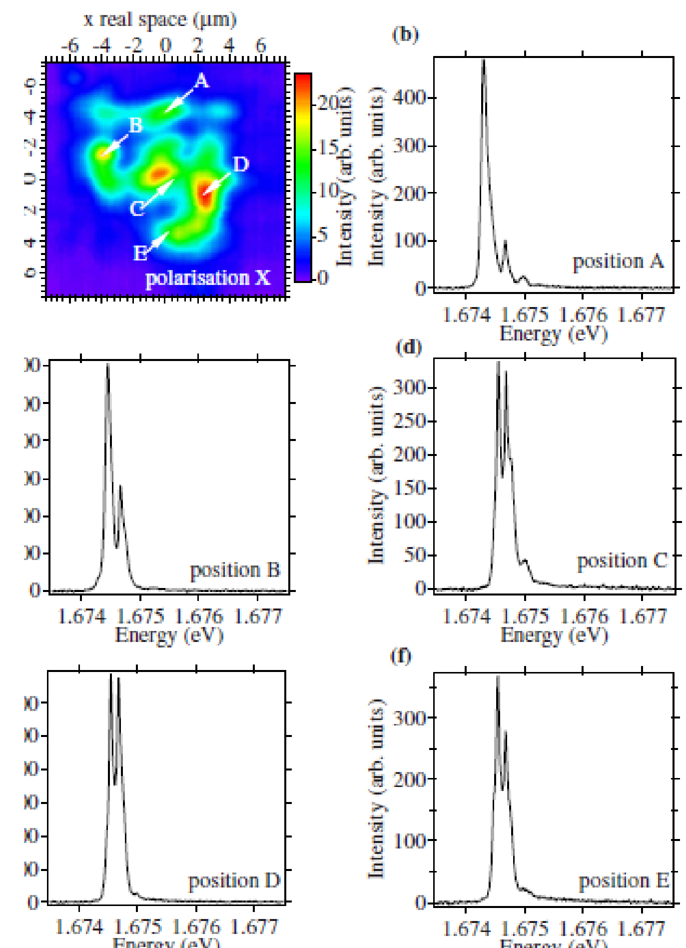
This is not the case!

Exciton-polaritons localized by disorder

D. N. Krizhanovskii, K. G. Lagoudakis, M. Wouters, B. Pietka, R. A. Bradley, K. Guda, D. M. Whittaker, M. S. Skolnick, B. Deveaud-Piedran, M. Richard, R. Andre, and Le Si Dang, Phys. Rev. B 80, 045317 (2009)

