Extremal characteristics of conditional models

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Abstract

Conditionally specified models are often used to describe complex multivariate data. Such models assume implicit structures on the extremes. So far, no methodology exists for calculating extremal characteristics of conditional models since the copula and marginals are not expressed in closed forms. We consider bivariate conditional models that specify the distribution of X and the distribution of Y conditional on X. We provide tools to quantify implicit assumptions on the extremes of this class of models. In particular, these tools allow us to approximate the distribution of the tail of Y and the coefficient of asymptotic independence η in closed forms. We apply these methods to a widely used conditional model for wave height and wave period. Moreover, we introduce a new condition on the parameter space for the conditional extremes model of Heffernan and Tawn (2004), and prove that the conditional extremes model does not capture η, when $\eta < 1$.

Keywords: Multivariate Extremes, Conditional Extremes, Laplace Approximation, Ocean Engineering

1 Introduction

Extreme value theory is a topic of growing interest because of its many important applications in for example risk management (Embrechts et al., 1999) or ocean engineering (Castillo et al., 2005). For instance, in the design or assessment of offshore facilities it is crucial to understand the distribution of extreme sea states. Such extreme sea states are quantified in terms of extreme wave heights, wave periods possibly associated with resonant frequencies, and extreme wind speeds. In risk management, it is important to identify which stocks are likely to suffer extreme losses simultaneously, and to which extent this might happen. In general, we need to use well-estabilished extreme value methods to model such events. Traditionally, such multivariate extreme value methods are composed of marginal models and a dependence copula, each having parametric forms for the tails.

In other areas of statistics, however, it is common to use conditional models for high-dimensional data. Intuitively, this is the most sensible approach. We observe X that partially explains Y. So, we define a model for X and a model for Y conditional on X. There exist many examples in the literature of models within this conditional framework with applications in extremes, e.g., the conditional extreme value model (Heffernan and Tawn, 2004; Fougeres and Soulier, 2012), the Weibull-log normal distribution (Haver and Winterstein, 2009, henceforth the Haver-Winterstein distribution), and hierarchical models (Eastoe, 2019). Although conditional models are easy to interpret, it can be rather difficult to study the extremes of both Y and (X, Y) within this class. Recently, Engelke and Hitz (2020) developed graphical models for extremes. However, we do not know of any literature that links existing conditional models directly to extremal dependence measures.

There are two extremal dependence measures that are key in identifying and measuring the degree of asymptotic dependence or asymptotic independence (Coles et al., 1999). Identifying the correct asymptotic dependence class is important since extrapolation of models from different classes is different. To define asymptotic dependence, we first define $\chi \in [0, 1]$, with

$$\chi := \lim_{p \uparrow 1} \chi(p) := \lim_{p \uparrow 1} \mathbb{P}\left\{Y > F_Y^{-1}(p) \mid X > F_X^{-1}(p)\right\}, \tag{1}$$

where F_X and F_Y denote the marginal distribution functions of X and Y. We say that these random variables are asymptotically dependent if $\chi > 0$, i.e., when the joint probability that both random variables are large is of the same magnitude as when one is large. If the coefficient of asymptotic dependence $\chi = 0$, we say that the variables are asymptotically independent. In this case, χ does not give us information on the level of asymptotic independence. So, we additionally define the coefficient of asymptotic independence $\eta \in (0, 1]$ (Ledford and Tawn, 1996) to satisfy for $u \to \infty$

$$\mathbb{P}\left\{X > F_X^{-1}\left[F_E(u)\right],\ Y > F_Y^{-1}\left[F_E(u)\right]\right\} \sim \mathcal{L}\left(e^u\right) e^{-u/\eta}, \tag{2}$$

where $F_E(u) = 1 - \exp(-u)$ is the distribution function of a standard exponential, and where $\mathscr{L}$ is a slowly varying function. Here, we write $f(x) \sim g(x)$ as $x \to \infty$ when $f(x)/g(x) \to 1$ as $x \to \infty$. We rewrite definition (2) as

$$\eta := \lim_{p \uparrow 1} \eta(p) := \lim_{p \uparrow 1} \frac{\log(1-p)}{\log\left[(1-p)\chi(p)\right]}. \tag{3}$$

If the variables are asymptotically dependent, then $\eta = 1$; if the variables are asymptotically independent, then $\eta \in (0, 1)$ or $\eta = 1$ and $\mathcal{L}(u) \to 0$ as $u \to \infty$. The coefficient of asymptotic independence η describes the rate of decay to zero of the joint exceedance probability $\mathbb{P}\{X > F_X^{-1}(p),\ Y > F_Y^{-1}(p)\}$ as p tends to 1, see Ledford and Tawn (1996).

It is relatively straightforward to calculate the two extremal dependence measures for distributions when the joint distribution function is specified parametrically, e.g., a bivariate extreme value distribution (Ledford and Tawn, 1996), or when the joint density function is specified parametrically (Nolde and Wadsworth, 2021), e.g., a multivariate normal distribution. In this paper, we consider models specified within the conditional framework. For these cases, it is not straightforward to calculate η analytically, and numerical estimation can be difficult since convergence of $\eta(p)$ to η can be exceptionally slow. We set up methodology to calculate η in closed form within this framework and demonstrate the techniques on two widely used examples specified below. We support these limiting results using numerical integration.

First, we consider the model described in Haver and Winterstein (2009), used to explain the dependence between extreme significant wave height and their associated wave periods. Secondly, we investigate the model of Heffernan and Tawn (2004). This is a conditional model which describes the distribution of $Y \mid X$ for large X, where both X and Y are on standard margins. As the Heffernan-Tawn model focusses on the averages and deviations of $Y \mid X$ for large X, and not necessarily on the tails of $Y \mid X$ for large X, it cannot be expected to model η correctly. Indeed, we will show that η of (X, Y) can be different to η from the associated exact Heffernan-Tawn model. More theoretical examples, like $Y \mid X := X^\beta Z$ and $Y \mid X := |Z|^{|X|}$ where Z is some random variable independent of X, can be found in the Ph.D. thesis of Tendijck (2023).

The layout of the article is as follows. In Section 2, we demonstrate novel techniques for calculating the coefficient of asymptotic independence η and illustrate the techniques with some examples. In Sections 3 and 4, we apply these techniques to the Haver-Winterstein model and the Heffernan-Tawn model, respectively. Proofs are found in the Appendix and Supplementary Material.

2 Methodology

2.1 Motivation

We aim to investigate the extremal properties of the bivariate distribution of (X, Y), for which the distribution of X and the distribution of $Y \mid X$ are specified. In particular, we aim to investigate the tail of the distribution of Y and joint extremes of X and Y via the coefficient of asymptotic independence η. Deriving such extremal quantities in closed form within this class is not trivial. In this section, we provide a set of tools, derived from the Laplace approximation, to calculate such properties for any conditional model.

First, we consider the tail of the distribution of Y. Because the distributions of X and $Y \mid X$ are specified, it is natural to write

$$1 - F_Y(y) := \mathbb{P}(Y > y) = \int_{-\infty}^{\infty} \mathbb{P}(Y > y \mid X = x) f_X(x) \,\mathrm{d}x,$$

where f_X is the density of X. In general, this integral is analytically intractable. In Section 2.2, we present the tools with which we can derive the asymptotic properties of this integral as y tends to the upper end point of the distribution of Y.

To derive the coefficient of asymptotic independence, we additionally need the inverse distribution $F_Y^{-1}(p)$ for values of p close to 1, and

$$\mathbb{P}(X > F_X^{-1}(p),\ Y > F_Y^{-1}(p)) = \int_{F_X^{-1}(p)}^{\infty} \mathbb{P}(Y > F_Y^{-1}(p) \mid X = x) f_X(x) \,\mathrm{d}x.$$

This integral is also intractable in general; the tools from Section 2.2 can again be applied to derive the asymptotic decay to 0 as p tends to 1.

2.2 Extension to the Laplace approximation

Here we present our theory to calculate asymptotic rates of decay of integrals, that can be used to compute extremal properties, such as η, of conditional models. We first recall the Laplace approximation, a technique commonly used in Bayesian inference for approximating intractable integrals. This asymptotic approximation forms the basis of our main result. We then state that result, and illustrate key differences with the Laplace approximation by comparing examples.

Proposition 1 (Laplace approximation) *Let $a < b$. Suppose $g : [a, b] \to \mathbb{R}$ is twice continuously differentiable and assume there exists a unique $x^* \in (a, b)$ such that $g(x^*) = \max_{x \in [a,b]} g(x)$ and $g''(x^*) < 0$. Then*

$$\int_a^b e^{ng(x) - ng(x^*)} \,\mathrm{d}x \cdot \sqrt{n(-g''(x^*))} \sim \sqrt{2\pi}$$

as $n \to \infty$.

The main disadvantage of the Laplace approximation is that it can only be used to approximate integrals where the integrands are of the form $f(x)^n$, where $f(x) = e^{g(x)}$ is a positive function. However, we are interested in calculating integrals with integrand $f_n(x) = e^{g_n(x)}$, for some sequence of functions $\{g_n\}_{n\in\mathbb{N}}$. Now we extend the Laplace approximation under the assumptions that: (i) the analogue x_n^* of x^* is allowed to depend on n; (ii) x_n^* can be equal to either a or b; (iii) $g_n''(x_n^*)$ does not need to be negative.

