Extreme ocean environments

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Acknowledgement and overview

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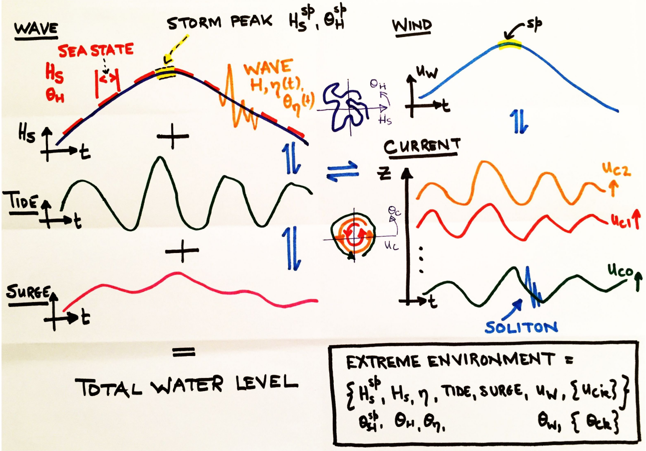
- UK Metoffice : Rob Shooter

Overview

- Motivation
- Marginal extremes
- Multivariate conditional extremes

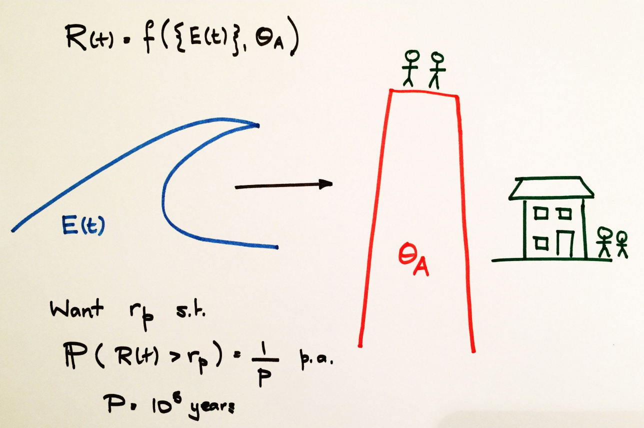
Motivation

Modelling ocean storm environment



- Multiple coupled physical processes
- Rare, extreme events

Modelling structural risk



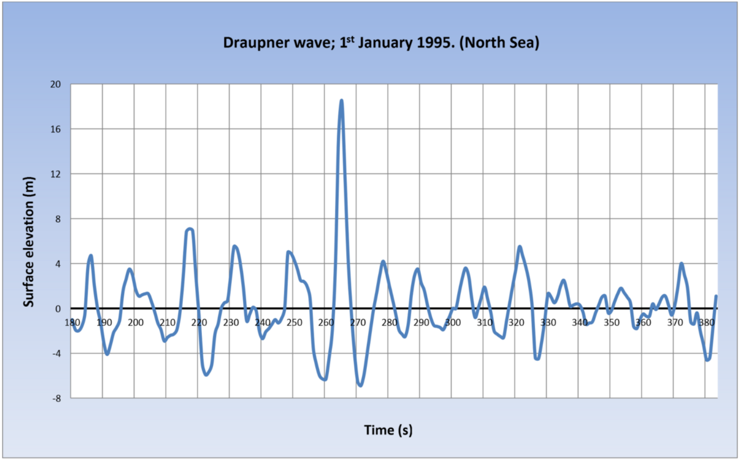
- Ocean environment is harsh
- Marine structures at risk of failure
- Reliability standards must be met

Spectacular scale

Offshore Portugal, 24m wave height, November 2017 (The Guardian)

- Nazaré is a great source of huge coastal waves

Spectacular scale



Laser readings, 1 January 1995. Wave 25.6m, crest 18.5m (Statoil / Equinor)

- Maximum recorded wave height > 30m (multiple events, various sources)
- Maximum recorded significant wave height : 19.0m (buoy, North Atlantic, 4 Feb 2013, WMO)

Wave-impact damage

Norwegian Dream, Atlantic, 2007
(gcaptain.com)

Ike, Gulf of Mexico, 2008
(Joe Richard)

Optimal design

Set-up

- A marine system with "strength" specifications $\boldsymbol{\mathcal{S}}$
- An ocean environment $\boldsymbol{X}$ dependent on covariates $\boldsymbol{\Theta}$
- A structural "loading" $\boldsymbol{Y}$ as a result of environment X and covariates $\boldsymbol{\Theta}$
- System utility (or risk) $U(\boldsymbol{Y}|\boldsymbol{\mathcal{S}})$ for loading $\boldsymbol{Y}$ and specification $\boldsymbol{\mathcal{S}}$
- Desired U typically specified in terms of annual probability of failure
- $\boldsymbol{Y}|\boldsymbol{X}, \boldsymbol{\Theta}$ and $\boldsymbol{X}|\boldsymbol{\Theta}$ (and U?) subject to uncertainty $\boldsymbol{Z}$
- $\boldsymbol{Z}, \boldsymbol{\Theta}, \boldsymbol{X}, \boldsymbol{Y}$ are multidimensional random variables

Optimal design

- Estimate a model $f_{\boldsymbol{X}|\boldsymbol{\Theta},\boldsymbol{Z}}$ for the environment
- Estimate a model $f_{\boldsymbol{Y}|\boldsymbol{X},\boldsymbol{\Theta},\boldsymbol{Z}}$ for environment-structure interaction
- Estimate a model $f_{\boldsymbol{\Theta}|\boldsymbol{Z}}$ for the covariates

$$\mathbb{E}[U|\boldsymbol{\mathcal{S}}] = \int_z \int_y \int_x \int_\theta U(\boldsymbol{y}|\boldsymbol{\mathcal{S}}, \boldsymbol{Z}) f_{\boldsymbol{Y}|\boldsymbol{X},\boldsymbol{\Theta},\boldsymbol{Z}}(\boldsymbol{y}|\boldsymbol{x}, \boldsymbol{\theta}, \boldsymbol{z}) f_{\boldsymbol{X}|\boldsymbol{\Theta},\boldsymbol{Z}}(\boldsymbol{x}|\boldsymbol{\theta}, \boldsymbol{z}) f_{\boldsymbol{\Theta}|\boldsymbol{Z}}(\boldsymbol{\theta}|\boldsymbol{z}) \, d\boldsymbol{\theta} \, d\boldsymbol{x} \, d\boldsymbol{y} \, d\boldsymbol{z}$$