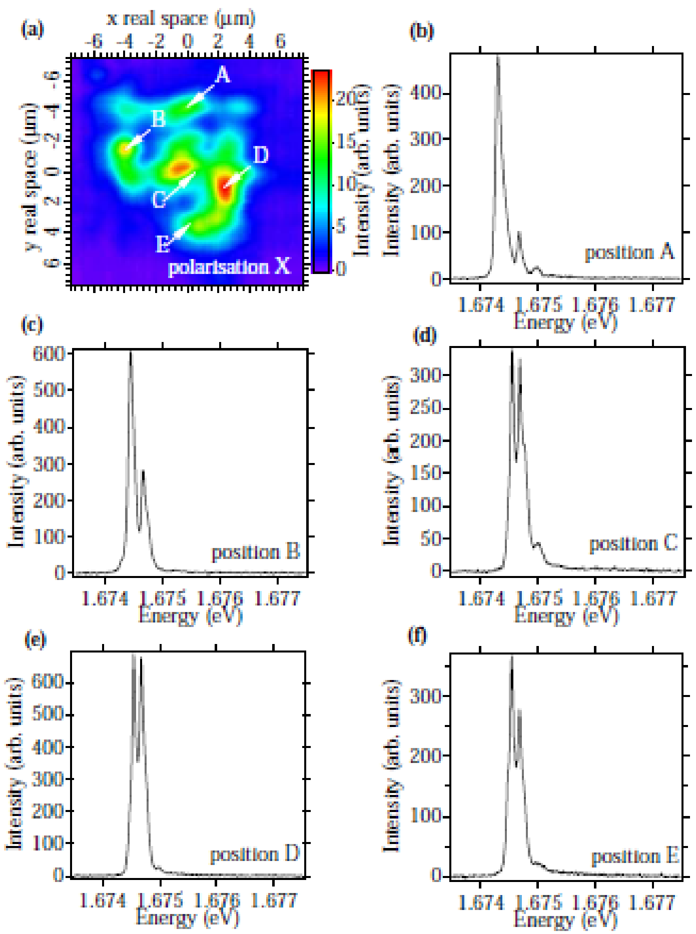


6 frequency modes

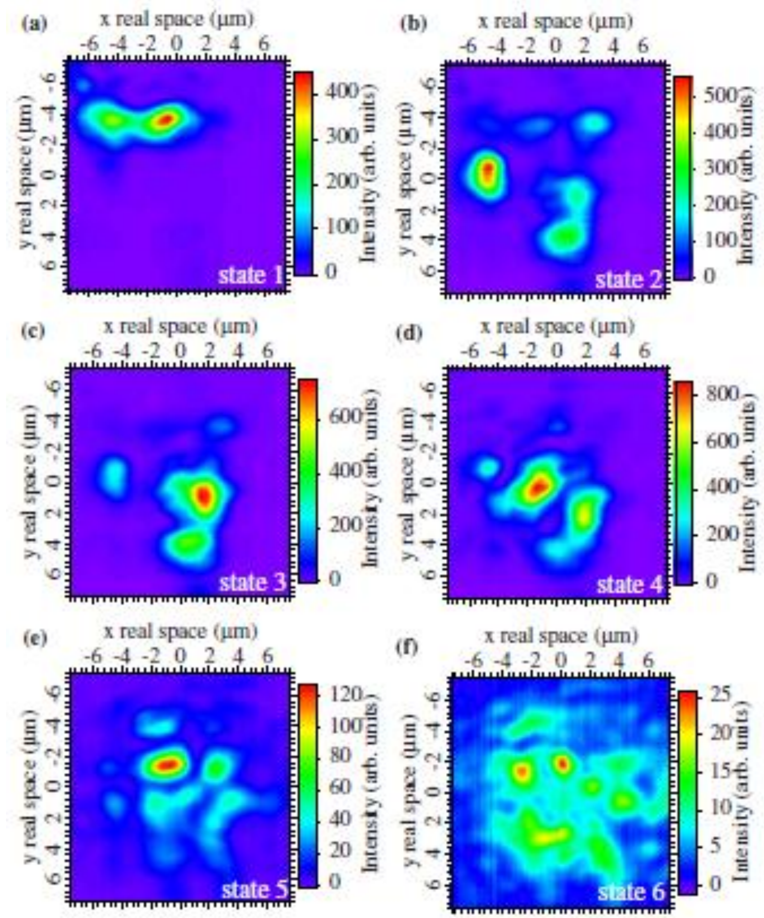


real space: 5 spots

Exciton-polaritons localized by disorder



Frequency spectra
of each spot



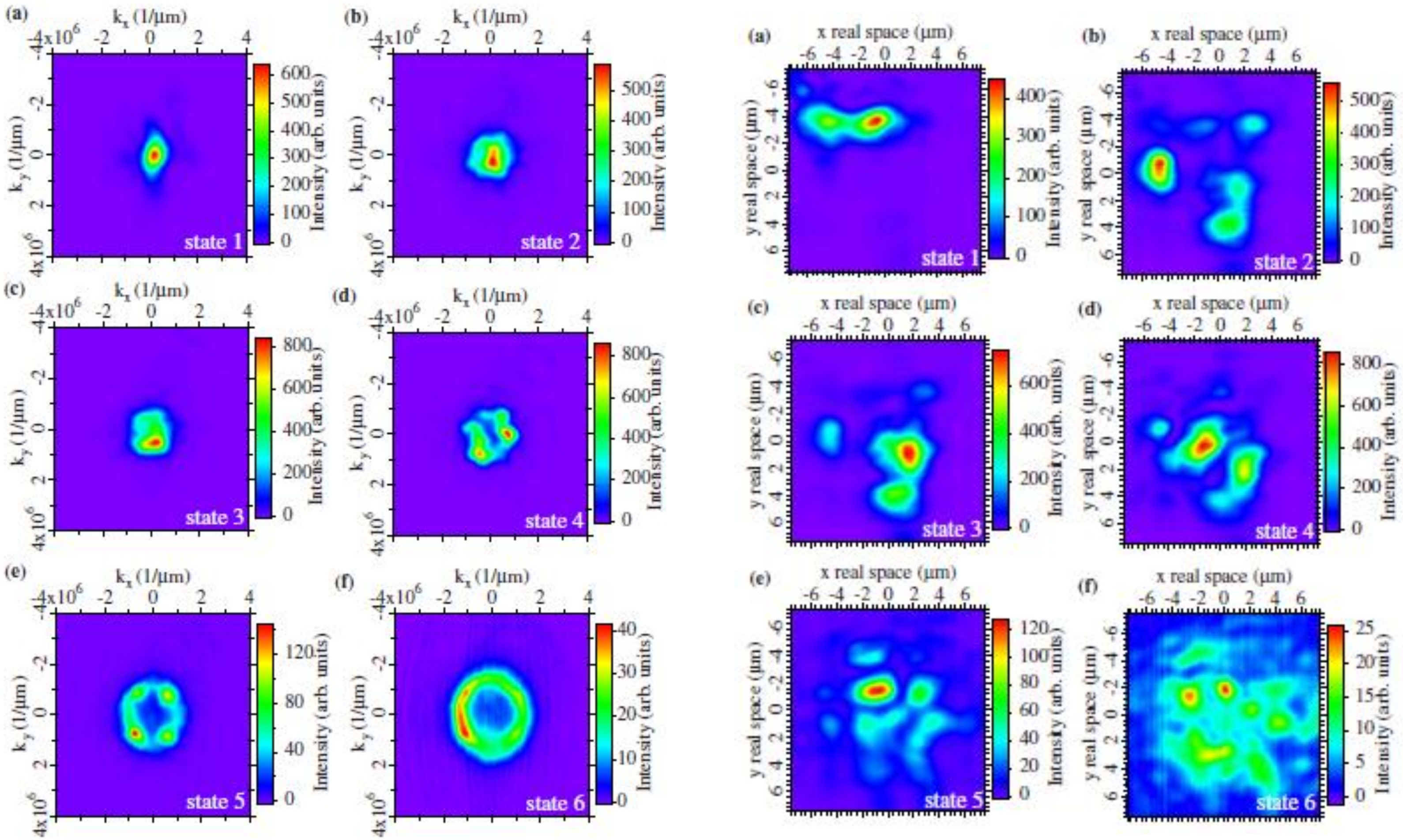
Spatial distribution of
each frequency mode

Each frequency is radiated by several spots;
Each spot radiates several frequencies !

Exciton-polaritons localized by disorder

D. N. Krizhanovskii et. Al., Phys. Rev. B 80, 045317 (2009)

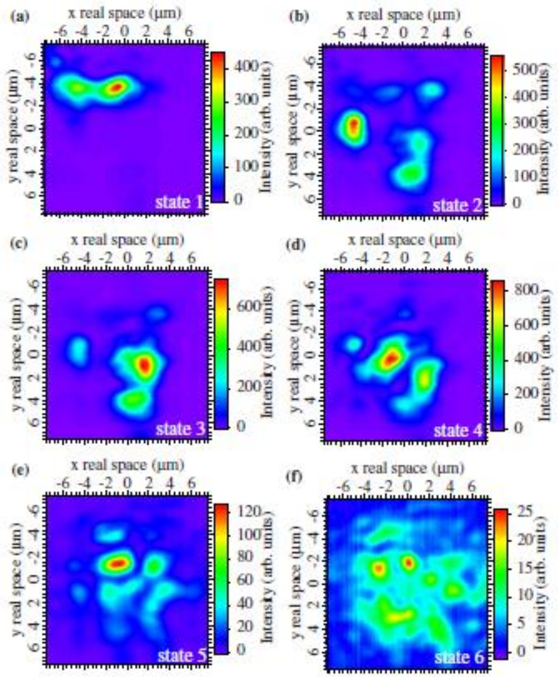
Imaging at the energy of each mode:



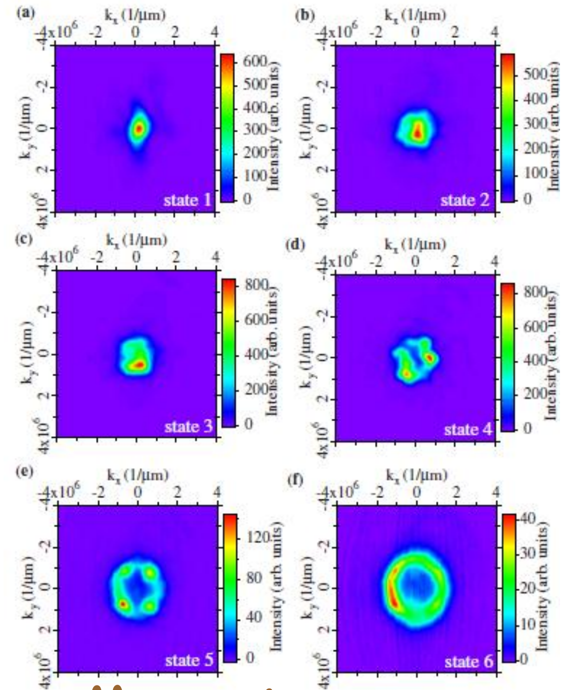
Momentum space Real space

D. N. Krizhanovskii et. Al., Phys. Rev. B 80, 045317 (2009)

Separated images of the modes



Real space



Momentum space

- Different spots – localized states of exciton-polaritons.
- More modes than localized states? Several condensates on one site?

- Some modes have **minimum** at $\vec{k}_{\parallel} = 0$ (at the ground state!)
- No $\vec{k}_{\parallel} \to -\vec{k}_{\parallel}$ symmetry?

One center: Coherent states $\hat{a}|z\rangle = z|z\rangle$

- One one-particle state,
- Coherent many-particle states
- Complex number $z = |z|e^{i\varphi}$ - eigenvalue of the annihilation operator $\hat{a}$
- Occupation number is $|z|^2$, while φ is the phase

Need to take into account:

- **Pumping** and **dissipation**
- **Nonlinearity**=interaction between the bosons

QM description – density matrix $\rho(z, z^*)$

Fokker-Planck equation ⇨ Langevin equation

Fokker-Planck eq-n for the density matrix $\rho\left(z,z^*\right)$

1. No pumping, no dissipation. Hamiltonian $H\left(z,z^*\right)=H\left(\left|z\right|^2\right)$

$$\frac{\partial\rho}{\partial t}=i\left(\frac{\partial\rho}{\partial z}\frac{\partial H}{\partial z^*}-\frac{\partial\rho}{\partial z^*}\frac{\partial H}{\partial z}\right)=-2\,\mathrm{Im}\left\{\frac{\partial}{\partial z}\left(\rho\frac{\partial H}{\partial z^*}\right)\right\}$$

2. Pumping W . Dissipation Γ ⇨ Hamiltonian function $h\left(z,z^*\right)$

$$h\left(z,z^*\right)=H\left(z,z^*\right)-i\Pi\left(z,z^*\right)$$

$$\boxed{\frac{\partial\rho}{\partial t}=-2\,\mathrm{Im}\left\{\frac{\partial}{\partial z}\left(\rho\frac{\partial h}{\partial z^*}\right)\right\}+W\frac{\partial^2\rho}{\partial z\partial z^*}}$$

Fokker-Planck eq-n for the density matrix $\rho\left(z, z^*\right)$

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$$h\left(z, z^*\right) = H\left(z, z^*\right) - i\Pi\left(z, z^*\right)$$

$$\boxed{\frac{\partial \rho}{\partial t} = -2\,\mathrm{Im}\left\{\frac{\partial}{\partial z}\left(\rho\frac{\partial h}{\partial z^*}\right)\right\} + W\frac{\partial^2 \rho}{\partial z \partial z^*}}$$

3. Hamiltonian of weakly interacting bosons with coupling constant α at a state with one-particle energy ω

$$\boxed{H\left(z, z^*\right) = \omega\left|z\right|^2 + \tfrac{1}{4}\alpha\left|z\right|^4}$$

Fokker-Planck eq-n for the density matrix $\rho\left(z,z^*\right)$

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4. Dissipative function

$$\boxed{\Pi\left(z,z^*\right)=\tfrac{1}{2}g\left|z\right|^2}\qquad g\equiv\left(\Gamma-W\right)$$

Fokker-Planck eq-n for the density matrix $\rho\left(z, z^{*}\right)$

Weakly interacting bosons

$$\frac{\partial \rho}{\partial t} = -2\,\mathrm{Im}\left\{\frac{\partial}{\partial z}\left(\rho \frac{\partial (H - i\Pi)}{\partial z^{*}}\right)\right\} + W\frac{\partial^{2}\rho}{\partial z \partial z^{*}}$$

$$H\left(z, z^{*}\right) = \omega |z|^{2} + \tfrac{1}{4}\alpha |z|^{4}$$

$$\Pi\left(z, z^{*}\right) = \tfrac{1}{2} g |z|^{2}$$

ω - one-particle energy

α - coupling constant

$$g \equiv (\Gamma - W)$$

W - pumping.