Proposition 2 *Let $I \subseteq \mathbb{R}$ be connected with non-zero Lebesgue mass, $k_0 \geq 1$ an integer, and $g_n \in C^{k_0}(I)$ a sequence of real-valued (at least) k_0-times continuously differentiable functions defined on I. For $1 \leq i \leq k_0$, we define $g_n^{(i)}$ as the ith derivative of g_n. We assume that for all $n \in \mathbb{N}$, there exists a unique $x_n^* \in I$ such that $g_n(x_n^*) > g_n(x)$ for all $x \in I \backslash \{x_n^*\}$. Moreover, we assume that k_0 is the smallest integer such that $g_n^{(k_0)}(x_n^*) < 0$ and $\lim_{n\to\infty} g_n^{(i)}(x_n^*)[-g_n^{(k_0)}(x_n^*)]^{-i/k_0} = 0$ for all $1 \leq i < k_0$. Additionally, assume that there exists a δ such that for all $|x| < \delta$*

$$\lim_{n\to\infty} \frac{g_n^{(k_0)}\left\{x_n^* + x\left[-g_n^{(k_0)}(x_n^*)\right]^{-\frac{1}{k_0}}\right\}}{g_n^{(k_0)}(x_n^*)} < \frac{3}{2}.$$

Then, for $n > N$, there exists a constant $C_1 > 0$ such that

$$\int_I e^{g_n(x)-g_n(x_n^*)}\,\mathrm{d}x \cdot \left[-g_n^{(k_0)}(x_n^*)\right]^{\frac{1}{k_0}} \geq C_1.$$

The proof of Proposition 2 can be found in Appendix A. One disadvantage of our extension is that it only gives an asymptotic lower bound. In many practical applications, an upper bound can be found directly using inequalities like that in equation (8).

2.3 Examples

We demonstrate the use of Proposition 2 in three cases. Firstly, let $g_n(x) = -nx^p$ for $n \in \mathbb{N}$, $p \in \mathbb{Z}_{\geq 1}$ and $I = [0,\infty)$. It is then valid to apply Proposition 2 with $x_n^* = 0$ and $k_0 = p$. Applying the proposition yields a constant $C_1 > 0$ such that as $n \to \infty$

$$n^{\frac{1}{p}} \int_0^\infty e^{-nx^p}\,\mathrm{d}x \geq C_1.$$

This lower bound is tight as we now verify for $p \geq 2$, since for $p = 1$ the statement holds trivially. For $p \geq 2$, we use the variable transformation $y = nx^p$ to give as $n \to \infty$

$$n^{\frac{1}{p}} \int_0^\infty e^{-nx^p}\,\mathrm{d}x = \frac{1}{p}\int_0^\infty y^{-\frac{1}{p}-1} e^{-y}\,\mathrm{d}y = \Gamma\left(\frac{1}{p}+1\right).$$

After recognizing that the integral over $[0,\infty)$ is equal to half of the integral over $\mathbb{R}$, we see that Proposition 1 is also applicable, but only in the special case $p = 2$. In this case, Proposition 1 additionally gives as $n \to \infty$

$$\int_0^\infty e^{-nx^2}\,\mathrm{d}x = \frac{1}{2}\int_{-\infty}^\infty e^{-nx^2}\,\mathrm{d}x \sim \frac{\sqrt{\pi}}{2\sqrt{n}}.$$

Secondly, let $g_n(x) = -x - nx^2$ and $I = [0, \infty)$. Now Proposition 1 is not applicable since no function $g(x)$ exists for which $g_n(x) = ng(x)$ holds. Note that Proposition 2 is also not applicable with $k_0 = 1$, since x_n^* has to be equal to 0 and for $x \neq 0$

$$\lim_{n\to\infty} \frac{g_n'(0 + x \cdot n)}{g_n'(0)} = \lim_{n\to\infty} 1 + 2n^2 x = \infty,$$

contradicting one of the assumptions. Proposition 2 is applicable with $k_0 = 2$, yielding a constant $C_2 > 0$ such that as $n \to \infty$

$$\sqrt{n}\int_{-\infty}^\infty e^{-x-nx^2}\,\mathrm{d}x \geq C_2.$$

Similar to our first example, this lower bound is tight since we can also directly calculate as $n \to \infty$

$$\sqrt{n}\int_{-\infty}^\infty e^{-x-nx^2}\,\mathrm{d}x = \sqrt{n}\int_{-\infty}^\infty e^{-n\left(x+\frac{1}{2n}\right)^2+\frac{1}{4n}}\,\mathrm{d}x \sim \sqrt{\pi}.$$

Finally, let $\alpha_n > 0$, $\beta_n > 0$ for $n \in \mathbb{N}$ and assume $\liminf \alpha_n > 0$. Define $g_n(x) = \alpha_n \log x - \beta_n x$. Using an argument similar to that in the second example, we see that Proposition 1 is not applicable. However Proposition 2 is applicable with $k_0 = 2$, yielding a constant $C_3 > 0$ such that as $n \to \infty$

$$\alpha_n^{-\alpha_n - \frac{1}{2}} \beta_n^{\alpha_n+1} e^{\alpha_n} \int_0^\infty x^{\alpha_n} e^{-\beta_n x}\,\mathrm{d}x \geq C_3.$$

This bound is also tight, which can be seen from recognizing the density of a gamma distribution in the expression above, and applying limit results for the gamma function.

3 Haver-Winterstein model

Haver and Winterstein (2009) introduce the Haver-Winterstein (HW) distribution for significant wave height H_S and wave period T_p in the North Sea. Their model is set up in the conditional framework: they specify a class of distributions for H_S and a class of distributions for $T_p \mid H_S$. Variations of this approach have been widely applied in ocean engineering with over 150 citations, 25 of which correspond to 2021, see for example Drago et al. (2013). However we are not aware of any literature quantifying χ and η in closed form for the HW distribution; we now show how to calculate these.

The HW distribution is formulated as

$$f_X(x) = \begin{cases} \frac{1}{\sqrt{2\pi}\alpha x} \exp\left\{-\frac{(\log x-\theta)^2}{2\alpha^2}\right\}, & \text{for } 0 < x \leq u, \\ \frac{k}{\lambda^k} x^{k-1} \exp\left\{-\left(\frac{x}{\lambda}\right)^k\right\}, & \text{for } x > u. \end{cases} \quad (4)$$

where $u, \alpha, k, \lambda > 0$ and $\theta \in \mathbb{R}$. In particular, the parameters are constrained such that f_X is continuous at u and integrates to 1. Secondly, they take $Y \mid X$ to be conditionally log-normal

$$f_{Y|X}(y \mid x) = \frac{1}{\sqrt{2\pi}\sigma(x) y} \exp\left\{-\frac{(\log y - \mu(x))^2}{2\sigma(x)^2}\right\}, \quad \text{for } x, y > 0, \quad (5)$$

where $\mu(x) := \mu_0 + \mu_1 x^{\mu_2}$ and $\sigma(x) := [\sigma_0 + \sigma_1 \exp(-\sigma_2 x)]^{1/2}$ with $\mu_0 \in \mathbb{R},\ \mu_1, \mu_2, \sigma_0, \sigma_1, \sigma_2 > 0$.

Model parameter estimates (Haver and Winterstein, 2009) from data observed in the northern North Sea are given in the Supplementary Material. For ease of presentation, we make two assumptions about the parameter space of the HW distribution that are consistent with parameter estimates $(\hat{\mu}_2, \hat{k}) = (0.225, 1.55)$ from Haver and Winterstein (2009). Specifically, we make the following restrictions: $0 < \mu_2 < 0.5$ and $2\mu_2 < k$. These assumptions reduce the number of cases to be considered significantly whilst including realistic domains for the parameters as considered by practioners.

We now show how to use Proposition 2 to calculate the extremal dependence measures χ and η for the bivariate random vector (X, Y) distributed according to the HW distribution in the restricted parameter space. Calculation of η is split into two steps. In the first step, we calculate the distribution function F_Y of Y and in the second we evaluate the rate of decay of joint probabilities $\mathbb{P}\{X > F_X^{-1}[F_E(u)], Y > F_Y^{-1}[F_E(u)]\}$ as u tends to infinity.

We have

$$\mathbb{P}(Y > y) = \int_0^\infty \mathbb{P}(Y > y \mid X = x) f_X(x)\,\mathrm{d}x = \int_0^\infty \overline{\Phi}\left(\frac{\log y - \mu(x)}{\sigma(x)}\right) f_X(x)\,\mathrm{d}x, \quad (6)$$

where $\overline{\Phi}$ is the survival function of a standard Gaussian. This integral is analytically intractable but we can calculate its limiting leading order behaviour in closed form. Proposition 2 gives a lower bound and an upper bound of the same order as the lower bound is then found directly. For ease of notation, we denote the integrand by

$$g_y(x) := \overline{\Phi}\left(\frac{\log y - \mu(x)}{\sigma(x)}\right) f_X(x) \quad (7)$$

for $x > 0$. In Figure 1, we plot g_y for various values of y. From the figure, we note that g_y has two local maxima for suffiiciently large y. These are x_y^*, which converges to zero, and x_y^{**}, which diverges to infinity. This observation implies

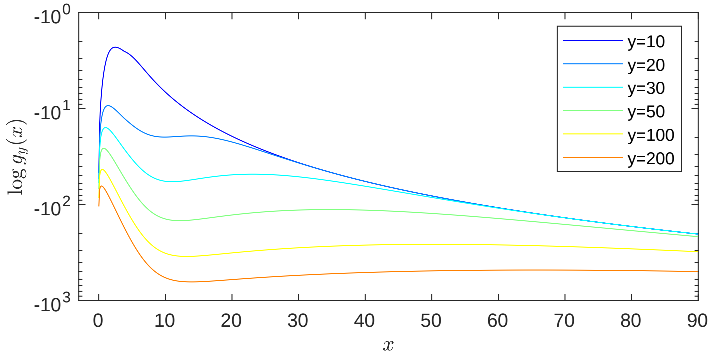

Figure 1 The function $\log g_y$ from equation (7) for $y = 10, 20, 30, 40, 50, 100$ with parameters as reported in Haver and Winterstein (2009), see Supplementary Material.

that we cannot apply Proposition 2 directly in this case. We therefore proceed as follows: (i) calculate x_y^* and x_y^{**}; (ii) partition the interval of integration into intervals I_1 and I_2, where $x_y^* \in I_1$ and $x_y^{**} \in I_2$, such that the conditions of Proposition 2 hold for both intervals, and then apply the proposition on each interval; (iii) combine the two lower bounds found to get a lower bound for integral (6); (iv) derive a limiting upper bound for integral (6) of the same order as the lower bound.