$\Rightarrow$ solve for $\boldsymbol{\mathcal{S}}$ to achieve required (safety) utility

Return values : conventional engineering practice

- Estimating $\mathbb{E}[U|\boldsymbol{\mathcal{S}}]$ is difficult
- Design to extreme quantile of marginal annual distribution of one X instead

$$F_A(x) = \int_{\boldsymbol{Z}} \int_{\boldsymbol{\theta}} \int_k F_{X|\boldsymbol{\Theta},\boldsymbol{Z}}(x|\boldsymbol{\theta},\boldsymbol{Z}) f_{C|\boldsymbol{\Theta},\boldsymbol{Z}}(k|\boldsymbol{\theta},\boldsymbol{z}) f_{\boldsymbol{\Theta}|\boldsymbol{Z}}(\boldsymbol{\theta}|\boldsymbol{z})\, dk\, d\boldsymbol{\theta}\, d\boldsymbol{z}$$

where $f_{C|\boldsymbol{\Theta},\boldsymbol{Z}}$ is the annual rate of occurrence of events given covariate Θ.
- Set the return value x_T (for $T = 1000$ years say) such that

$$F_A(x_T) = 1 - \frac{1}{T}$$

- Specify conditional return values for other Xs given $X = x_T$
- Potentially as a function of covariates
- Ambiguous ordering of expectation operators ... a can of worms!

A model for the (non-stationary multivariate extreme) environment

- Expected utility and return values are dominated by extreme environments
- Have to estimate tails of distributions well

- Focus on a simple $\boldsymbol{Z}$-free 2-D environment with stationary dependence

$$F_{\boldsymbol{X}|\boldsymbol{\Theta},\boldsymbol{Z}}(\boldsymbol{x}|\boldsymbol{\theta},\boldsymbol{z}) = C\Big(F_{X_1|\Theta}(x_1|\theta), F_{X_2|\Theta}(x_2|\theta)\Big) \text{ for simplicity, so}$$

$$\begin{aligned} f_{\boldsymbol{X}|\boldsymbol{\Theta},\boldsymbol{Z}}(\boldsymbol{x}|\boldsymbol{\theta},\boldsymbol{z}) &= f_{X_1,X_2|\Theta}(x|\theta) \\ &= f_{X_1|\Theta}(x_1|\theta) f_{X_2|\Theta}(x_2|\theta) \times c\Big(F_{X_1|\Theta}(x_1|\theta), F_{X_2|\Theta}(x_2|\theta)\Big) \text{ typically} \end{aligned}$$

- Marginal models (non-stationary, extreme) $f_{X_1|\Theta}(x_1|\theta)$, $f_{X_2|\Theta}(x_2|\theta)$
- Multivariate model on standard marginal scale (stationary, “extreme”) $c(u_1, u_2)$

Marginal extremes

- Theory : Beirlant et al. [2004]
- Method : Dey and Yan [2016]

Generalised extreme value distribution

- F_X^n is the distribution of the maximum of n independent draws of X
- If F_X^n "looks like" $F_X^{n'}$, we say F_X is max-stable

- More formally, F_X is max-stable if there exist sequences of constants a_n, b_n, and non-degenerate G_ξ such that

$$\lim_{n\to\infty} F_X^n\left(a_n x + b_n\right) = G_\xi(x)$$

- We say $F_X \in D(G_\xi)$ or that F_X lies in the max-domain of attraction of G_ξ
- The Fisher–Tippett–Gnedenko theorem states that G_ξ is the generalised extreme value distribution with parameter ξ

$$G_\xi(y) = \exp\left(-\left(1+\xi y\right)^{-1/\xi}\right), \quad \xi \in \mathbb{R}$$

- For sufficiently large n, it makes sense to model block maxima of n independent identically-distributed draws of X using G_ξ (with $(x-\mu)/\sigma$ in place of y above)

Generalised Pareto distribution

- Now suppose we have an exceedance X of high threshold ψ
- The Pickands-Balkema-De Haan theorem states

$$\begin{aligned}\lim_{\psi\to\infty} \mathbb{P}[X \leq x | X > \psi] &= \lim_{\psi\to\infty} \frac{F_X(x)}{1-F_X(\psi)} \\ &= \mathrm{GP}(x|\xi,\sigma,\psi) \\ &= 1-\left(1+\frac{\xi}{\sigma}(x-\psi)\right)_+^{-1/\xi}, \quad \xi \in \mathbb{R}\end{aligned}$$

Theory

- Derived from max-stability of F_X
- Threshold-stability property
- "Poisson $\times$ GP $=$ GEV"

Practicalities

- How to isolate independent threshold exceedances from observed time-series?
- How to specify extreme threshold ψ?
- ξ, σ, ψ functions of covariates
- Davison and Smith [1990]

Marginal extremes in practice

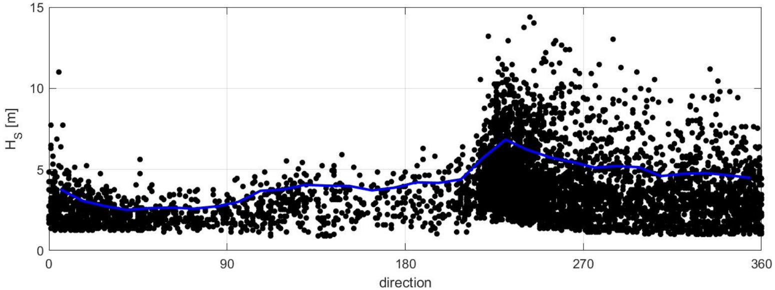
- Motivation : Chavez-Demoulin and Davison [2005]

- Practicalities : Jonathan and Ewans [2013], Feld et al. [2019]
- Semi-parametric : Randell et al. [2016], Zanini et al. [2020]
- ... lots more

- Non-stationary marginal extremes

Motivation

- Environmental extremes vary smoothly with multidimensional covariates
- Generic modelling framework for different covariate representations
- Statistical and computational efficiency for n-D covariates
- Full (Bayesian) uncertainty quantification



Typical data for northern North Sea. Storm peak H_S on direction, with $\tau = 0.8$ extreme value threshold.
Rate and size of occurrence varies with direction.