Γ - dissipation

$g = g\left(|z|^{2}\right)$ – monotonically increasing function e.g. due to the depletion of the reservoir

Fokker-Planck eq-n for the density matrix $\rho\left(z, z^*\right)$

Weakly interacting bosons

$$\boxed{\frac{\partial \rho}{\partial t} = -2\,\mathrm{Im}\left\{ \frac{\partial}{\partial z}\left(\rho \frac{\partial \left(H - i\Pi \right)}{\partial z^*} \right) \right\} + W \frac{\partial^2 \rho}{\partial z \partial z^*}}$$

$$\boxed{H\left(z, z^*\right) = \omega |z|^2 + \tfrac{1}{4}\alpha |z|^4}$$

$$\boxed{\Pi\left(z, z^*\right) = \tfrac{1}{2} g |z|^2}$$

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α - coupling constant

$$g \equiv \left(\Gamma - W\right)$$

W - pumping.

Γ - dissipation

$g = g\left(|z|^2\right)$ – monotonically increasing function e.g. due to the depletion of the reservoir

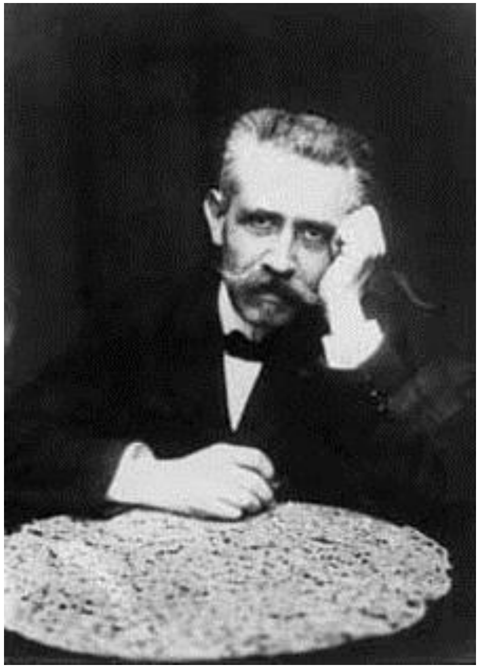
$g(0) = 0$ - threshold

$g\left(|z|_s^2\right) = 0 \qquad |z|_s^2$ – stable number of bosons above the threshold

One center: Langevin equation

$$\frac{\partial \rho}{\partial t} = W \frac{\partial^2}{\partial z \partial z^*} - 2\,\mathrm{Im}\,\frac{\partial}{\partial z}\left(\rho \frac{\partial h}{\partial z^*}\right)$$

$$h\left(z, z^*\right) = \left[\left(\omega + \tfrac{1}{4}\alpha |z|^2\right) - \frac{ig}{2}\right]|z|^2$$



$$\frac{\partial z}{\partial t} - i\,\frac{\partial h\left(z, z^*\right)}{\partial z^*} = f\left(t\right)$$

$f\left(z\right)$ Gaussian random white noise

$$\left\langle f\left(t\right)\right\rangle = 0;$$

$$\left\langle f\left(t\right) f\left(t'\right)\right\rangle = W\delta\left(t - t'\right)$$

One center - conclusions

$$\frac{\partial z}{\partial t} + (g + 2i\Omega) z = f(t)$$

1. $g = \Gamma - W$ Γ- dissipation; W - pumping

2. $\boxed{\Omega \equiv \partial H / \partial |z|^2 = \omega + \frac{1}{2}\alpha |z|^2}$ Frequency of the emitted light Blue shift

α - interaction constant; ω - one-particle energy

3. $f(z)$ Gaussian random white noise $\langle f(t) \rangle = 0;$ $\langle f(t) f(t') \rangle = W\delta(t - t')$

4. Threshold: $\Gamma = W \Leftrightarrow g = 0$

Below the threshold: no noise - no light

Above the threshold: need nonlinearity in dissipation or pumping $g = g\left(|z|^2\right) \approx g + A|z|^2$

One center – conclusions

$$\boxed{\frac{\partial z}{\partial t} + (g + 2i\Omega) z = f(t)}$$

$g = \Gamma - W$ Γ- dissipation; W - pumping

$$\boxed{\Omega \equiv \partial H / \partial |z|^2 = \omega + \tfrac{1}{2}\alpha |z|^2}$$

Frequency of the emitted light
Blue shift

α – interaction constant; ω – one-particle energy

$f(z)$ Gaussian random white noise

$$\langle f(t) \rangle = 0;$$

$$\langle f(t) f(t') \rangle = W\delta(t - t')$$

Note: This is the classical equation for a nonlinear oscillator with dissipation in the presence of the noise

$\mathrm{Re}\, z$ - coordinate $\mathrm{Im}\, z$ - momentum

N>1 centers:

$$z \leftarrow \{z_\mu\} \equiv \vec{Z}; \quad \mu = 1, 2, \ldots, N$$

$$h(\vec{Z}, \vec{Z}^*) = \sum_\mu (\omega_\mu + \Omega_\mu - ig_\mu) \frac{|z_\mu|^2}{2} + \sum_{\mu \neq \nu} (J_{\mu,\nu} - i\gamma_{\mu,\nu}) \frac{z_\mu^* z_\nu}{2}$$

set of independent centers

Generic bilinear coupling

$$\Omega_\mu = \omega_\mu + \tfrac{1}{2} \alpha_\mu |z_\mu|^2$$

Both matrices $\gamma_{\mu,\nu}$ and $J_{\mu,\nu}$ are Hermitian

$$\frac{\partial z_\mu}{\partial t} + (g + 2i\Omega_\mu) \frac{z_\mu}{2} + \sum_{\mu \neq \nu} (\gamma_{\mu,\nu} + iJ_{\mu,\nu}) \frac{z_\nu}{2} = f_\mu(t)$$

System of the Langevin equations

N>1 centers: $z \leftarrow \{z_\mu\} \equiv \vec{Z}; \quad \mu = 1, 2, \ldots, N$

$$h\left(\vec{Z}, \vec{Z}^*\right) = \tfrac{1}{2}\sum_{\mu}\left(g_\mu + 2i\Omega_\mu\right)\left|z_\mu\right|^2 + \tfrac{1}{2}\sum_{\mu \neq \nu}\left(\gamma_{\mu,\nu} + iJ_{\mu,\nu}\right) z_\mu^* z_\nu$$

$$\Omega_\mu = \omega_\mu + \tfrac{1}{2}\alpha_\mu \left|z_\mu\right|^2$$

$$\frac{\partial z_\mu}{\partial t} + \left(g + 2i\Omega_\mu\right)\frac{z_\mu}{2} + \sum_{\mu \neq \nu}\left(\gamma_{\mu,\nu} + iJ_{\mu,\nu}\right)\frac{z_\nu}{2} = f_\mu(t)$$

System of coupled nonlinear oscillators

Two types of coupling:

Hermitian "Josephson" coupling

Anti-Hermitian Dissipative coupling

Both matrices $\gamma_{\mu,\nu}$ and $J_{\mu,\nu}$ are Hermitian

1. N>1 centers: $z \leftarrow \{z_\mu\} \equiv \vec{Z}; \quad \mu = 1, 2, \ldots, N$

$$\frac{\partial z_\mu}{\partial t} + \left(g + 2i\Omega_\mu\right)\frac{z_\mu}{2} + \sum_{\mu \neq \nu}\left(\gamma_{\mu,\nu} + iJ_{\mu,\nu}\right)\frac{z_\nu}{2} = f_\mu(t)$$

$$\Omega_\mu = \omega_\mu + \tfrac{1}{2}\alpha_\mu \left|z_\mu\right|^2$$

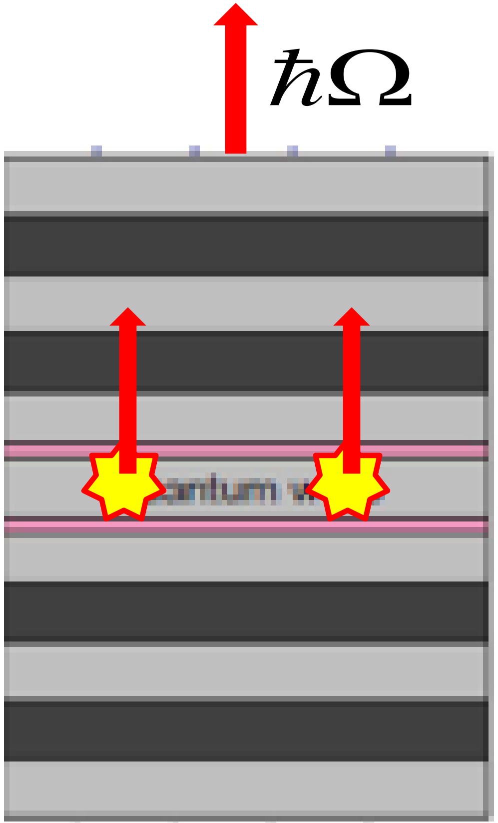
System of coupled nonlinear oscillators

$J_{\mu,\nu}$ Josephson coupling-tunneling

$\gamma_{\mu,\nu}$ Dissipative coupling

Physics of the off-diagonal decay:

Interference of the radiated photons



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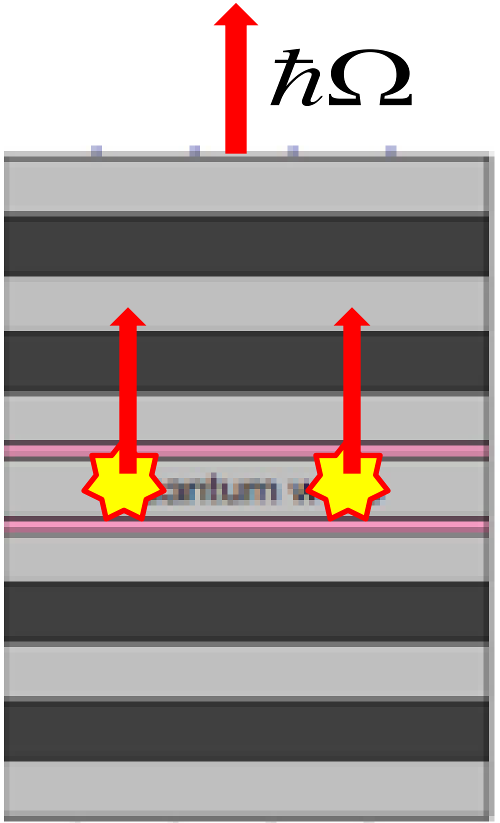

Weak lasing regime $0 < g < \gamma$

The increment is positive when the centers are out of phase

The system is stabilized without nonlinearities in dissipation

N>1 centers:

$$z \leftarrow \{z_\mu\} \equiv \vec{Z}; \quad \mu = 1, 2, \dots, N$$

$$\frac{\partial z_\mu}{\partial t} + \left(g + 2i\Omega_\mu\right)\frac{z_\mu}{2} + \sum_{\mu \neq \nu}\left(\gamma_{\mu,\nu} + iJ_{\mu,\nu}\right)\frac{z_\nu}{2} = f_\mu(t)$$

$$\Omega_\mu = \omega_\mu + \tfrac{1}{2}\alpha_\mu \left|z_\mu\right|^2$$

$$\langle f_\mu \rangle = 0; \quad \langle f_\mu(t)\, f_\mu^*(t') \rangle = W_\mu \delta(t - t')\, \delta_{\mu\nu}$$

System of N coupled classical nonlinear oscillators in the presence of the pumping and noise.

It is also a discrete version of the Gross-Pitaevskii equation (nonlinear Schrodinger equation)

$$i\hbar \frac{\partial \Psi(\vec{r}, t)}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r}) + g\left|\Psi(\vec{r}, t)\right|^2\right)\Psi(\vec{r}, t)$$

+ non-Hermitian terms + "thermal" noise

N>1 centers: $z \leftarrow \{z_\mu\} \equiv \vec{Z}; \quad \mu = 1, 2, \ldots, N$

$$\frac{\partial z_\mu}{\partial t} + \left(g + 2i\Omega_\mu\right)\frac{z_\mu}{2} + \sum_{\mu \neq \nu}\left(\gamma_{\mu,\nu} + iJ_{\mu,\nu}\right)\frac{z_\nu}{2} = f_\mu(t)$$

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System of N coupled classical nonlinear oscillators in the presence of the pumping and noise.

It is also a discrete version of the Gross-Pitaevskii eq-n (1961) + non-Hermitian terms + "thermal" noise

Wouters and Carusotto equation

(M. Wouters and I. Carusotto, Phys. Rev. Lett. 99, 140402 (2007).

- Include Z - dependence of the pumping W
- No noise term
- No dissipative coupling $\gamma_{\mu\nu} = 0$

$N>1$ centers: $z \leftarrow \{z_\mu\} \equiv \vec{Z}; \quad \mu = 1, 2, \ldots, N$

$$\frac{\partial z_\mu}{\partial t} + \left(g + 2i\Omega_\mu\right)\frac{z_\mu}{2} + \sum_{\mu\neq\nu}\left(\gamma_{\mu,\nu} + iJ_{\mu,\nu}\right)\frac{z_\nu}{2} = f_\mu(t)$$

$$\Omega_\mu = \omega_\mu + \tfrac{1}{2}\alpha_\mu \left|z_\mu\right|^2$$

$$\left\langle f_\mu \right\rangle = 0; \quad \left\langle f_\mu(t)\, f_\mu^*(t')\right\rangle = W_\mu \delta(t-t')\,\delta_{\mu\nu}$$

Neglect noise

N>1 centers: $z \leftarrow \{z_\mu\} \equiv \vec{Z}; \quad \mu = 1, 2, \ldots, N$

$$\frac{\partial z_\mu}{\partial t} + \left(g + 2i\omega_\mu + \alpha \left|z_\mu\right|^2\right)\frac{z_\mu}{2} + \sum_{\mu \neq \nu}\left(\gamma_{\mu,\nu} + iJ_{\mu,\nu}\right)\frac{z_\nu}{2} = 0$$

$\vec{Z} = 0$ is always a solution - trivial solution.

Are there other nontrivial solutions?

Condensation:
Stationary (up to the total phase) nontrivial solution

$$\vec{Z}(t) = e^{i\Phi(t)}\vec{Z}(0)$$

Without dissipative coupling:

- $g>0$ – only trivial solution
- $g<0$ – need to take into account $g(z_\mu)$ – dependence
- $g=0$ – threshold

N>1 centers: $z \leftarrow \{z_\mu\} \equiv \vec{Z}; \quad \mu = 1, 2, \ldots, N$

$$\frac{\partial z_\mu}{\partial t} + \left(g + 2i\omega_\mu + \alpha |z_\mu|^2\right)\frac{z_\mu}{2} + \sum_{\mu \neq \nu}\left(\gamma_{\mu,\nu} + iJ_{\mu,\nu}\right)\frac{z_\nu}{2} = 0$$

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the phases of Z_μ are locked

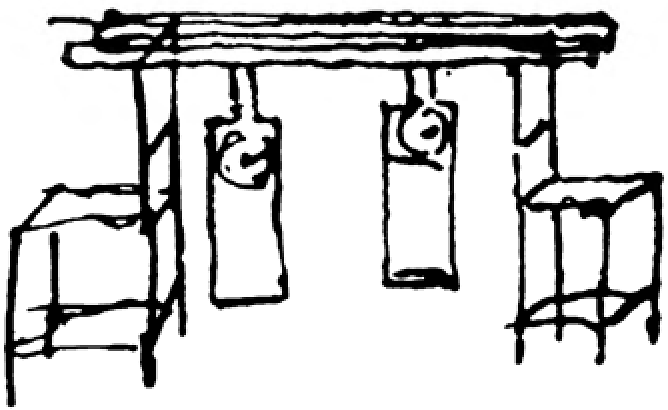
BEC $\Longleftrightarrow$ Mode locking

Mode locking: Pendulum Clock

Christiaan Huygens
(1629-1695)

1656 - patented first pendulum clock

1665 – discovered the phenomenon of synchronization



Invisible oscillations of the wooden beam

The second pendulum always had the same frequency and opposite phase as compared with the first one

Q: What about two centers of Bose condensation?