In the Supplementary Material, we derive that as $y \to \infty$

$$x_y^* \sim \left(\frac{\sigma_1 \sigma_2 \cdot \log y}{2\mu_1\mu_2(\sigma_0 + \sigma_1)} \right)^{-\frac{1}{1-\mu_2}} \quad \text{and} \quad x_y^{**} \sim \left(\frac{\lambda^k \mu_1 \mu_2 \cdot \log y}{k\sigma_0} \right)^{\frac{1}{k-\mu_2}}.$$

From Figure 1, we recognize that $g_y(x_y^*) > g_y(x_y^{**})$ as $y \to \infty$. We show that this holds analytically in the Supplementary Material when $2\mu_2 < k$. We now apply Proposition 2 and find that $k_0 = 2$ is appropriate. The proposition then gives a lower bound for integral (6) around x_y^* as $y \to \infty$ of

$$\mathbb{P}(Y > y) \geq \exp\left\{ -\frac{\log^2 y}{2(\sigma_0 + \sigma_1)} + O(\log y) \right\}.$$

Finally, since $g_y(x_y^*) > g_y(x_y^{**})$, it is straightforward to show as $y \to \infty$ that

$$\mathbb{P}(Y > y) \leq \exp\left\{ -\frac{\log^2 y}{2(\sigma_0 + \sigma_1)} + O(\log y) \right\}$$

using the inequality

$$\mathbb{P}(Y > y \mid X = x) f_X(x) \leq g_y(x_y^*) \mathbb{1}\{x \in [0, x_y^{**}]\} + f_X(x) \mathbb{1}\{x > x_y^{**}\}. \quad (8)$$

We now can calculate η and show that $\chi = 0$. To that end, we first need to calculate the inverse probability integral transform, transforming Y to standard exponential margins; i.e., we need $F_Y^{-1}[F_E(u)]$. Next, we need to evaluate the asymptotic behaviour of $\mathbb{P}\{Y > F_Y^{-1}[F_E(u)], X > F_X^{-1}[F_E(u)]\}$ as $u \to \infty$. To evaluate $F_Y^{-1} \circ F_E$, we first calculate for $y \to \infty$

$$F_E^{-1}(F_Y(y)) = -\log(1 - F_Y(y)) = \frac{\log^2 y}{2(\sigma_0 + \sigma_1)} + O(\log y).$$

We invert this expression by solving $F_E^{-1}(F_Y(y)) = u$ for $\log y$. This yields $\log y = \sqrt{2\sigma_0 + \sigma_1)u} + O(1)$ as $y \to \infty$. We can now write down an asymptotic expression for $\chi(u)$ as $u \to \infty$

$$\begin{aligned}\chi(u) &:= \mathbb{P}\left\{F_E^{-1}\left[F_Y(Y)\right] > u,\ F_E^{-1}\left[F_X(X)\right] > u\right\} \\ &= \mathbb{P}\left\{\log Y > \sqrt{2}(\sigma_0 + \sigma_1)\sqrt{u} + O(1),\ (X/\lambda)^k > u\right\} \\ &= \int_{\lambda u^{1/k}}^{\infty} \overline{\Phi}\left(\frac{\sqrt{2}(\sigma_0 + \sigma_1)\sqrt{u} + O(1) - \mu(x)}{\sigma(x)} \mid X = x\right) \cdot \frac{kx^{k-1}}{\lambda^k} \exp\left\{-\left(\frac{x}{\lambda}\right)^k\right\} \mathrm{d}x.\end{aligned}$$

In the Supplementary Material, we show that Proposition 2 is applicable for this integral with $k_0 = 1$ and $x_u^* = \lambda u^{1/k}$. Moreover, we derive directly an upper bound of the same order, obtaining

$$\chi(u) = \exp\left\{-\left(2 + \frac{\sigma_1}{\sigma_0}\right) u + O\left(u^{1/2+\mu_2/k}\right)\right\}$$

as $u \to \infty$. Hence, $\chi = 0$ and

$$\eta = \left(2 + \frac{\sigma_1}{\sigma_0}\right)^{-1}.$$

In particular, for the parameter estimates from Haver and Winterstein (2009), the value of $\eta \in (0, 1/2)$ implies that the distribution exhibits negative asymptotic independence (Ledford and Tawn, 1996).

4 Heffernan-Tawn model

In multivariate extreme value theory, the conditional extreme value model of Heffernan and Tawn (2004), henceforth denoted the HT model, is widely studied and applied to extrapolate multivariate data. The HT model has been cited over 600 times, and is applied e.g. in oceanography (Ross et al., 2020), finance (Hilal et al., 2011), and spatio-temporal extremes (Simpson and Wadsworth, 2021). The HT model is a limit model and its form is motivated by derived limiting forms from numerous theoretical examples. Keef et al. (2013) assume that for (X, Y) on standard Laplace margins there exist parameters $\alpha \in [-1, 1]$,

$\beta < 1$ and a non-degenerate distribution function H such that for $x > 0$, $z \in \mathbb{R}$

$$\lim_{u\to\infty} \mathbb{P}\left(\frac{Y-\alpha X}{X^\beta} \le z,\ X-u > x \mid X > u\right) = \exp(-x)H(z). \tag{9}$$

In the limit of $u \to \infty$, this formulation implies that $(Y - \alpha X)X^{-\beta}$ and $(X - u)$ are independent conditional on $X > u$, and are distributed as H and a standard exponential, respectively. As is common practice in extreme value theory, the HT model assumes that the corresponding limiting family in (9) holds exactly at a finite level. Thus the HT model is specified for $x > u$, where u is a sufficiently high threshold such that the limit representation in (9) is considered a good approximation. Let (X, Y) be a random vector such that X and Y both have standard Laplace margins. Moreover, let $\alpha, \beta \in [0, 1)$ and assume that for $x > u > 0$

$$\mathbb{P}(Y > y \mid X = x) = \overline{H}\left(\frac{y-\alpha x}{x^\beta}\right) \tag{10}$$

holds for all $y \in \mathbb{R}$ where $\overline{H} = 1-H$ is some non-degenerate survival function. In this case, we say that (X, Y) are distributed according to an exact version of the HT model. We consider two cases for H, corresponding to finite and infinite upper end points. If H has a finite upper end point z^H, calculations for η are trivial. Indeed, when $X = x$, Y cannot be larger than $\alpha x + x^\beta z^H$. In particular, as $u \to \infty$, $Y > u$ is equivalent to $X > u/\alpha + O(u^\beta)$. Hence as $u \to \infty$

$$\begin{aligned}\mathbb{P}(X > u, Y > u) &\sim \mathbb{P}\left\{X > u, X > u/\alpha + O(u^\beta)\right\}\\ &\sim \mathbb{P}\left\{X > u/\alpha + O(u^\beta)\right\}\\ &= \exp\left\{-u/\alpha + O(u^\beta)\right\}.\end{aligned}$$

Therefore, $\eta = \alpha$ when $\alpha > 0$ and otherwise does not exist.

Now assume that H has an infinite upper end point. To make calculations tractable, we parameterise $\overline{H}$ as

$$\overline{H}(z) = \exp\left\{-\gamma z^\delta + o\left(z^\delta\right)\right\} \mathbb{1}\{z > 0\} + \mathbb{1}\{z \le 0\} \tag{11}$$

for $\gamma > 0$, $\delta \ge 1$. For simplicity, we do not consider potential negative arguments for $\overline{H}$ since the precise form of its lower tail is not relevant to the current work. Parameterisation (11) covers most non-trivial cases for the upper tail including Gaussian, Weibull and exponential tails; see examples in Heffernan and Tawn (2004). Moreover if the tail of $\overline{H}$ is heavier than that of the exponential, Y cannot possibly possibly follow a standard Laplace distribution. This links to the restricton $\delta \ge 1$. For illustration, we set $o(z^\delta) = 0$ in equation (11). The resulting Weibull survival function is a suitable choice for $\overline{H}$, since it has an extreme value tail index of 0, but a varying tail thickness controlled by δ.