Model for size of occurrence

- Sample of storm peaks Y over threshold ψ_θ, with 1-D covariate θ
- Extreme value threshold ψ_θ assumed known
- Y assumed to follow generalised Pareto distribution with shape ξ_θ, (modified) scale ν_θ

$$f_{GP}(y|\xi_\theta, \nu_\theta) = \frac{1}{\sigma_\theta}\left(1 + \frac{\xi_\theta}{\sigma_\theta}(y - \psi_\theta)\right)_+^{-1/\xi_\theta - 1}$$

- $\nu_\theta = \sigma_\theta(1 + \xi_\theta)$
- $y > \psi_\theta$, $\psi_\theta \in \mathbb{R}$
- Shape parameter $\xi_\theta \in \mathbb{R}$ and scale parameter $\nu_\theta > 0$

- Non-stationary Poisson model for rate of occurrence, with rate $\rho_\theta \geq 0$

Covariate representations

- Index set $\mathcal{I}_\theta = \{\theta_s\}_{s=1}^{m}$ on periodic covariate domain $\mathcal{D}_\theta$
- Each observation belongs to exactly one θ_s
- On $\mathcal{I}_\theta$, assume

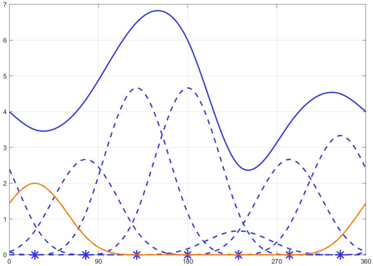
$$\begin{aligned} \eta_s &= \sum_{k=1}^{n} B_{sk}\beta_k, s = 1, 2, ..., m, \text{ or} \\ \boldsymbol{\eta} &= \boldsymbol{B}\boldsymbol{\beta} \text{ in vector terms} \end{aligned}$$

- $\eta \in (\xi, \nu)$ (and similar for ρ)
- $\boldsymbol{B} = \{B_{sk}\}_{s=1;k=1}^{m;n}$ basis for $\mathcal{D}_\theta$
- $\boldsymbol{\beta} = \{\beta_k\}_{k=1}^{n}$ basis coefficients
- Inference reduces to estimating n_ξ, n_ν, $\boldsymbol{B}_\xi$, $\boldsymbol{B}_\nu$, $\boldsymbol{\beta}_\xi$, $\boldsymbol{\beta}_\nu$ (and roughnesses λ_ξ, λ_ν)
- P-splines, BARS and Voronoi are different forms of $\boldsymbol{B}$

P-splines

- n regularly-spaced knots on $\mathcal{D}_\theta$
- $\boldsymbol{B}$ consists of n B-spline bases
  - Order d
  - Each using $d+1$ consecutive knot locations
  - Local support
  - Wrapped on $\mathcal{D}_\theta$
  - Cox - de Boor recursion formula

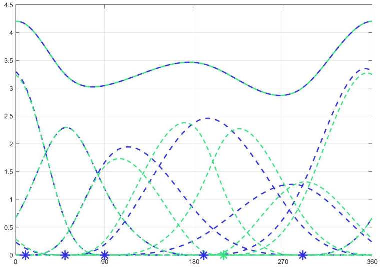
- n is fixed and "over-specified"
- Knot locations $\{r_k\}_{k=1}^{n}$ fixed
- Local roughness λ of $\boldsymbol{\beta}$ penalised



Periodic P-splines

BARS basis

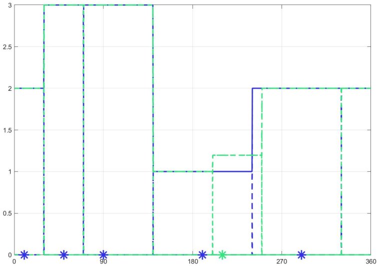
- n irregularly-spaced knots on $\mathcal{D}_\theta$
- $\boldsymbol{B}$ consists of n B-spline bases
- Knot locations $\{r_k\}_{k=1}^{n}$ can change
- Number of knots n can change



Periodic BARS knot birth and death

Voronoi partition

- n irregularly-spaced centroids on $\mathcal{D}_\theta$
  - Define n neighbourhoods or "cells"
- $\boldsymbol{B}$ consists of n basis functions
  - Piecewise constant on $\mathcal{D}_\theta$
  - $= 1$ "within cell", $= 0$ "outside"
- Centroid locations $\{r_k\}_{k=1}^n$ can change
- Number of centroids n can change
- Trivial extension to n-D



Periodic Voronoi centroid birth and death

Prior for $\boldsymbol{\beta}$ (all representations)

$$\text{prior density of } \boldsymbol{\beta} \quad \propto \quad \exp\left(-\frac{1}{2}\boldsymbol{\beta}'\boldsymbol{P}\boldsymbol{\beta}\right)$$

- $\boldsymbol{P} = \lambda \boldsymbol{D}'\boldsymbol{D}$, $\boldsymbol{D}$ is a $n \times n$ (wrapped) differencing matrix
- P-splines: $\boldsymbol{D}$ represents first-difference; prior equivalent to local roughness penalty
- BARS and Voronoi: $\boldsymbol{D}$ is $\boldsymbol{I}_n$; prior is "ridge-type" for Bayesian regression

Prior for λ (all representations)

$$\lambda \quad \sim \quad \text{gamma}$$

Prior for n (BARS and Voronoi)

$$n \quad \sim \quad \text{Poisson}$$

Prior for $r_k, k = 1, 2, ..., n$ (BARS and Voronoi)

$$r_k \quad \sim \quad \text{uniform}$$

Inference for GP

Parameter set Ω

- P-splines: $\Omega = \{\boldsymbol{\beta}_\xi, \lambda_\xi, \boldsymbol{\beta}_\nu, \lambda_\nu\}$ with $n_\xi, \boldsymbol{r}_\xi, n_\nu$ and $\boldsymbol{r}_\nu$ pre-specified
- BARS and Voronoi: $\Omega = \{n_\xi, \boldsymbol{r}_\xi, \boldsymbol{\beta}_\xi, \lambda_\xi, n_\nu, \boldsymbol{r}_\nu, \boldsymbol{\beta}_\nu, \lambda_\nu\}$
- $\boldsymbol{r} = \{r_k\}_{k=1}^n$, $\boldsymbol{\beta} = \{\beta_k\}_{k=1}^n$