1. N>1 centers: $z \leftarrow \{z_\mu\} \equiv \vec{Z}; \quad \mu = 1, 2, \ldots, N$

Without the noise:

$$\frac{\partial z_\mu}{\partial t} + \left(g + 2i\Omega_\mu\right)\frac{z_\mu}{2} + \sum_{\mu \neq \nu}\left(\gamma_{\mu,\nu} + iJ_{\mu,\nu}\right)\frac{z_\nu}{2} = 0$$

$$\Omega_\mu = \omega_\mu + \frac{\alpha_\mu}{2}\left|z_\mu\right|^2$$

In linear limit $\alpha_\mu = 0$ the system resembles "random laser":

- many localized one-particle states, which could serve as the condensation centers for photons
- the photons choose the smallest decay rate rather than the lowest energy

Difference:

Interaction=nonlinearity allows synchronization of the different centers of condensation

N=2 condensation centers: $z_1(t),\ z_2(t)$

2x2 matrices $\gamma_{\mu\nu}$ and $J_{\mu\nu}$:

In general
$$\gamma_{\mu\nu} = \gamma_x \hat{\sigma}^{(x)} + \gamma_y \hat{\sigma}^{(y)}$$
$$J_{\mu\nu} = J_x \hat{\sigma}^{(x)} + J_y \hat{\sigma}^{(y)}$$

$\hat{\sigma}^{(i)}_{\mu\nu}$ — Pauli matrices

We assume time reflection symmetry, i.e. $\gamma_y = J_y = 0$

$$\gamma_x \equiv \gamma; \qquad J_x \equiv J$$

$z_1(t),\ z_2(t)$ 4 real variables. However the total phase is irrelevant for our discussion

Remaining 3 variables:

Occupations of the two centers and the phase difference

$$|z_1|^2,\ |z_2|^2, \quad \varphi \equiv -i \ln\left(\frac{z_1 z_2^*}{|z_1||z_2|} \right)$$

N=2 centers: Nontrivial stationary solutions

$z_{1,2} = 0$ is always a solution – trivial stationary point

At $g < \gamma$ two nontrivial solutions appear. One of them is stable (s), another – unstable (u)

N=2 centers: $z_1(t),\ z_2(t)$

2x2 matrices $\gamma_{\mu\nu}$ and $J_{\mu\nu}$:

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$z_1(t),\ z_2(t)$ 4 real variables. However the total phase is irrelevant for our discussion

Remaining 3 variables – pseudo spin:

$$S_i(t) \equiv \sum_{\mu,\nu} \hat{\sigma}^{(i)}_{\mu\nu} z^*_\mu(t) z_\nu(t)$$

N=2 centers:

Pseudo spin

$$S_i(t) \equiv \sum_{\mu,\nu} \hat{\sigma}^{(i)}_{\mu\nu} z^*_\mu(t) z_\nu(t)$$

In components:

$$S_x = \tfrac{1}{2}\left(z_1^* z_2 + z_2^* z_1\right)$$
$$S_y = \tfrac{1}{2}\left(z_1^* z_2 - z_2^* z_1\right)$$
$$S_z = \tfrac{1}{2}\left(|z_2|^2 - |z_1|^2\right)$$

$$S^2 = \tfrac{1}{4}\left(|z_2|^2 + |z_1|^2\right)^2$$

Equations:

$$\dot{S}_x = -gS_x - \gamma S - (\omega + \alpha_+ S_z - \alpha_- S) S_y$$
$$\dot{S}_y = -gS_y + JS_z + (\omega + \alpha_+ S_z - \alpha_- S) S_x$$
$$\dot{S}_z = -gS_z - JS_y$$

$$\dot{S} = -gS - \gamma S_x$$

$$\omega \equiv \omega_1 - \omega_2$$
$$\alpha_\pm \equiv \tfrac{1}{2}(\alpha_1 \pm \alpha_2)$$

N=2 centers: Nontrivial stationary points

Stationary points:

$$-gS_x - \gamma S - (\omega + \alpha_+ S_z - \alpha\_S)S_y = 0$$
$$-gS_y + JS_z + (\omega + \alpha_+ S_z - \alpha\_S)S_x = 0$$
$$-gS_z - JS_y = 0$$

$S = 0$ is always a solution - trivial stationary point

At $g < \gamma$ two more solutions appear - nontrivial stationary points provided that $S > 0$
In polar coordinates they are

$$\varphi = \pi - \arctan r; \quad \cos\vartheta = -\frac{Jr}{\gamma}$$
$$S = -\frac{\gamma}{g}\frac{g\omega + (J^2 + g^2)r}{\alpha\_\gamma + \alpha_+ Jr}$$

$$r \equiv \pm\sqrt{\frac{\gamma^2 - g^2}{J^2 + g^2}}$$

! $\varphi \neq \pi n$!

N=2 centers: Nontrivial stationary points

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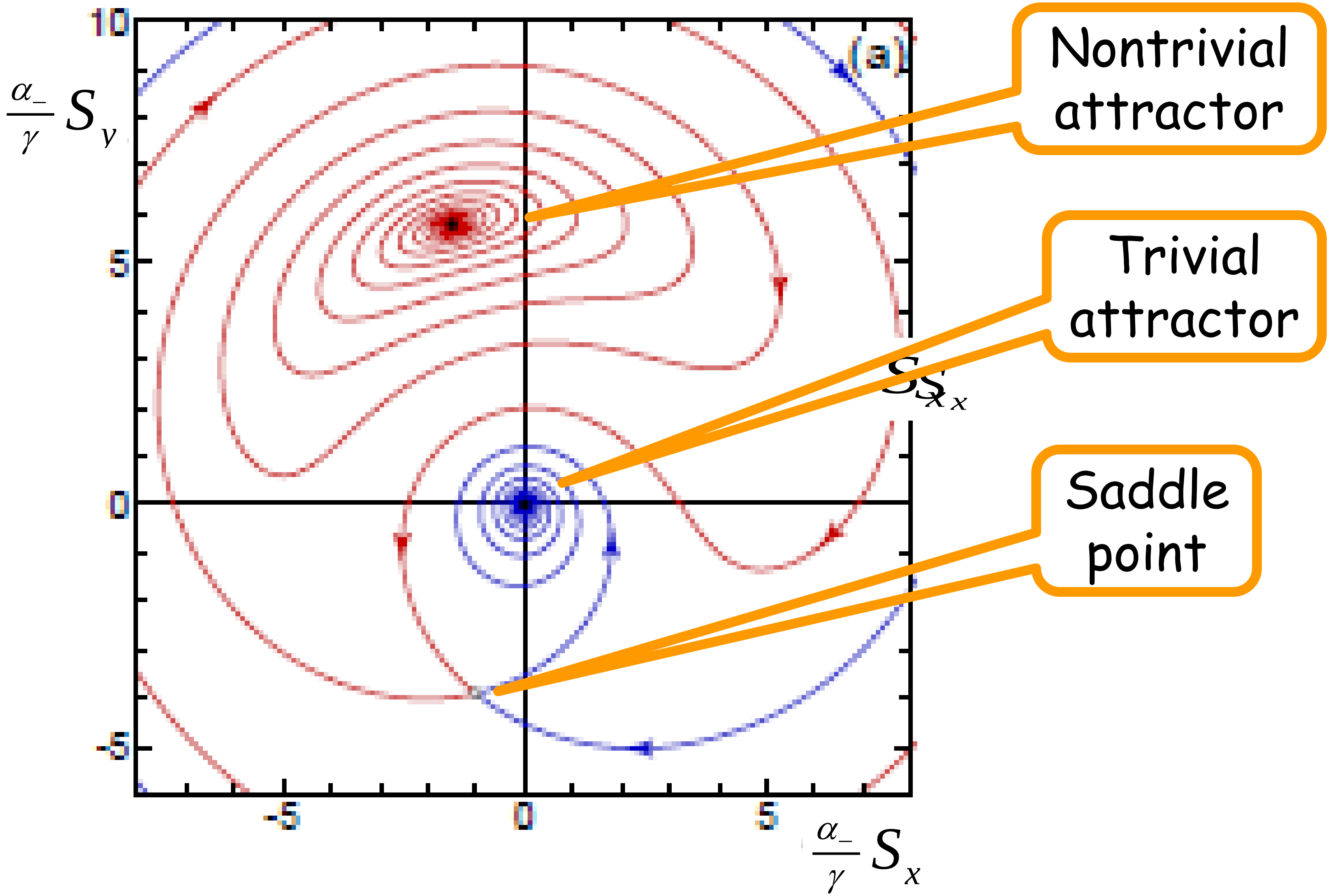
In polar coordinates they are

$$\boxed{\varphi=\pi-\arctan r;\quad \cos\vartheta=-\frac{Jr}{\gamma} \qquad S=-\frac{\gamma}{g}\frac{g\omega+\left(J^2+g^2\right)r}{\alpha_-\gamma+\alpha_+Jr}}$$