α	β	γ	δ	η
$(0,1)$	$[0,1)$	$(0,\infty)$	$((1-\beta)^{-1},\infty)$	α
$(0,1)$	$(0,1)$	$(0,\infty)$	$(1-\beta)^{-1}$	$\left(\frac{\gamma(1-\alpha c)^{\delta}}{c^{\delta-1}}+c\right)^{-1}$
$(0,1)$	0	$(1/\alpha,\infty)$	1	α
$(0,1)$	0	$(0,1/\alpha]$	1	$1/(\gamma+1-\gamma\alpha)$
0	$(0,1)$	$(0,\infty)$	$((1-\beta)^{-1},\infty)$	Not defined
0	$(0,1)$	$(0,(1-\beta)/\beta]$	$(1-\beta)^{-1}$	$1/(\gamma+1)$
0	$(0,1)$	$[(1-\beta)/\beta,\infty)$	$(1-\beta)^{-1}$	$\gamma^{-1/\delta}(\delta-1)^{1-1/\delta}/\delta$

Table 1 Values of η for model (10) with $\overline{H}$ as in (11) for different ranges of parameter combinations, where $c=\max\{1,c_0\}\in[1,1/\alpha)$ for c_0 given in equation (12).

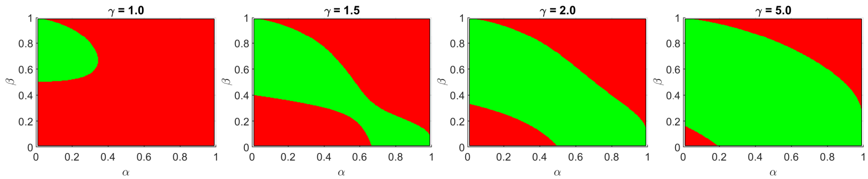

Figure 2 Visualisation of c_0 from equation (12) for $\gamma=1,\ 1.5,\ 2,\ 5$ and $\delta=(1-\beta)^{-1}$. The region corresponding to $c_0\in(0,1)$ is shown in red; the region corresponding to $c_0\in(1,1/\alpha)$ is shown in green.

Proposition 3 *If (X,Y) follows distribution (10) with H as in (11) with $o(z^{\delta})=0$, then $\delta\geq(1-\beta)^{-1}$.*

The proof of Proposition 3 is found in Appendix A. Following similar arguments to those used in the proof of Proposition 3, we calculate χ and η for any combination of the parameters $(\alpha,\beta,\delta,\gamma)$ in their specified parameter space. We collect results in Table 1. In the Supplementary Material, we only give details of the η calculations when $\alpha,\beta\in(0,1)$, $\gamma>0$ and $\delta=(1-\beta)^{-1}$. For the other five cases in Table 1, we state results without proof. In particular, the argument underpinning the η calculation when $\delta>(1-\beta)^{-1}$ is similar to the argument used when $\overline{H}$ has a finite upper end point. In this case, $\eta=\alpha$ when $\alpha>0$ and when $\alpha=0$, η is not defined.

In Table 1, it is convenient to refer to $c=\max\{1,c_0\}\in[1,1/\alpha)$ where $c_0\in(0,1/\alpha)$ satisfies

$$\gamma(1-\alpha c_0)^{\delta-1}(\delta-1+\alpha c_0)=c_0^{\delta}. \tag{12}$$

To give some intuition on the value of c, in Figure 2 we sketch the region of the parameter space corresponding to $c=1$ (in red) for different values of γ. Finally in Figure 3 we visualise η for a set of different parameter combinations with $\delta=(1-\beta)^{-1}$.

We note the following interesting findings. The parameter η is non-decreasing with increasing α and with increasing β. Parameter combinations $(\alpha,\beta,\gamma,\delta)$ exist for which $\alpha,\beta>0$ but $\eta<0.5$. Hence, there are cases for

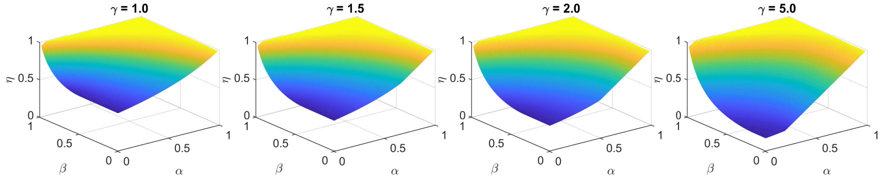

Figure 3 The value of η as a function of α, β and γ with $\delta = (1-\beta)^{-1}$ from the HT model (10) and (11).

which Y increases with X but the extremes of (X, Y) are negatively associated as measured by η (Ledford and Tawn, 1996).

Finally we note that the Heffernan-Tawn model is not η invariant, i.e., when the HT model occurs in the limit of the distribution of (X, Y), then η for (X, Y) is not necessarily the same as η for the associated exact HT model. To illustrate this, let (X, Y) follow an inverted bivariate extreme value distribution with a logistic dependence structure (Ledford and Tawn, 1996) on Laplace margins with parameter $\xi \in (0, 1]$, such that

$$\mathbb{P}(X > x,\ Y > y) = \exp\left\{-\left[t_x^{1/\xi} + t_y^{1/\xi}\right]^{\xi}\right\}, \tag{13}$$

where $t_x := \log 2 - \log[2 - \exp(x)]$ for $x < 0$ and $t_x := \log 2 + x$ for $x > 0$, with t_y similarly defined. It is straightforward to derive that in the limit, the Heffernan-Tawn model (10) is applicable to (X, Y) with $\overline{H}$ as in equation (11) and $o(z^{\delta}) = 0$. Specifically,

$$\lim_{x\to\infty} \mathbb{P}\left(YX^{\xi-1} > z \mid X = x\right) = \exp\left(-\xi z^{1/\xi}\right).$$

Now let (X_{HT}, Y_{HT}) be distributed following our exact version of the HT model associated with (X, Y). That is, for $X_{HT} < u$, we have $(X_{HT}, Y_{HT}) = (X, Y)$. For $X_{HT} \geq u$, $X_{HT} - u$ is standard exponentially distributed, and $Y_{HT} \mid X_{HT}$ follows model (10) with $\overline{H}$ as in (11) with parameters $(\alpha,\ \beta,\ \gamma,\ \delta) = (0,\ 1 - \xi,\ \xi,\ 1/\xi)$ and $o(z^{\delta}) = 0$. In this case $\gamma < (1 - \beta)/\beta$, and Table 1 implies that the coefficient of asymptotic independence η_{HT} of (X_{HT}, Y_{HT}) is equal to $1/(\xi+1)$. In contrast, it is straightforward to derive directly from definition (13) that η of (X, Y) is equal to $2^{-\xi}$. Hence $\eta_{HT} \neq \eta$ when $\xi \in (0, 1)$.

Finally we illustrate numerically the differences between η, η_{HT} and their finite level counterparts $\eta(p)$ and $\eta_{HT}(p)$ for $p \in (0, 1)$. For definiteness, we let (X, Y) follow distribution (13) with $\xi = 0.35$. We simulate a sample $\{(x_i, y_i) : i = 1, \ldots, n\}$ of size $n = 10,000$. First we empirically estimate $\eta(p)$ from equation (3) for $p \in (0, 1)$ and calculate pointwise 95% confidence intervals using the binomial distribution. Next we note that $\eta(p) = \eta$ for $p \in (0.5, 1)$. Finally we calculate the corresponding $\eta_{HT}(p)$ for p near 1 using numerical integration.

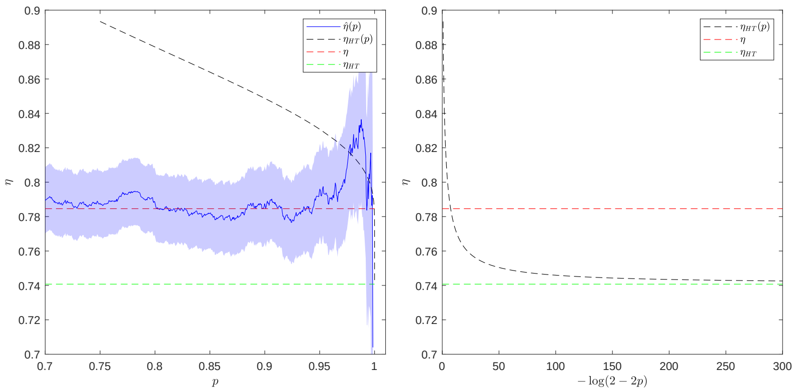

Figure 4 Coefficients of asymptotic independence η (red dashed) for distribution (13) with $\xi = 0.35$, and the corresponding value for the exact limiting HT model η_{HT} (green dashed), and its finite level counterpart $\eta_{HT}(p)$ (black dashed). Empirical estimates $\hat{\eta}(p)$ for a sample of size 10,000 with pointwise confidence intervals are shown in blue. Left and right hand panels are the same except for the scale of the x-axis, set on the right to illustrate the behaviour of $\eta_{HT}(p)$ for p near 1.

Results are shown in Figure 4. Left and right hand plots are the same except for the scale of the x-axis, illustrating the behaviour of $\eta_{HT}(p)$ for p near 1. Reassuringly, the true η of the underlying model (red dashed) falls within the 95% confidence interval for its empirical counterpart $\hat{\eta}(p)$ (blue). Further, $\eta_{HT}(p)$ (black dashed) converges to η_{HT} (green dashed). We note that $\eta_{HT}(p)$ varies as a function of p and only seems to asymptote for $p > 1 - \exp(-50)/2 \approx 1 - 9.6 \cdot 10^{-23}$. Finally, since $\eta_{HT} < \eta$, we would expect that $\eta_{HT}(p)$ would underestimate η, but it turns out this is only the case for $p > 1 - \exp(-7.5)/2 \approx 0.9997$.