Inference

- Gibbs sampling when full conditionals available
- Otherwise Metropolis-Hastings (MH) within Gibbs, using suitable proposal mechanisms, mMALA where possible
- Reversible-jump for $n, \boldsymbol{r}$ (satisfy dimension-jumping detailed balance)

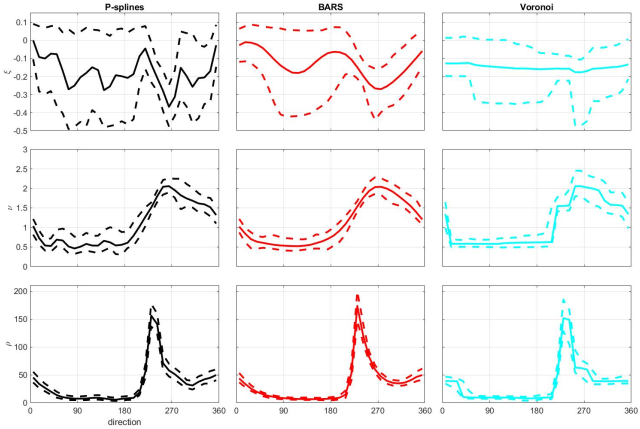
Basic conditional structure for non-dimension-jumping

$$\begin{aligned}
f(\boldsymbol{\beta}_\eta|\boldsymbol{y}, \Omega \setminus \boldsymbol{\beta}_\eta) &\propto f(\boldsymbol{y}|\boldsymbol{\beta}_\eta, \Omega \setminus \boldsymbol{\beta}_\eta) \times f(\boldsymbol{\beta}_\eta|\boldsymbol{\lambda}_\eta) \\
f(\lambda_\eta|\boldsymbol{y}, \Omega \setminus \lambda_\eta) &\propto f(\boldsymbol{\beta}_\eta|\lambda_\eta) \times f(\lambda_\eta) \\
f(\boldsymbol{r}_\eta|\boldsymbol{y}, \Omega \setminus \boldsymbol{r}_\eta) &\propto f(\boldsymbol{y}|\boldsymbol{r}_\eta, \Omega \setminus \boldsymbol{r}_\eta) \times f(\boldsymbol{r}_\eta),
\end{aligned}$$

- $\eta \in (\xi, \nu)$ (and ρ)

Posterior parameter estimates for ξ, ν and ρ for northern North Sea

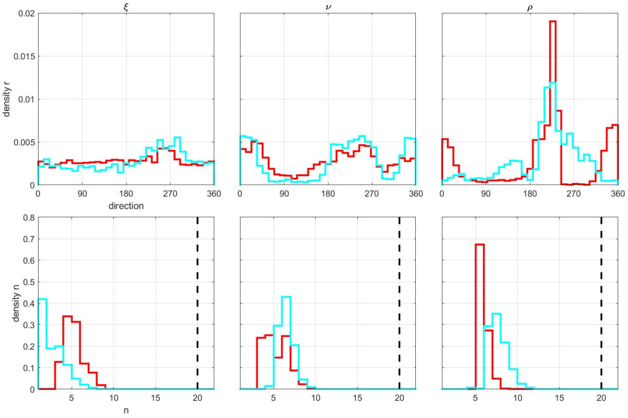
- Note colour scheme
- Rate ρ and ν very similar
- Voronoi gives almost constant ξ
- Voronoi piecewise constant
- Land shadow effects
- General agreement



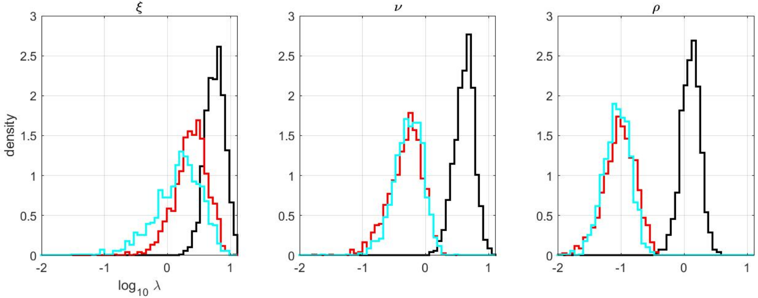
Posterior densities for locations r and numbers n

- Prior uniform knot placement for $\boldsymbol{r}$
- Knot placement uniform for ξ, clear effect for ρ

- n close to 1 for Voronoi ξ
- General agreement
- Effect of different priors on n checked



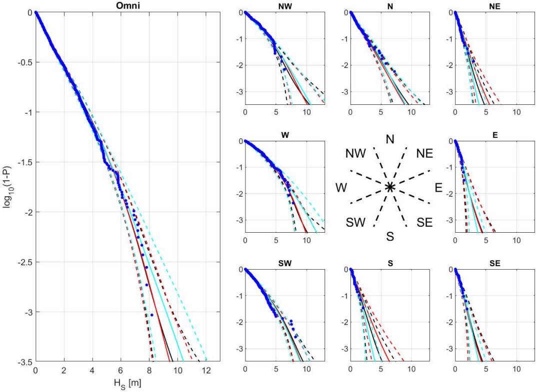
Posterior densities for penalty coefficients λ



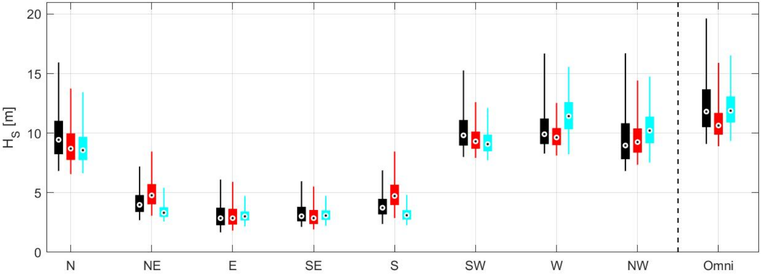
- Prior density is gamma(1,1) ($f(x) \propto \exp(-x)$, $x \geq 0$)
- Ridge penalties for BARS and Voronoi, but roughness for P-splines
- λ somewhat lower for Voronoi, but also this has smaller n
- General consistency