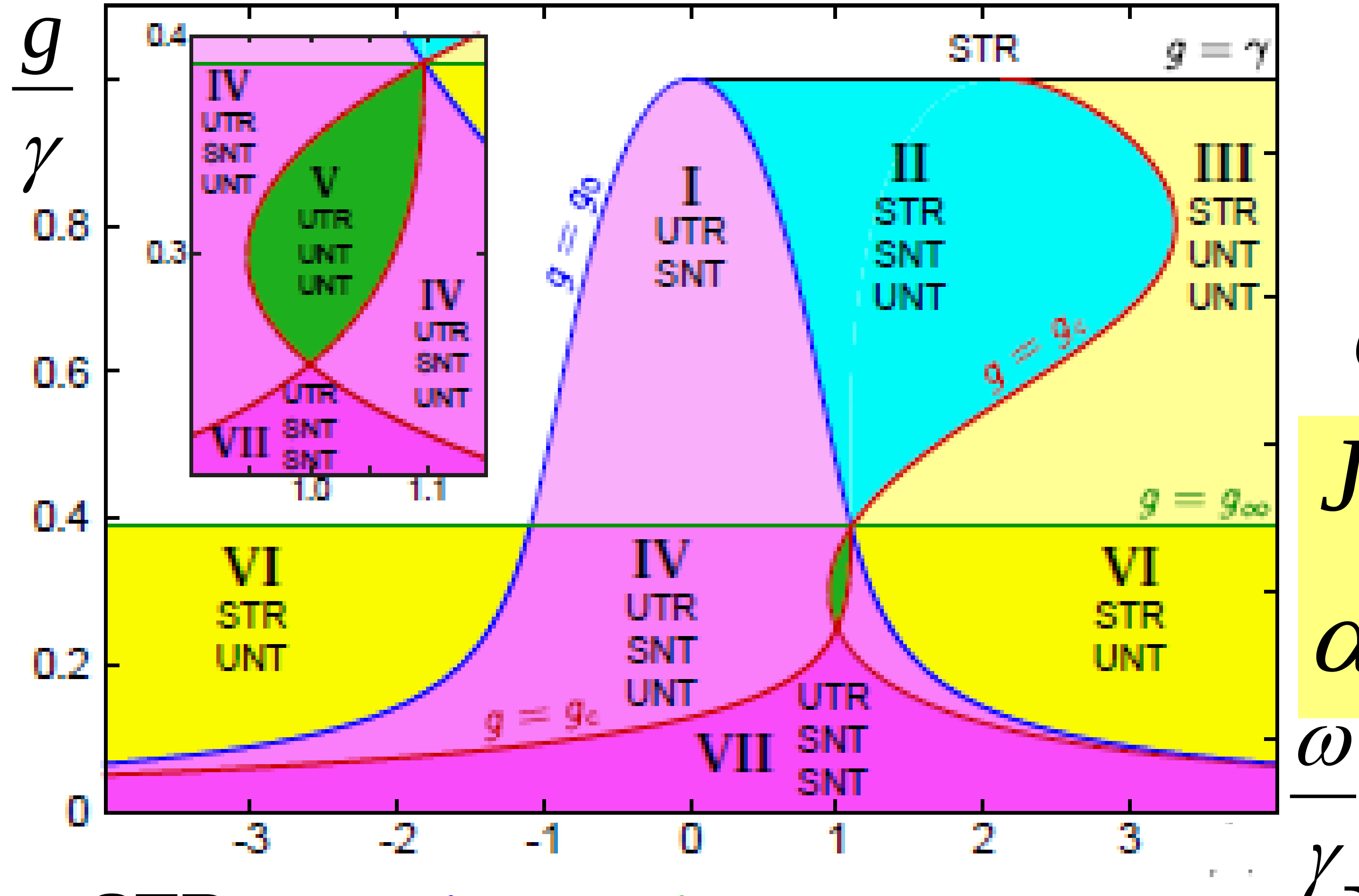
$$r\equiv\pm\sqrt{\frac{\gamma^2-g^2}{J^2+g^2}}$$

Note: $\varphi\neq 0,\pi \Longrightarrow$ time reversal symmetry breaking, which translates into the breaking of the $\vec{k}_\perp\to-\vec{k}_\perp$ symmetry



$J = 0; \quad \omega = 5\gamma; \quad g = 0.25\gamma$

Stability diagram



g Distance from the "threshold"

γ dissipative coupling

ω original energy mismatch

$J = 0.25\gamma$

$\alpha_- = 0.5\alpha_+$

STR – stable trivial stationary point

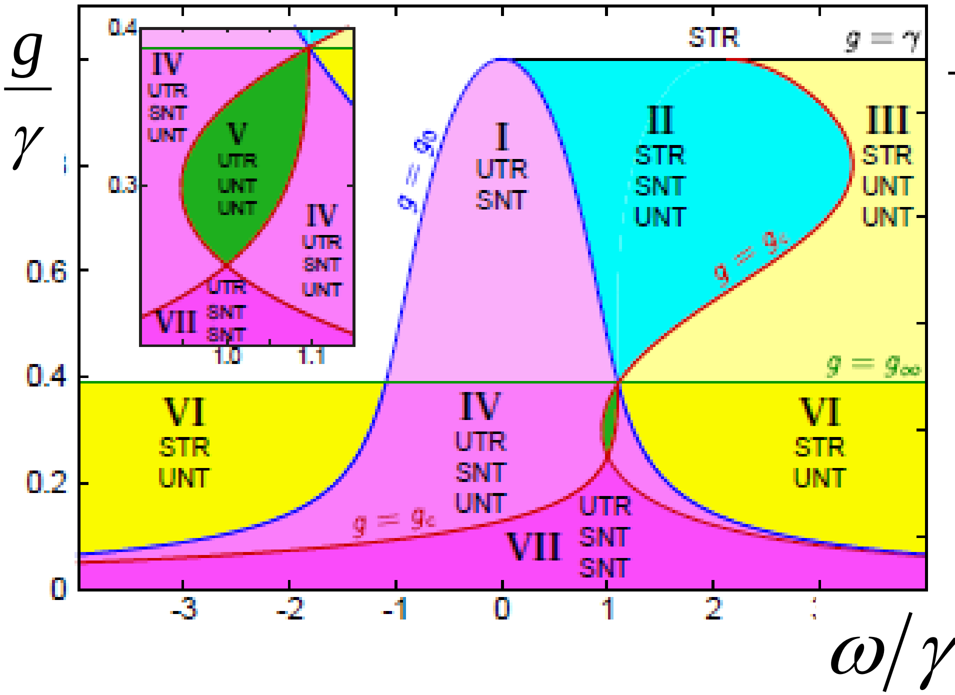
UTR – unstable trivial stationary point

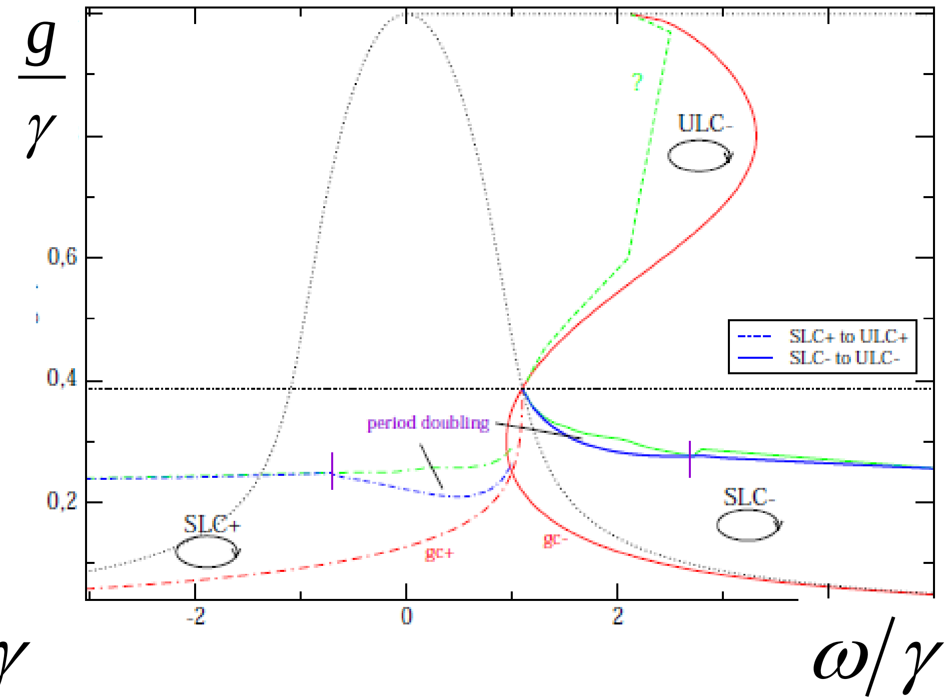
$\Big\}\ \vec{S} = 0$

SNT – stable nontrivial stationary point

UNT – unstable nontrivial stationary point

$\Big\}\ \vec{S} \neq 0$





SLC – stable limiting cycle ULC – unstable limiting cycle

Nonlinear Zoo:

- Hopf bifurcations – subcritical and supercritical
- Fold bifurcations
- Period doubling instabilities
- . . .