Supplementary information. In the Supplementary Material, we give details of the mathematical derivations corresponding to the case studies.

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Appendix A Proofs

Proof of Proposition 2. We prove that as $n \to \infty$, there exists a constant $C_1 > 0$ such that

$$\mathscr{I}_n := \int_I e^{g_n(x) - g_n(x_n^*)} \, \mathrm{d}x \cdot \left(-g_n^{(k_0)}(x_n^*)\right)^{\frac{1}{k_0}} \geq C_1.$$

To bound $\mathscr{I}_n$ from below, we first simplify its expression by applying the variable transformation $y = t_n(x) := (x - x_n^*)\left(-g_n^{(k_0)}(x_n^*)\right)^{1/k_0}$ and defining

$$h_n(y) := g_n\left(x_n^* + y\left(-g_n^{(k_0)}(x_n^*)\right)^{-\frac{1}{k_0}}\right), \quad \text{for } y \in I_n' := \{t_n(x) : \ x \in I\}.$$

Then, the integral $\mathscr{I}_n$ becomes

$$\mathscr{I}_n = \int_{I_n'} e^{h_n(y) - h_n(0)}\, \mathrm{d}y.$$

We note that for all $n \in \mathbb{N}$, we have $0 \in I_n'$, $h_n \in C^{k_0}(I_n')$, and $h_n(0) > h_n(y)$ for all $y \in I_n' \setminus \{0\}$. Moreover, we have for $y \in I_n'$, $i = 1, \dots, k_0$,

$$h_n^{(i)}(y) = g_n^{(i)}\left(x_n^* + y\left(-g_n^{(k_0)}(x_n^*)\right)^{-1/k}\right) \cdot \left(-g_n^{(k_0)}(x_n^*)\right)^{-i/k_0}.$$

Hence, $h_n^{(k_0)}(0) = -1$ and $\lim_{n\to\infty} h_n^{(i)}(0) = 0$ for all $1 \le i < k_0$. Using Taylor's theorem, there exists a function $\xi(y)$ taking on a value between 0 and y such that

$$h_n(y) - h_n(0) = \sum_{i=1}^{k_0 - 1} \frac{y^i}{i!} h_n^{(i)}(0) + \frac{y^{k_0}}{k_0!} h_n^{(k_0)}(\xi(y)).$$

Let $\varepsilon > 0$. Because $\lim_{n\to\infty} h_n^{(i)}(0) = 0$ for all $i < k_0$, we can find an $N_0 \in \mathbb{N}$ such that for all $n > N_0$, we have $\max_{i=1,\dots,k_0-1} |h_n^{(i)}(0)| < \varepsilon$. Moreover, from the assumptions of the proposition, we can find a $\delta > 0$ and an $N_1 \in \mathbb{N}$ such that for all $n > N_1$, $h_n^{(k_0)}(y) > -3/2$ for $y \in (-\delta, \delta) \cap I_n'$. For $n > \max\{N_0, N_1\}$,

$$h_n(y) - h_n(0) > -|y|\epsilon - \frac{|y|^2}{2!}\epsilon - \cdots - \frac{|y|^{k_0-1}}{(k_0-1)!}\varepsilon - \frac{3|y|^{k_0}}{2k_0!} > -\varepsilon e^{\delta} - \frac{3|y|^{k_0}}{2k_0!}$$

for $y \in (-\delta, \delta) \cap I_n'$. Hence, we derive a lower bound

$$\mathscr{I}_n \ge e^{-\varepsilon e^{\delta}} \int_{I_n' \cap (-\delta, \delta)} e^{-\frac{3|y|^{k_0}}{2k_0!}}\, \mathrm{d}y =: C_1.$$

From the connectedness of I and $0 \in I_n'$, we conclude that $I_n' \cap (-\delta, \delta)$ has positive mass under the Lebesgue measure. Hence, $C_1 \in (0, \infty)$. □

Proof of Proposition 3. Let (X, Y) be a random vector such that X and Y both have standard Laplace margins. Moreover, assume that there exist $-1 \le \alpha \le 1$, $0 \le \beta < 1$ and $u > 0$ such that for $x > u$

$$\mathbb{P}(Y > y \mid X = x) = \overline{H}\left(\frac{y - \alpha x}{x^{\beta}}\right)$$

holds for all $y \in \mathbb{R}$ with

$$\overline{H}\,(z) = \exp(-\gamma z^{\delta})\mathbb{1}\{z > 0\} + \mathbb{1}\{z \leq 0\},$$

where $\gamma, \delta > 0$. We now derive that $\delta \geq (1-\beta)^{-1}$ must hold. Since Y is distributed as a standard Laplace, we have for $y > 0$

$$\begin{aligned}\frac{\exp(-y)}{2} &= \mathbb{P}(\alpha X + X^{\beta} Z \geq y,\ X \geq u)\mathbb{P}(X \geq u) + \mathbb{P}(Y \geq y,\ X < u)\\ &\geq \mathbb{P}(\alpha X + X^{\beta} Z \geq y,\ X \geq u) \geq \mathbb{P}(X^{\beta} Z \geq y,\ X \geq u)\\ &= \int_u^{\infty} \mathbb{P}\left(Z \geq \frac{y}{x^{\beta}}\right) f_X(x)\,\mathrm{d}x = \frac{1}{2}\int_u^{\infty} \exp\left(-\frac{\gamma y^{\delta}}{x^{\beta\delta}} - x\right)\mathrm{d}x =: \tilde{\mathscr{I}}_y.\end{aligned}$$

We will show that $2\exp(y)\tilde{\mathscr{I}}_y > 1$ as $y \to \infty$ if $\delta < (1-\beta)^{-1}$, which thus would contradict with the marginal distribution of Y. This result holds trivially for $\beta = 0$. So, for now, we let $\beta > 0$. We will prove this asymptotic inequality by applying Proposition 2, with $k_0 = 2$, to bound $\tilde{\mathscr{I}}_y$ from below.

First define $I := [u, \infty)$ as the integration domain, and

$$g_y(x) := \exp\left(-\frac{\gamma y^{\delta}}{x^{\beta\delta}} - x\right)\mathbb{1}\{x \in I\}, \quad \text{and} \quad h_y(x) := \left(-\frac{\gamma y^{\delta}}{x^{\beta\delta}} - x\right)\mathbb{1}\{x \in I\}.$$

Next we find the mode x_y^* of $g_y(x)$. We assume that x_y^* lies in the interior of I such that $h_y'(x_y^*) = 0$, which implies that $\beta\delta\gamma y^{\delta}(x_y^*)^{-\beta\delta-1} = 1$. So, $x_y^* = (\beta\delta\gamma)^{\frac{1}{\beta\delta+1}}\, y^{\frac{\delta}{\beta\delta+1}}$, which lies in the interior of I for sufficiently large y. We now compute

$$g_y(x_y^*) = \exp\left(-\frac{\gamma y^{\delta}}{(x_y^*)^{\beta\delta}} - x_y^*\right) = \exp\left(-A y^{\frac{\delta}{\beta\delta+1}}\right)$$

with $A := \gamma\,(\beta\delta\gamma)^{-\frac{\beta\delta}{\beta\delta+1}} + (\beta\delta\gamma)^{\frac{1}{\beta\delta+1}}$. Secondly,

$$h_y''(x_y^*) = -\beta\delta(\beta\delta+1)(x_y^*)^{-\beta\delta-2}\gamma y^{\delta} = -(\beta\delta+1)\,(\beta\delta\gamma)^{-\frac{1}{\beta\delta+1}}\, y^{-\frac{\delta}{\beta\delta+1}}.$$

Using these expression, we can now check that the assumptions from Proposition 2 with $k_0 = 2$ are satisfied. First we note that $h_y'(x_y^*)(-h_y''(x_y^*))^{-1/2} = 0$.