Fit diagnostic

- Empirical tail (blue)
- Posterior means and 95% credible intervals for quantile levels from different models
- General consistency



Directional posterior predictive distribution of $T = 1000$-year maximum



- Box-whiskers with 2.5%, 25%, 50%, 75% and 97.5% percentiles
- Uncertainties larger for P-splines?
- General consistency

- This is more-or-less what the engineer needs to design a “compliant” structure

Multivariate extremes

- Theory : Beirlant et al. [2004]
- Copulas : Joe [2014]
- Method : Dey and Yan [2016]

Multivariate extreme value distribution, MEVD

- $\boldsymbol{X}_i = (X_{i1}, ..., X_{ij}, ..., X_{ip}), i = 1, ..., n$ iid p-vectors, distribution F
- $M_{n,j} = \max_i X_{ij}$, component-wise maximum
- Then for $Z_{n,j} = (M_{n,j} - b_{n,j})/a_{n,j}$, normalised with scaling constants:

$$\mathbb{P}(\boldsymbol{Z} \leq \boldsymbol{z}) = F^n(\boldsymbol{a}_n \boldsymbol{z} + \boldsymbol{b}_n) \to G(\boldsymbol{z}) \quad \text{as} \quad n \to \infty$$

- Non-degenerate $G(\boldsymbol{z})$ must be max-stable, so $\forall k \in \mathbb{N}$, $\exists\ \boldsymbol{\alpha_k} > \boldsymbol{0}, \boldsymbol{\beta_k}$ s.t.

$$G^k(\boldsymbol{\alpha}_k \boldsymbol{z} + \boldsymbol{\beta}_k) = G(\boldsymbol{z})$$

- We say $F \in D(G)$
- Margins $G_1, ..., G_p$ are unique GEV, but $G(\boldsymbol{z})$ is not unique

- The component-wise maximum is not "observed" (especially as $n \to \infty$)

MEVD on common margins

- On uniform margins, we have extreme value copula: $C(\boldsymbol{u}) = C^k(\boldsymbol{u}^{1/k})$
- On standard Fréchet margins ($G_j(z) = \exp\left(-z^{-1}\right)$), with pseudo-polars $(r, \boldsymbol{w})$

$$
\begin{aligned}
G(\boldsymbol{z}) &= \exp\left(-V(\boldsymbol{z})\right), \quad \text{for exponent measure } V \\
\text{with } V(\boldsymbol{z}) &= \int_\Delta \max_j\{\frac{w_j}{z_j}\}\, S(d\boldsymbol{w}), \quad \text{on } \Delta = \{\boldsymbol{w} \in \mathbb{R}^p : ||\boldsymbol{w}|| = 1\} \\
\text{and } 1 &= \int_\Delta w_j\, S(d\boldsymbol{w}), \quad \forall j, \text{ for angular measure } S
\end{aligned}
$$

- Max-stability : $V(r\boldsymbol{z}) = r^{-1}V(\boldsymbol{z})$, homogeneity order -1
- Rich spatial extensions to max-stable processes, MSPs
- Multivariate generalised Pareto distribution, MGPD
- Condition of multivariate regular variation, MRV

$$
\frac{1 - F(t\boldsymbol{x})}{1 - F(t\mathbf{1})} \to \lambda(\boldsymbol{x}) \text{ as } t \to \infty, \boldsymbol{x} \in \mathbb{R}^p
$$

useful to prove that $F \in D(G)$ for some MEVD G

Extremal dependence (2D, uniform margins)

$$\chi(u) = \frac{\mathbb{P}(U > u, V > u)}{\mathbb{P}(U > u)} = \frac{\bar{C}(u,u)}{1-u} \quad \to \chi \text{ as } u \to 1$$

- $\chi = 1$ perfect dependence
- $\chi \in (0,1)$ asymptotic dependence, AD
- $\chi = 0$ perfect independence

$$\bar{\chi}(u) = 2\frac{\log \mathbb{P}(U > u)}{\log \mathbb{P}(U > u, V > u)} - 1 = 2\frac{\log(1-u)}{\log \bar{C}(u,u)} - 1 \quad \to \bar{\chi} \text{ as } u \to 1$$

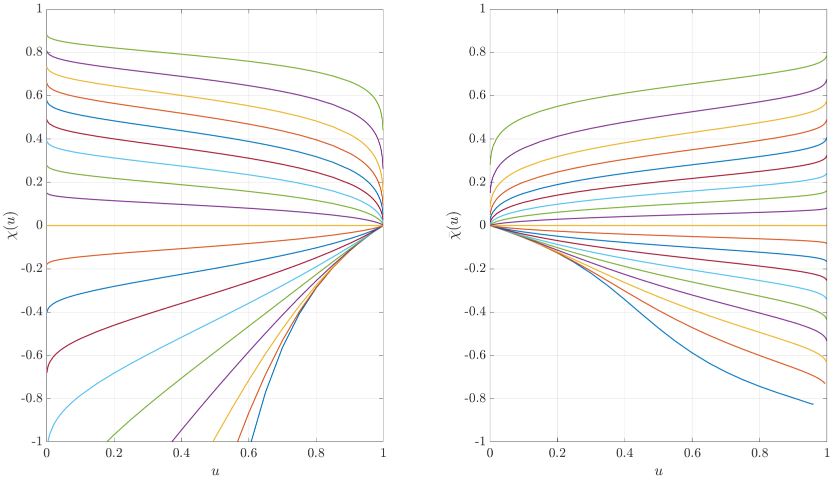
- $\bar{\chi} = 1$ perfect dependence and AD
- $\bar{\chi} \in (0,1)$ asymptotic independence, AI
- $\bar{\chi} = 0$ perfect independence
- See η for motivation

$$\theta(u) = \frac{\log \mathbb{P}(U \leq u, V \leq u)}{\log \mathbb{P}(U \leq u)} = \frac{\log C(u,u)}{\log u} \quad \to \theta \text{ as } u \to 1$$

- $\theta = 2 - \chi$

- MEVDs do not admit asymptotic independence

Extremal dependence (bivariate Gaussian)



$\chi(u)$ and $\bar{\chi}(u)$ for bivariate Gaussian ($\Rightarrow \chi = 0, \bar{\chi} = \rho$)
Colours are correlations ρ on -0.9, -0.8, ..., 0.9
(Recreated from Coles et al. 1999)