Thanks to Sergej Flach and Kristian Rayanov, MPIPKS, Dresden

Nonlinear Zoo:

- Hopf bifurcations – subcritical and supercritical
- Fold bifurcations
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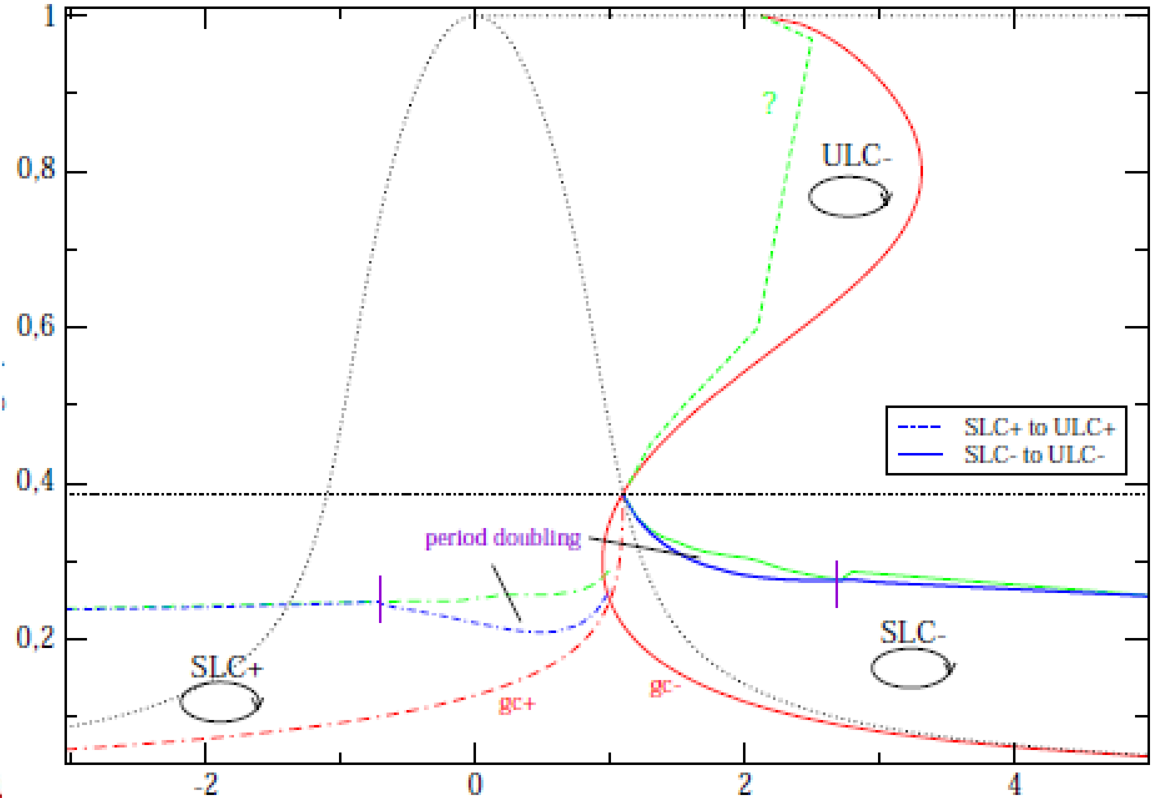
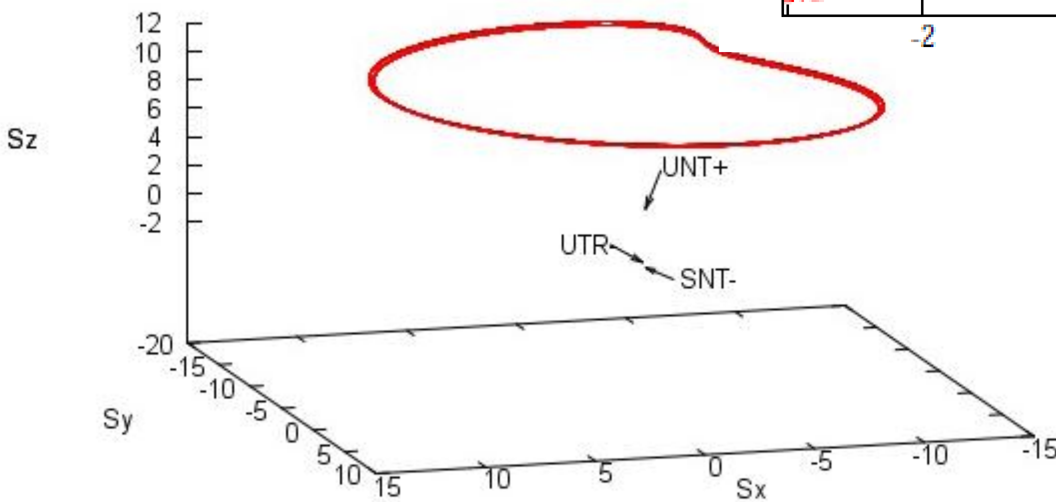
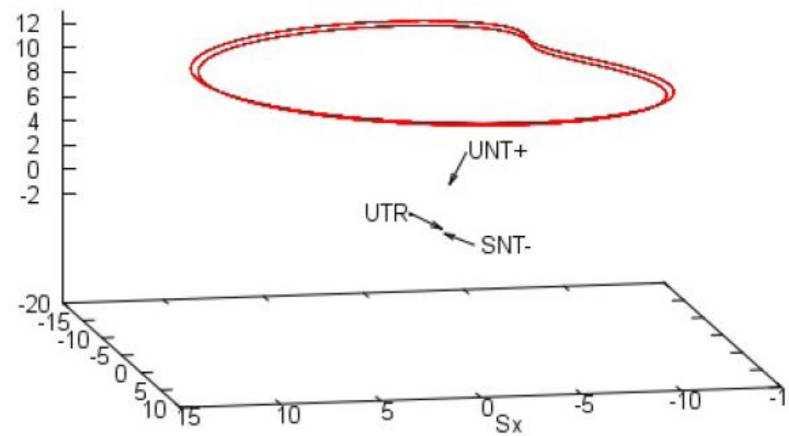




$\omega/\gamma = 0.3 \quad g/\gamma = 0.233$



$\omega/\gamma = 0.3 \quad g/\gamma = 0.235$

N=2 centers: Nontrivial stationary points

$S=0$ is always a solution – trivial stationary point

At $g<\gamma$ more solutions - nontrivial stationary points.

Identical Condensation Centers

$$\omega=0 \quad \alpha_-=0 \quad \alpha_+=\alpha$$

$$\varphi=\pi\pm\arctan\left(\sqrt{\frac{\gamma^2-g^2}{J^2+g^2}}\right);$$

$$S=\frac{\gamma}{g}\frac{\left(J^2+g^2\right)}{\alpha|J|}$$

Stable provided that

$$g<J<\gamma$$

$$|\varphi-\pi|\le\frac{\pi}{2}$$

$$J<g<\gamma$$ ⟹ Limiting Cycle

Radiation in the stable states

1. Nontrivial stationary point – single line

2. Limiting cycle – sequence of the lines

3. Trivial stationary state – two lines

4. Line-shapes and photon statistics are determined by the noise

5. Switching between different stable states due to the noise – Kramers problem. Coexistence of the signals.

6. Phase locking ⇨ time-reversal symmetry violation ⇨ $\vec{k}_{||} \rightarrow -\vec{k}_{||}$ asymmetry

$N=2n$ identical condensation centers

For nearest neighbor couplings $g < J < \gamma$

$$\varphi = \pi \pm \arctan\left(\sqrt{\frac{\gamma^2 - g^2}{J^2 + g^2}}\right);$$

$$S = \frac{\gamma}{g}\frac{\left(J^2 + g^2\right)}{\alpha|J|}$$

n such pairs!