Next let $C > 0$ and $|x| \le C$, then

$$\begin{aligned}
&\lim_{y\to\infty} \frac{h_y''\left(x_y^* + \frac{x}{\sqrt{-h_y''(x_y^*)}}\right)}{h_y''(x_y^*)} \\
&= \lim_{y\to\infty} \frac{-\beta\delta(\beta\delta+1)\left((\beta\delta\gamma)^{\frac{1}{\beta\delta+1}} y^{\frac{\delta}{\beta\delta+1}} + \frac{x}{\sqrt{(\beta\delta+1)(\beta\delta\gamma)^{-\frac{1}{\beta\delta+1}} y^{-\frac{\delta}{\beta\delta+1}}}}\right)^{-\beta\delta-2} \gamma y^{\delta}}{-(\beta\delta+1)\,(\beta\delta\gamma)^{-\frac{1}{\beta\delta+1}} y^{-\frac{\delta}{\beta\delta+1}}} \\
&= \lim_{y\to\infty} \frac{\left(y^{\frac{\delta}{\beta\delta+1}} + \frac{x}{\sqrt{(\beta\delta+1)(\beta\delta\gamma)^{\frac{1}{\beta\delta+1}} y^{-\frac{\delta}{\beta\delta+1}}}}\right)^{-\beta\delta-2} y^{\delta}}{y^{-\frac{\delta}{\beta\delta+1}}} \\
&= \lim_{y\to\infty} \left(1 + \frac{x}{\sqrt{(\beta\delta+1)\,(\beta\delta\gamma)^{\frac{1}{\beta\delta+1}}\, y^{\frac{\delta}{\beta\delta+1}}}}\right)^{-\beta\delta-2} \\
&= 1,
\end{aligned}$$

which is sufficient to show that for each $\tilde{x}$, Proposition 2 is applicable with $k_0 = 2$ on interval $I_{\tilde{x}} := \left[x_y^* - \frac{\tilde{x}}{\sqrt{-h_y''(x_y^*)}}, x_y^* + \frac{\tilde{x}}{\sqrt{-h_y''(x_y^*)}}\right]$. Hence for each $\tilde{x}$, there exists a constant $C_1(\tilde{x}) > 0$ such that as $y \to \infty$

$$\begin{aligned}
y^{-\frac{\delta/2}{\beta\delta+1}} \exp\left(A y^{\frac{\delta}{\beta\delta+1}}\right) \cdot \tilde{\mathscr{I}}_y &\ge y^{-\frac{\delta/2}{\beta\delta+1}} \exp\left(A y^{\frac{\delta}{\beta\delta+1}}\right) \cdot \int_{I_{\tilde{x}}} g_y(x)\,\mathrm{d}x \\
&= \frac{C_1(\tilde{x})\,(\beta\delta\gamma)^{\frac{1}{2(\beta\delta+1)}}}{\sqrt{\beta\delta+1}}.
\end{aligned}$$

Using the inequality $2\exp(y)\tilde{\mathscr{I}}_y \le 1$ as $y \to \infty$, we must have

$$y^{-\frac{\delta/2}{\beta\delta+1}} \exp\left(A y^{\frac{\delta}{\beta\delta+1}}\right) \cdot \frac{1}{2}\exp(-y) \ge \frac{C_1(\tilde{x})\,(\beta\delta\gamma)^{\frac{1}{2(\beta\delta+1)}}}{\sqrt{\beta\delta+1}} \tag{A1}$$

as $y \to \infty$. Since $0 \le \beta < 1$, we note that if $\delta < (1-\beta)^{-1}$ then inequality (A1) does not hold. So, we derive that $\delta \ge (1-\beta)^{-1}$. □

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Extremal characteristics of conditional models

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S1 Introduction

We give an overview of the content in the Supplementary Material. In Section S2.1, we give parameter estimates of the Haver-Winterstein (HW) distribution as referred to in Section 3. In Sections S2.2-S2.6, we give the details of the calculations that support the arguments in Section 3. Finally, in Section S3 one can find the mathematical derivations of the results stated in of Section 4.

S2 Supplementary Material

S2.1 HW model parameters

Parameter	α	θ	u	k	λ	
	0.573	0.893	3.803	1.550	2.908	
Parameter	μ_0	μ_1	μ_2	σ_0	σ_1	σ_2
	1.134	0.892	0.225	0.005	0.120	0.455

Table S1 Parameters of the joint probability density function of significant wave height H_S (m) and wave period T_p (s) for the Northern North Sea (Haver and Winterstein, 2009).

S2.2 Details on calculations for the HW distribution

Let (X, Y) follow the HW model, see Section 3, with $0 < \mu_2 < 0.5$ and $2\mu_2 < k$. The goal is to calculate the asymptotic behaviour of joint probabilities $\mathbb{P}(X > F_X^{-1}(p),\ Y > F_Y^{-1}(p))$ when p tends to 1 where F_X and F_Y denote the distribution functions of the random variables X and Y, respectively. First, we evaluate the distribution function of Y at large values such that we can calculate $F_Y^{-1}(p)$. After, we compute joint probabilities, like $\mathbb{P}(X_E > u, Y_E > u)$, where X_E and Y_E denote X and Y, respectively, on exponential margins, i.e., $X_E = -\log(1 - F_X(X))$ and $Y_E = -\log(1 - F_Y(Y))$.

We write down an analytical expression for the survival function $\overline{F}_Y$ of Y

$$\overline{F}_Y(y) := \mathbb{P}(Y > y) = \int_0^\infty \overline{\Phi}\left(\frac{\log y - \mu(x)}{\sigma(x)}\right) f_X(x)\,\mathrm{d}x. \tag{S1}$$

where $\mu(x)$ and $\sigma(x)$ are defined in the main paper. We remark that we need to evaluate $\overline{F}_Y$ at large y. To that end, we denote $p_y(x) := (\log y - \mu(x))/\sigma(x)$, and the integrand

$$g_y(x) := \overline{\Phi}\left(p_y(x)\right) f_X(x). \tag{S2}$$

As seen in Figure 1, the integrand g_y has two local maxima for y large enough. Hence, Proposition 2 is not directly applicable. However, we can use the proposition to indirectly prove a lower bound for the intergal (S1). Next, it is straightforward to find an upper bound for the integral with the same rate of decay as the proven lower bound.

We follow the following sets of steps: (a) we prove that there exist (at least) two local maxima x_y^* and x_y^{**}, and find expressions for both. If there are more then x_y^* is the one with the smallest argument, and x_y^{**} is the one with the second smallest argument; (b) we show that we can apply part of Proposition 2 to the smaller of the two local maxima, which gives a lower bound for the integral; (c) we define an upperbound $\tilde{g}_y$ for the integrand g_y, compute the integral of $\tilde{g}_y$, and show that this integral has the same rate of decay as the lower bound; (d) finally, we combine the results to get an asymptotic expression for $\overline{F}_Y(y)$ as $y \to \infty$.

We need to start by working out the expressions for the local maxima. We do this by considering all possible options, which yields five (types of) local extrema $0 < x_0 < x_1 < x_2 < x_3 < x_4 < \infty$ that satisfy the following: (i) as $y \to \infty$, $p_y(x_0) \to \infty$ holds and $x_0 \to 0$; (ii) as $y \to \infty$, $p_y(x_1) \to \infty$ holds and $x_1 \to c \in (0, \infty)$; (iii) as $y \to \infty$, $p_y(x_2) \to \infty$ holds and $x_2 \to \infty$; (iv) as $y \to \infty$, $p_y(x_3) \to c \in \mathbb{R}$ holds and $x_3 \to \infty$; (v) as $y \to \infty$, $p_y(x_4) \to -\infty \in \mathbb{R}$ holds and $x_4 \to \infty$. It is straightforward to show that x_3 and x_4 cannot exist. However, this argument is unnecessary for the purpose of this section.

Finally, we calculate $\overline{F}_Y(y)$ using Proposition 2. In particular, we will get a lower bound by applying Proposition 2 around the local maximum x_0 and we derive an upper bound directly.

S2.3 Finding local extrema

We consider cases (i), (ii) and (iii). These cases have in common that $p_y(x_*) \to \infty$ for $x_* \in \{x_0, x_1, x_2\}$. We will write x_* rather than either x_0, x_1, x_2 to not distinguish between arguments that are applicable to all three cases. To find an expression for x_* in closed form, we define $h_y(x) := \log g_y(x)$ and we solve $h'_y(x_*) = 0$. First, we calculate $h'_y(x)$,

$$\begin{aligned} h'_y(x) &= \frac{\mathrm{d}}{\mathrm{d}x} \left(\log \overline{\Phi}\left(p_y(x)\right) + \log f_X(x) \right) \\ &= \frac{-\varphi\left(p_y(x)\right)}{\overline{\Phi}\left(p_y(x)\right)} \cdot p'_y(x) + \frac{f'_X(x)}{f_X(x)}. \end{aligned}$$

Since $p_y(x_*) \to \infty$, we can simplify this expression by using Mills' ratio, which says that

$$\frac{\overline{\Phi}(x)}{\varphi(x)} = \frac{1}{x} - \frac{1}{x^3} + O(x^{-5})$$

as $x \to \infty$, which implies $\varphi(x)/\overline{\Phi}(x) = x + x^{-1} + O(x^{-3})$ as $x \to \infty$. Moreover, we can write

$$\begin{aligned} p'_y(x) &= \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\log y - \mu(x)}{\sigma(x)} \right) \\ &= -(\log y - \mu(x)) \cdot \frac{\sigma'(x)}{\sigma(x)^2} - \frac{\mu'(x)}{\sigma(x)} \\ &= -p_y(x) \cdot \frac{\sigma'(x)}{\sigma(x)} - \frac{\mu'(x)}{\sigma(x)}. \end{aligned}$$