Beyond component-wise maxima

- Many (almost all?) environmental extremes problems involve asymptotic independence, at least in part
- Need to move away from MEVDs

- On Fréchet margins ($F(z) = \exp\left(-z^{-1}\right)$), assume

$$\frac{\mathbb{P}(Z_1 > z, Z_2 > z)}{\left(\mathbb{P}(Z > z)\right)^{1/\eta}} = \mathcal{L}(z)$$

  where $\mathcal{L}$ is slowly varying : $\mathcal{L}(xz)/\mathcal{L}(z) \to 1$ as $z \to \infty$
- $\bar{\chi} = 2\eta - 1$
- Ledford and Tawn [1996], Ledford and Tawn [1997]
- e.g. use non-extreme value copulas or inverted EV copulas

- $\mathbb{P}(Z_1 > z | Z_2 > z) \approx Cz^{1-1/\eta}$ from above
- Idea: assume a max-stable-like normalisation for conditional extremes

Conditional extremes

- $\boldsymbol{X} = (X_1, ..., X_j, ..., X_p)$
- Each X and Y have standard Laplace margins ($f(x) = \exp(-|x|)/2$, $x \in \mathbb{R}$)
- Seek a model for $\boldsymbol{X}|(Y = y)$ for $y > u$

- Assume we can find p-dimensional scaling $\boldsymbol{a} > \boldsymbol{0}$, $\boldsymbol{b}$ such that

$$\mathbb{P}(\boldsymbol{Z} \leq \boldsymbol{z}|Y = y) \quad \rightarrow \quad \text{n.d. } G(\boldsymbol{z}) \quad \text{as} \quad u \rightarrow \infty$$

$$\text{for} \quad \boldsymbol{Z} \quad = \quad \frac{\boldsymbol{X} - \boldsymbol{b}(y)}{\boldsymbol{a}(y)}$$

- Non-degenerate G is unknown, and estimated empirically

- Typical scaling is $\boldsymbol{a} = \boldsymbol{\alpha} y$ and $\boldsymbol{b} = y^{\boldsymbol{\beta}}$, $\boldsymbol{\alpha} \in [-1, 1]^p$, $\boldsymbol{\beta} \in (-\infty, 1]^p$
- So simply fit regression model

$$\boldsymbol{X}|(Y = y) = \boldsymbol{\alpha} y + y^{\boldsymbol{\beta}} \boldsymbol{Z}$$

- $\alpha = 1$, $\beta = 0$: perfect dependence and AD, and $\alpha \in (0, 1)$: AI

- Heffernan and Tawn [2004] find choices for α and β for popular bivariate cases
- Bivariate Gaussian : $\alpha = \rho^2$, $\beta = 1/2$

Conditional extremes in practice

- Non-stationary : Jonathan et al. [2014]
- Time-series : Winter and Tawn [2016], Tendijck et al. [2019]
- Mixture model : Tendijck et al. [2021]
- Spatial : Shooter et al. [2021b], Shooter et al. [2021a]
- ... lots more

- Multivariate spatial : Shooter et al. [2022]

Multivariate spatial conditional extremes (MSCE)

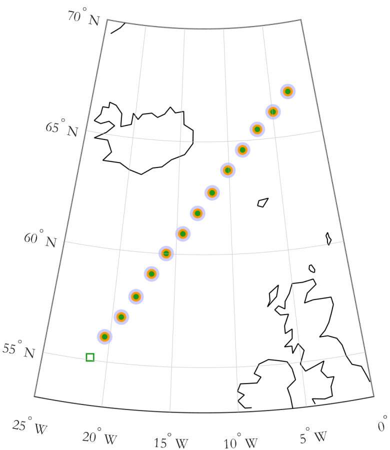
Motivation

- How useful are satellite observations of ocean waves and winds?
- Could they become the primary data source for decisions soon?
- What are the spatial characteristics of extremes from satellite observations?

Overview

- A look at the data
- Brief overview of methodology
- Results for joint spatial structure of extreme scatterometer wind speed, hindcast wind speed and hindcast significant wave height in the North Atlantic
- Implications for future practical applications

In a nut-shell



- Condition on large value x of first quantity X_{01} at one location $j = 0$ (green square)
- Estimate "conditional spatial profiles" for $m > 1$ quantities $\{X_{jk}\}_{j=1,k=1}^{p,m}$ at $p > 0$ other locations (green, orange and blue circles)

$$X_{jk} \sim \text{Lpl}$$

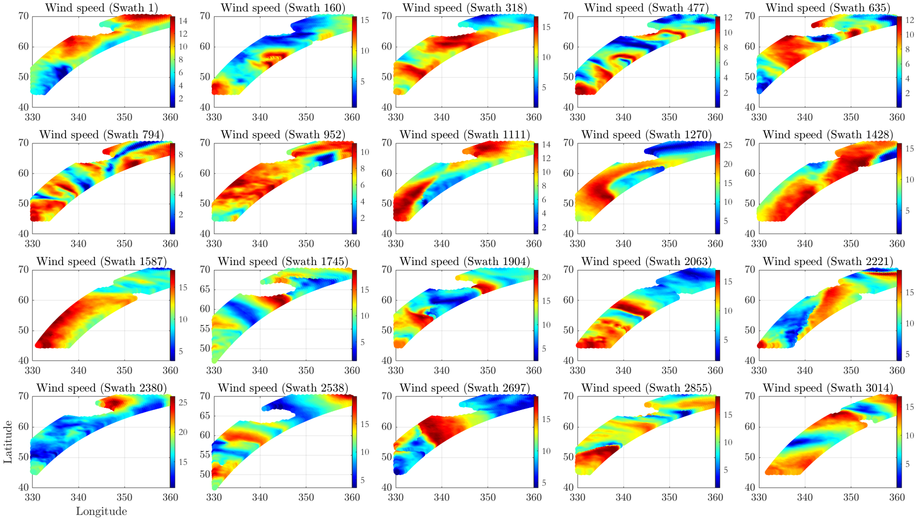
$$x > u$$

$$\boldsymbol{X} | \{X_{01} = x\} = \boldsymbol{\alpha} x + x^{\boldsymbol{\beta}} \boldsymbol{Z}$$

$$\boldsymbol{Z} \sim \text{DL}(\boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\delta}; \boldsymbol{\Sigma}(\boldsymbol{\lambda}, \boldsymbol{\rho}, \boldsymbol{\kappa}))$$