Symmetry breaking
Period doubling
Close to - π state

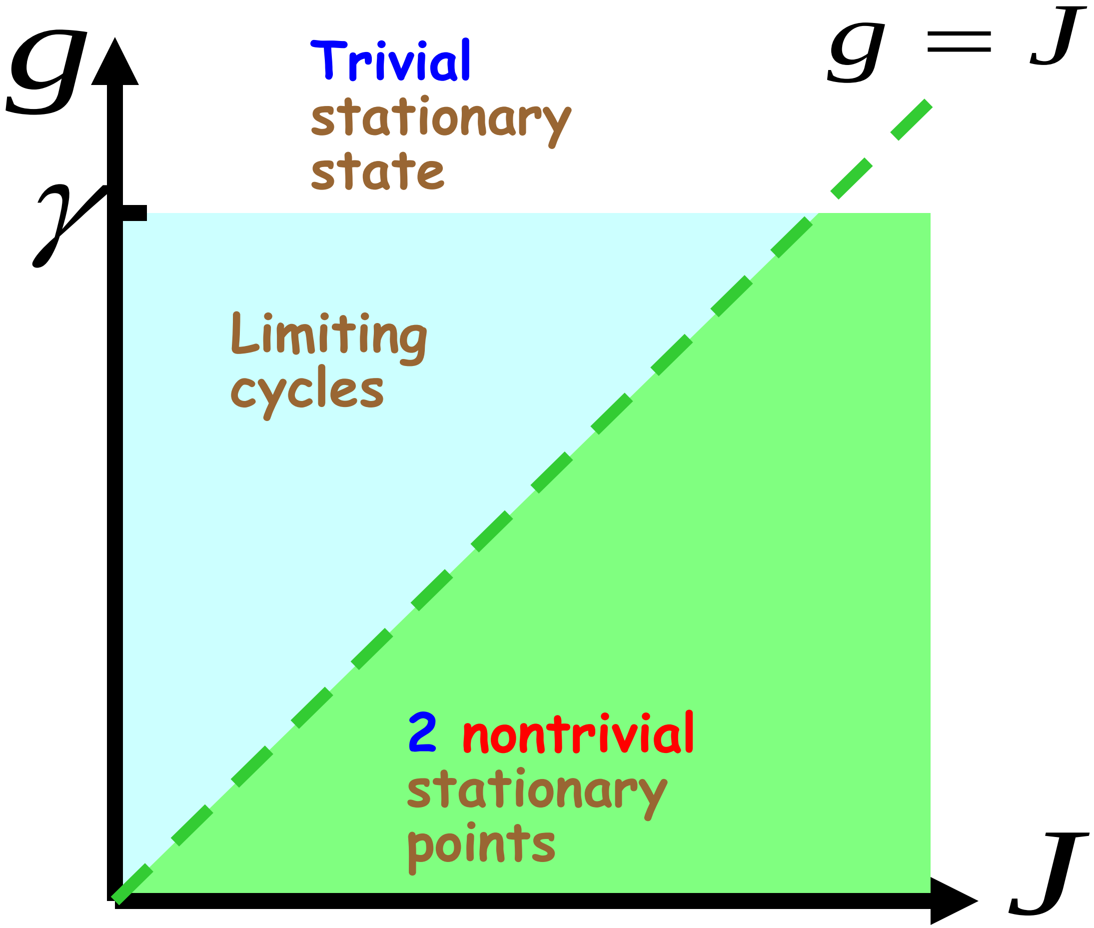
Ferromagnetic coupling + dissipative coupling

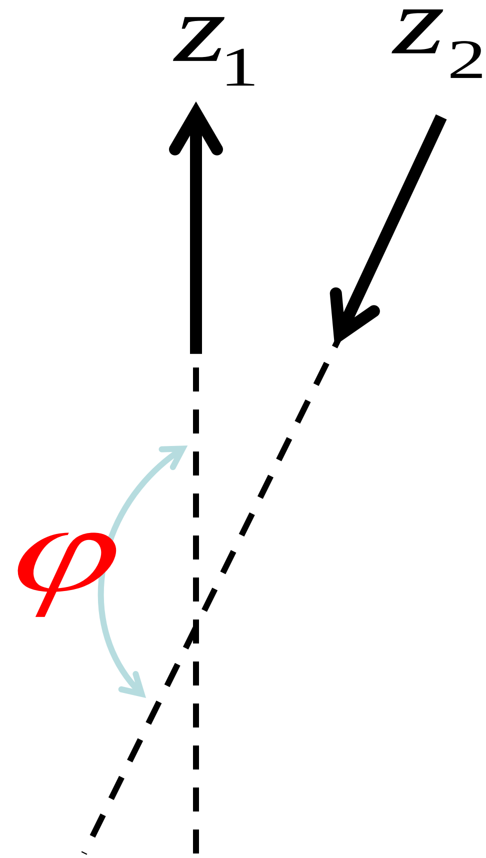
⇓

Almost "antiferromagnetic" state

Stable non-stationary states $J < g < \gamma$?

2 identical condensation centers:





$$\varphi = \pi \pm \arctan\left(\sqrt{\frac{\gamma^2 - g^2}{J^2 + g^2}}\right)$$

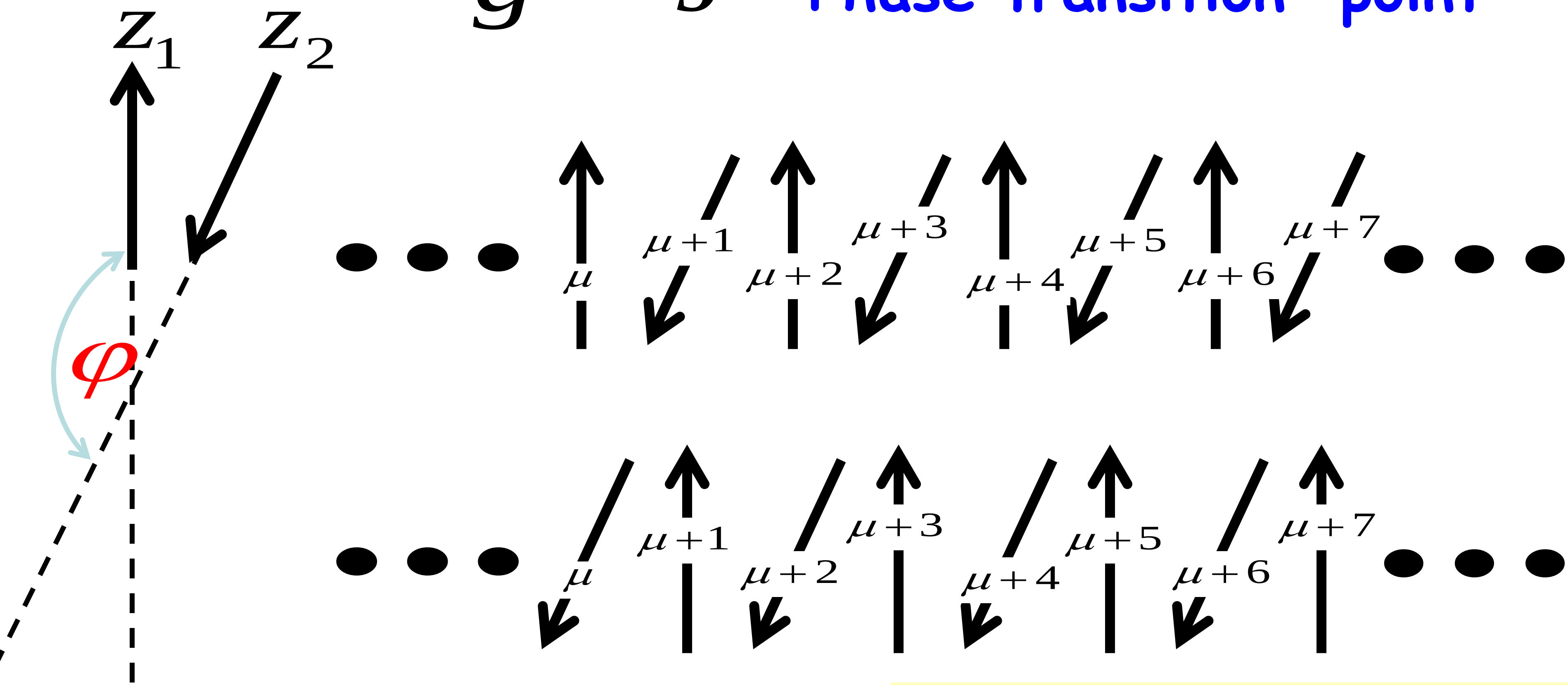
2 nontrivial
PT-symmetric
stationary
states

$$P : z_1 \leftrightarrow z_2$$

$$T : t \to -t$$

2N identical condensation centers; $g < J < \gamma$

$g = J$ "Phase transition" point



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Conclusions

- Driven system selects the most stable state rather than the state with the lowest energy
- Nontrivial stable states. System stabilizes itself without adjusting the reservoir. Weak lasing.
- Existing experiments can be naturally interpreted.
- In particular – natural explanation of the violation of the T-invariance
- Equations of motion: Condensation centers = coupled nonlinear oscillators
- Periodic structures: phase locking – driven states of matter. Dissipative coupling is crucial