So,

$$\begin{aligned} h'_y(x_*) = &- \left(p_y(x_*) + \frac{1}{p_y(x_*)} + O\left(p_y(x_*)^{-3}\right) \right) \cdot \left(-p_y(x_*) \cdot \frac{\sigma'(x_*)}{\sigma(x_*)} - \frac{\mu'(x_*)}{\sigma(x_*)} \right) \\ &+ \frac{f'_X(x_*)}{f_X(x_*)} \\ = &\; p_y(x_*)^2 \cdot \frac{\sigma'(x_*)}{\sigma(x_*)} + p_y(x_*) \cdot \frac{\mu'(x_*)}{\sigma(x_*)} + \frac{\sigma'(x_*)}{\sigma(x_*)} + \frac{\mu'(x_*)}{p_y(x_*)\sigma(x_*)} \\ &+ O\left(\frac{\sigma'(x_*)}{p_y(x_*)^2 \sigma(x_*)} + \frac{\mu'(x_*)}{p_y(x_*)^3 \sigma(x_*)} \right) + \frac{f'_X(x_*)}{f_X(x_*)} \end{aligned}$$

as $y \to \infty$. We now fill in the parametric forms for μ and σ. We can then simplify this expression to

$$\begin{aligned}
h_y'(x_*) =& \frac{(\log y - \mu_0 - \mu_1 x_*^{\mu_2})^2 \cdot -\frac{1}{2}\sigma_1\sigma_2 \exp\left(-\sigma_2 x_*\right)\left(\sigma_0 + \sigma_1 \exp(-\sigma_2 x_*)\right)^{-1/2}}{(\sigma_0 + \sigma_1 \exp(-\sigma_2 x_*))^{3/2}} \\
&+ \frac{(\log y - \mu_0 - \mu_1 x_*^{\mu_2})\mu_1\mu_2 x_*^{\mu_2 - 1}}{\sigma_0 + \sigma_1 \exp(-\sigma_2 x_*)} \\
&- \frac{\frac{1}{2}\sigma_1\sigma_2 \exp(-\sigma_2 x_*)(\sigma_0 + \sigma_1 \exp(-\sigma_2 x_*))^{-1/2}}{(\sigma_0 + \sigma_1 \exp(-\sigma_2 x_*))^{1/2}} \\
&+ \frac{\mu_1\mu_2 x_*^{\mu_2-1}}{\log y - \mu_0 - \mu_1 x_*^{\mu_2}} \\
&+ O\left(\frac{x_*^{\mu_2-1}}{(\log y - \mu_0 - \mu_1 x_*^{\mu_2})^3} + \frac{\exp(-\sigma_2 x_*)}{(\log y - \mu_0 - \mu_1 x_*^{\mu_2})^2}\right) + \frac{f_X'(x_*)}{f_X(x_*)} \\
=& - \frac{(\log y - \mu_0 - \mu_1 x_*^{\mu_2})^2 \cdot \sigma_1\sigma_2 \exp\left(-\sigma_2 x_*\right)}{2(\sigma_0 + \sigma_1 \exp(-\sigma_2 x_*))^2} \\
&+ \frac{(\log y - \mu_0 - \mu_1 x_*^{\mu_2})\mu_1\mu_2 x_*^{\mu_2 - 1}}{\sigma_0 + \sigma_1 \exp(-\sigma_2 x_*)} - \frac{\sigma_1\sigma_2 \exp(-\sigma_2 x_*)}{2(\sigma_0 + \sigma_1 \exp(-\sigma_2 x_*))} \\
&+ \frac{\mu_1\mu_2 x_*^{\mu_2-1}}{\log y - \mu_0 - \mu_1 x_*^{\mu_2}} + O\left(\frac{x_*^{\mu_2-1}}{(\log y - \mu_0 - \mu_1 x_*^{\mu_2})^3} + \frac{\exp(-\sigma_2 x_*)}{(\log y - \mu_0 - \mu_1 x_*^{\mu_2})^2}\right) \\
&+ \frac{f_X'(x_*)}{f_X(x_*)}.
\end{aligned}$$

Since, $h_y'(x_*) = 0$ for all y, we can let $y \to \infty$, to further simplify

$$\begin{aligned}
0 =& \lim_{y\to\infty} h_y'(x_*) \\
=& \lim_{y\to\infty} \Bigg(- \log^2 y \cdot \frac{\sigma_1\sigma_2 \exp(-\sigma_2 x_*)}{2(\sigma_0 + \sigma_1 \exp(-\sigma_2 x_*))^2} \\
&\qquad + \log y \cdot \left(\frac{(\mu_0 + \mu_1 x_*^{\mu_2})\sigma_1\sigma_2 \exp\left(-\sigma_2 x_*\right)}{(\sigma_0 + \sigma_1 \exp(-\sigma_2 x_*))^2} + \frac{\mu_1\mu_2 x_*^{\mu_2-1}}{\sigma_0 + \sigma_1 \exp(-\sigma_2 x_*)}\right) \\
&\qquad - \frac{(\mu_0 + \mu_1 x_*^{\mu_2})\mu_1\mu_2 x_*^{\mu_2-1}}{\sigma_0 + \sigma_1 \exp(-\sigma_2 x_*)} - \frac{\sigma_1\sigma_2 \exp(-\sigma_2 x_*)}{2(\sigma_0 + \sigma_1 \exp(-\sigma_2 x_*))} \\
&\qquad - \frac{(\mu_0 + \mu_1 x_*^{\mu_2})^2 \cdot \sigma_1\sigma_2 \exp\left(-\sigma_2 x_*\right)}{2(\sigma_0 + \sigma_1 \exp(-\sigma_2 x_*))^2} + \frac{\mu_1\mu_2 x_*^{\mu_2-1}}{\log y - \mu_0 - \mu_1 x_*^{\mu_2}} + \frac{f_X'(x_*)}{f_X(x_*)}\Bigg).
\end{aligned} \tag{S3}$$

We now split up the analysis into the three cases: (i) $x_* = x_0 \to 0$; (ii) $x_* = x_1 \to c \in (0, \infty)$; (iii) $x_* = x_2 \to \infty$.

Case (i): $x_* = x_0 \to 0$

In this case, there must exist a $y' > 0$ such that for all $y > y'$, $x_0(y) < u$. So, let $y > y'$, then

$$\frac{f_X'(x_0)}{f_X(x_0)} = -\frac{\log x_0 - \theta}{x_0 \alpha^2} - \frac{1}{x_0}.$$

Filling in $x_* = x_0$ simplifies equation (S3) to

$$\lim_{y\to\infty} \Bigg(-\log^2 y \cdot \frac{\sigma_1\sigma_2}{2(\sigma_0+\sigma_1)^2} + \log y \cdot \left(\frac{\mu_0\sigma_1\sigma_2}{(\sigma_0+\sigma_1)^2} + \frac{\mu_1\mu_2 x_0^{\mu_2-1}}{\sigma_0+\sigma_1} \right) - \frac{\mu_0\mu_1\mu_2 x_0^{\mu_2-1}}{\sigma_0+\sigma_1} - \frac{\sigma_1\sigma_2}{2(\sigma_0+\sigma_1)} - \frac{\mu_0^2\sigma_1\sigma_2}{2(\sigma_0+\sigma_1)^2} + \frac{\mu_1\mu_2 x_0^{\mu_2-1}}{\log y} - \frac{\log x_0 - \theta}{x_0\alpha^2} - \frac{1}{x_0} \Bigg) = 0. \tag{S4}$$

Because $0 < \mu_2 < 0.5$, the dominating terms within this limit are of the order $\log^2(y)$ and $\log y \cdot x_0^{\mu_2-1}$. Indeed, $(\log x_0)/x_0$ is dominated by both of these terms since, we must eventually have $x_0^{2\mu_2-2} > (\log x_0)/x_0$. So x_0 must satisfy as $y \to \infty$

$$-\log y \cdot \frac{\sigma_1\sigma_2}{2(\sigma_0+\sigma_1)} + \log y \cdot \mu_1\mu_2 \cdot x_0^{\mu_2-1} = O\left(\frac{\log x_0}{x_0 \log y}\right).$$

Finally, we derive the following asymptotic expression for x_0 as $y \to \infty$

$$x_0 = \left(\frac{\sigma_1\sigma_2}{2\mu_1\mu_2(\sigma_0+\sigma_1)}\right)^{-\frac{1}{1-\mu_2}} \cdot (\log y)^{-\frac{1}{1-\mu_2}} + O\left(\log^{-2}(y)\right). \tag{S5}$$

We will later show that $h_y''(x_0) < 0$. So, indeed x_0 corresponds to a local maximum.

Case (ii): $x_* = x_1 \to c \in (0, \infty)$

In this case, equation (S3) is equivalent to

$$\lim_{y\to\infty} -c_1 \log^2 y + c_2 \log y - c_3 = 0$$

where

$$0 < c_1 = \frac{\sigma_1\sigma_2\exp(-\sigma_2 c)}{2(\sigma_0+\sigma_1\exp(-\sigma_2 c))^2}$$
$$0 < c_2 = \frac{(\mu_0+\mu_1 c^{\mu_2})\sigma_1\sigma_2\exp(-\sigma_2 c)}{(\sigma_0+\sigma_1\exp(-\sigma_2 c))^2} + \frac{\mu_1\mu_2 c^{\mu_2-1}}{\sigma_0+\sigma_1\exp(-\sigma_2 c)}$$
$$0 < c_3 = \frac{(\mu_0+\mu_1 c^{\mu_2})\mu_1\mu_2 c^{\mu_2-1}}{\sigma_0+\sigma_1\exp(-\sigma_2 c)} + \frac{\sigma_1\sigma_2\exp(-\sigma_2 x_*)}{2(\sigma_0+\sigma_1\exp(-\sigma_2 x_*))} + \frac{(\mu_0+\mu_1 c^{\mu_2})^2 \cdot \sigma_1\sigma_2\exp(-\sigma_2 c)}{2(\sigma_0+\sigma_1\exp(-\sigma_2 c))^2} + \frac{\mu_1\mu_2 c^{\mu_2-1}}{\log y - \mu_0 - \mu_1 c^{\mu_2}} - \frac{f_X'(c)}{f_X(c)}.$$

We can now clearly see that equation (S3) cannot be valid under this assumption. We conclude that x_1 does not exist.