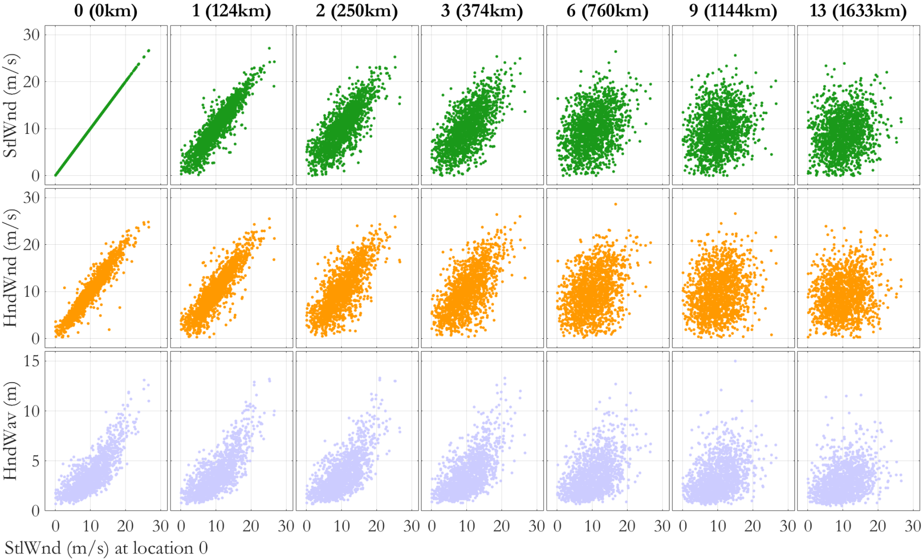
- MCMC to estimate $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\mu}$, $\boldsymbol{\sigma}$, $\boldsymbol{\delta}$ and $\boldsymbol{\rho}$, $\boldsymbol{\kappa}$, $\boldsymbol{\lambda}$
- $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\mu}$, $\boldsymbol{\sigma}$, $\boldsymbol{\delta}$ spatially smooth for each quantity
- Residual correlation $\boldsymbol{\Sigma}$ for conditional Gaussian field, powered-exponential decay with distance

Swath wind speeds



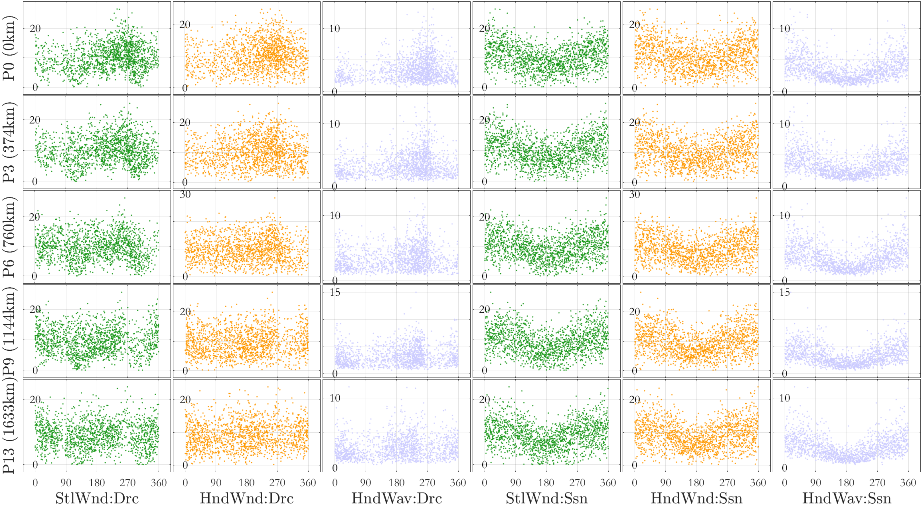
Daily descending METOP swaths. Satellite swath location changes over time. Spatial structure evident

Scatter plots on physical scale



Scatter plots of registered data : StlWnd (green), HndWnd (orange), HndWav(blue)

Covariate dependence



Directional and seasonal dependence. "Direction" is that from which fluid flows measured clockwise from North
StlWnd (green), HndWnd (orange), HndWav(blue)

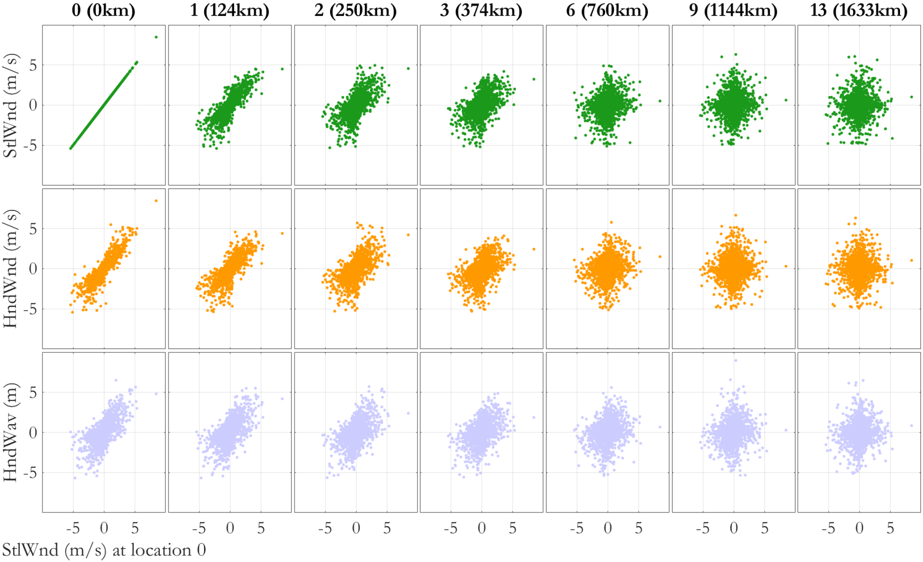
Marginal transformation to standard Laplace scale

Procedure

- Non-stationary piecewise constant directional-seasonal marginal extreme value model
- Pre-specified 8 directional bins ("octants") of equal width centred on cardinal and semi-cardinal directions
- Pre-specified "summer" and "winter" seasonal bins
- Generalised Pareto model for peaks over threshold
- Model parameters vary smoothly between bins, optimal roughness found using cross-validation
- Multiple extreme value thresholds with non-exceedance probabilities between 0.7 and 0.9 considered
- Bootstrapping for uncertainties
- Uncertainty in marginal model not propagated
- Independent marginal models for pair of variable (`StlWnd, HndWnd, HndWav`) and location (0,1,...,13)

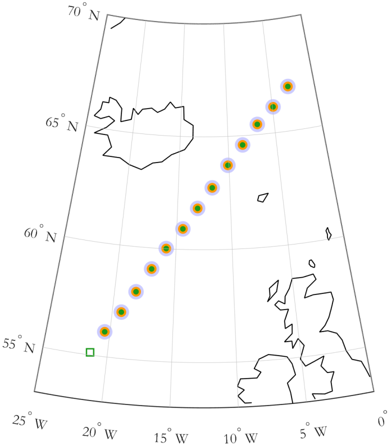
- Software : `github.com/ECSADES/ecsades-matlab`

Scatter plots on Laplace scale



Registered data on Laplace scale: StlWnd (green), HndWnd (orange), HndWav(blue)

In a nut-shell



- Condition on large value x of first quantity X_{01} at one location $j = 0$ (green square)
- Estimate "conditional spatial profiles" for $m > 1$ quantities $\{X_{jk}\}_{j=1,k=1}^{p,m}$ at $p > 0$ other locations (green, orange and blue circles)

$$X_{jk} \sim \text{Lpl}$$

$$x > u$$

$$\boldsymbol{X} | \{X_{01} = x\} = \boldsymbol{\alpha} x + x^{\boldsymbol{\beta}} \boldsymbol{Z}$$

$$\boldsymbol{Z} \sim \text{DL}(\boldsymbol{\mu}, \boldsymbol{\sigma}^2, \boldsymbol{\delta}; \boldsymbol{\Sigma}(\boldsymbol{\lambda}, \boldsymbol{\rho}, \boldsymbol{\kappa}))$$

- MCMC to estimate $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\mu}$, $\boldsymbol{\sigma}$, $\boldsymbol{\delta}$ and $\boldsymbol{\rho}$, $\boldsymbol{\kappa}$, $\boldsymbol{\lambda}$
- $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\mu}$, $\boldsymbol{\sigma}$, $\boldsymbol{\delta}$ spatially smooth for each quantity
- Residual correlation $\boldsymbol{\Sigma}$ for conditional Gaussian field, powered-exponential decay with distance

Inference

- Delta-Laplace residual margins

$$f_{Z_{j,k}}(z_{j,k}) = \frac{\delta_{j,k}}{2\kappa_{j,k}\sigma_{j,k}\Gamma\left(\frac{1}{\delta_{j,k}}\right)} \exp\left\{-\left|\frac{z-\mu_{j,k}}{\kappa_{j,k}\sigma_{j,k}}\right|^{\delta_{j,k}}\right\}, \quad \kappa_{j,k}^2 = \Gamma\left(1/\delta_{j,k}\right)/\Gamma\left(3/\delta_{j,k}\right)$$

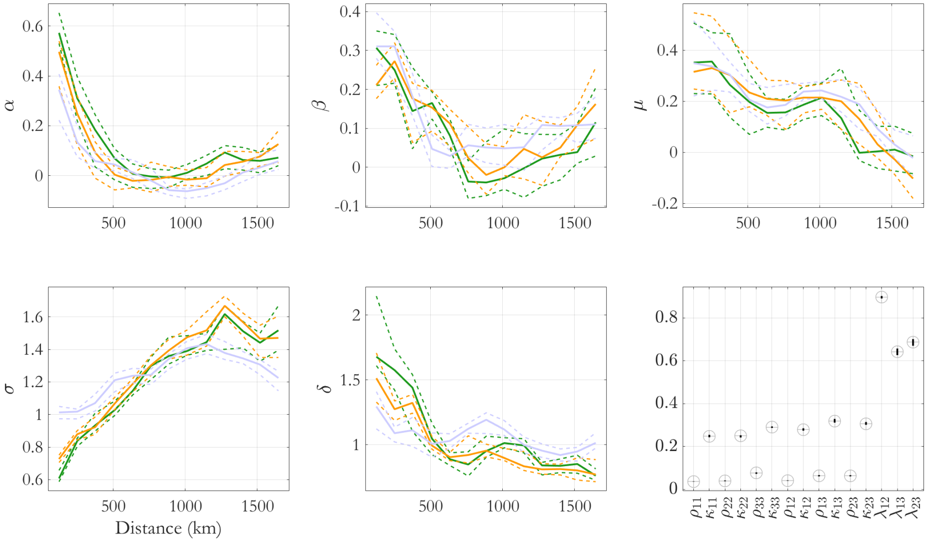
- Gaussian residual dependence

$$\Sigma^*_{\mathcal{A}^*(j,k)\mathcal{A}^*(j',k')} = \lambda_{k,k'}^{|k-k'|} \exp\left(-\left(\frac{\text{dist}(r_j, r_{j'})}{\rho_{k,k'}}\right)^{\kappa_{k,k'}}\right)$$

- Piecewise linear forms for all parameters with distance using n_{Nod} spatial nodes

- Adaptive MCMC, Roberts and Rosenthal [2009]
- Total of $m(5n_{\text{Nod}} + (3m+1)/2)$ parameters
- Rapid convergence, 10k iterations sufficient

Parameter estimates



Estimates for α, β, μ, σ and δ with distance, and residual process estimates ρ, κ and λ. Model fitted with $\tau = 0.75$
StlWnd (green), HndWnd (orange), HndWav(blue)
(Residual Gaussian field : ρ=scale, κ=exponent, λ=cross-correlation)

Summary

Why?

- Careful quantification of “rare-event” risk
- Characterise tails of (multivariate) distributions
- Limited observations
- Immediate real-world consequences

The next 10 years?

- Univariate : fuller covariate descriptions, exploit measurement scale / sub-asymptotics, UQ, provide real-world decision-support
- Multivariate : theoretical development, computational tractability, expansion in scope (time-series, spatial), serious real-world applications

An interesting field for research?

- Environmental extremes is a nice area if you like a mix of statistical theory, method, computation and serious physical science-based application

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Thanks for listening / Diolch am wrando!