Case (iii): $x_* = x_2 \to \infty$

Finally, let $x_* = x_2 \to \infty$. In this case, there must exist a $y'' > 0$ such that for all $y > y''$, $x_0(y) > u$. So, let $y > y''$, then

$$\frac{f'_X(x_0)}{f_X(x_0)} = \frac{k-1}{x_*} - \frac{k x_*^{k-1}}{\lambda^k}.$$

Now, equation (S3) is equivalent to

$$\begin{aligned} \lim_{y\to\infty} \Bigg(& -\log^2 y \cdot \frac{\sigma_1\sigma_2 \exp(-\sigma_2 x_2)}{2\sigma_0^2} \\ & + \log y \cdot \left(\frac{(\mu_0 + \mu_1 x_2^{\mu_2})\sigma_1\sigma_2 \exp\left(-\sigma_2 x_2\right)}{\sigma_0^2} + \frac{\mu_1\mu_2 x_2^{\mu_2-1}}{\sigma_0} \right) \\ & - \frac{(\mu_0 + \mu_1 x_2^{\mu_2})\mu_1\mu_2 x_2^{\mu_2-1}}{\sigma_0} - \frac{\sigma_1\sigma_2\exp(-\sigma_2 x_2)}{2\sigma_0} \\ & - \frac{(\mu_0 + \mu_1 x_2^{\mu_2})^2 \cdot \sigma_1\sigma_2 \exp\left(-\sigma_2 x_2\right)}{2\sigma_0^2} \\ & + \frac{\mu_1\mu_2 x_2^{\mu_2-1}}{\log y - \mu_0 - \mu_1 x_2^{\mu_2}} + \frac{k-1}{x_2} - \frac{k x_2^{k-1}}{\lambda^k} \Bigg) = 0. \end{aligned} \tag{S6}$$

The dominating terms in equation (S6) are of the order $\log^2 y$, $\log y \cdot x_2^{\mu_2-1}$ and x_2^{k-1}. So, we can simplify equation (S6) to

$$\lim_{y\to\infty} -\log^2 y \cdot \frac{\sigma_1\sigma_2\exp(-\sigma_2 x_2)}{2\sigma_0^2} + \log y \cdot \frac{\mu_1\mu_2 x_2^{\mu_2-1}}{\sigma_0} - \frac{k x_2^{k-1}}{\lambda^k} = 0. \tag{S7}$$

We note that the first and third terms have a negative sign, and the second has a positive sign. We note that we cannot simplify this further without considering the following two options as $y \to \infty$: (a) $\exp(-\sigma_2 x_2)\log^2 y \gg x_2^{k-1}$; (b) $\exp(-\sigma_2 x_2)\log^2 y \ll x_2^{k-1}$. Both of these cases will yield a solution to equation (S7) which we call x_{2a} and x_{2b} respectively.

Case (iii-a): $x_* = x_{2a} \to \infty$ **and** $\exp(-\sigma_2 x_{2a})\log^2 y \gg x_{2a}^{k-1}$

We derive from equation (S7) that x_{2a} must satisfy as $y \to \infty$

$$-\log y \cdot \frac{\sigma_1\sigma_2}{2\sigma_0}\exp(-\sigma_2 x_{2a}) + \mu_1\mu_2 x_{2a}^{\mu_2-1} = O\left(\frac{x_{2a}^{k-1}}{\log y}\right).$$

So, x_{2a} must satisfy as $y \to \infty$

$$x_{2a}^{\mu_2-1}\exp(\sigma_2 x_{2a}) = \log y \cdot \left(\frac{\sigma_1\sigma_2}{2\sigma_0\mu_1\mu_2} + O\left(\frac{x_{2a}^{k-1}}{\exp(-\sigma_2 x_{2a})\log^2 y}\right)\right).$$

Finally, we derive the following asymptotic expression for x_{2a} as $y \to \infty$

$$x_{2a} = \frac{\log\log y}{\sigma_2} + O(\log\log\log y). \tag{S8}$$

Case (iii-b): $x_* = x_{2b} \to \infty$ **and** $\exp(-\sigma_2 x_{2a})\log^2 y \ll x_{2a}^{k-1}$

We derive from equation (S7) that x_{2b} must satisfy as $y \to \infty$

$$\log y \cdot \frac{\mu_1\mu_2}{\sigma_0} - \frac{k x_2^{k-\mu_2}}{\lambda^k} = O\left(\log^2 y \exp(-\sigma_2 x_2) x_2^{1-\mu_2}\right).$$

So, x_{2b} must satisfy as $y \to \infty$

$$x_{2b} = \left(\frac{\lambda^k\mu_1\mu_2}{k\sigma_0}\right)^{\frac{1}{k-\mu_2}} \cdot (\log y)^{\frac{1}{k-\mu_2}} + O\left((\log y)^{\frac{1}{k-\mu_2}-\frac{k-2\mu_2}{k-\mu_2}}\right). \tag{S9}$$

S2.4 Identifiying local maxima and local minima

In the previous section, we have found expressions for local extrema, see equations (S5), (S8) and (S9). In this section, we will show by using the second derivative h_y'' that x_0 and x_{2b} correspond to local maxima and that x_{2a} corresponds to a local minimum.

We calculated before

$$h_y'(x) = \frac{-\varphi\left(p_y(x)\right)}{\overline{\Phi}\left(p_y(x)\right)} \cdot p_y'(x) + \frac{f_X'(x)}{f_X(x)}.$$

So,

$$h_y''(x) = -\frac{\varphi\left(p_y(x)\right)^2 p_y'(x)^2}{\overline{\Phi}\left(p_y(x)\right)^2} - \frac{\varphi'\left(p_y(x)\right) p_y'(x)^2}{\overline{\Phi}\left(p_y(x)\right)} - \frac{\varphi\left(p_y(x)\right) p_y''(x)}{\overline{\Phi}\left(p_y(x)\right)} - \frac{f_X'(x)^2}{f_X(x)^2} + \frac{f_X''(x)}{f_X(x)}.$$

We can simplify $h_y''(x_*)$ for $x_* \in \{x_0, x_{2a}, x_{2b}\}$ as $y \to \infty$ by using the identities $\varphi(p_y(x_*))/\overline{\Phi}(p_y(x_*)) \sim p_y(x_*) + p_y(x_*)^{-1}$ as $y \to \infty$ and $\varphi'(x) = -x\varphi(x)$. We get as $y \to \infty$

$$\begin{aligned} h_y''(x_*) &\sim -\left(p_y(x_*) + \frac{1}{p_y(x_*)}\right)^2 p_y'(x_*)^2 + \left(p_y(x_*) + \frac{1}{p_y(x_*)}\right) p_y(x_*) p_y'(x_*)^2 \\ &\qquad - \left(p_y(x_*) + \frac{1}{p_y(x_*)}\right) p_y''(x_*) - \frac{f_X'(x_*)^2}{f_X(x_*)^2} + \frac{f_X''(x_*)}{f_X(x_*)} \\ &\sim -p_y'(x_*)^2 - \frac{p_y'(x_*)^2}{p_y(x_*)^2} - p_y(x_*)p_y''(x_*) - \frac{p_y''(x_*)}{p_y(x_*)} - \frac{f_X'(x_*)^2}{f_X(x_*)^2} + \frac{f_X''(x_*)}{f_X(x_*)} \\ &\sim -p_y'(x_*)^2 - p_y(x_*)p_y''(x_*) - \frac{f_X'(x_*)^2}{f_X(x_*)^2} + \frac{f_X''(x_*)}{f_X(x_*)}. \end{aligned}$$

We work out $p_y'(x)^2$ and $p_y''(x)$ in terms of $p_y(x)$

$$p_y'(x)^2 = \left(-p_y(x) \cdot \frac{\sigma'(x)}{\sigma(x)} - \frac{\mu'(x)}{\sigma(x)}\right)^2 = p_y(x)^2 \cdot \frac{\sigma'(x)^2}{\sigma(x)^2} + p_y(x) \cdot \frac{2\sigma'(x)\mu'(x)}{\sigma(x)^2} + \frac{\mu'(x)^2}{\sigma(x)^2}$$

and

$$\begin{aligned} p_y''(x) &= \frac{\mathrm{d}^2}{\mathrm{d}x^2} \left(\frac{\log y - \mu(x)}{\sigma(x)}\right) \\ &= -\frac{\mu''(x)}{\sigma(x)} + 2 \cdot \frac{\mu'(x)\sigma'(x)}{\sigma(x)^2} + (\log y - \mu(x)) \cdot \left(\frac{2\sigma'(x)^2}{\sigma(x)^3} - \frac{\sigma''(x)}{\sigma(x)^2}\right) \\ &= p_y(x) \cdot \left(\frac{2\sigma'(x)^2}{\sigma(x)^2} - \frac{\sigma''(x)}{\sigma(x)}\right) + 2 \cdot \frac{\mu'(x)\sigma'(x)}{\sigma(x)^2} - \frac{\mu''(x)}{\sigma(x)}. \end{aligned}$$

So, as $y \to \infty$

$$\begin{aligned} h_y''(x_*) &\sim -p_y(x_*)^2 \cdot \frac{\sigma'(x_*)^2}{\sigma(x_*)^2} - p_y(x_*) \cdot \frac{2\sigma'(x_*)\mu'(x_*)}{\sigma(x_*)^2} - \frac{\mu'(x_*)^2}{\sigma(x_*)^2} \\ &\qquad - p_y(x_*) \left(p_y(x_*) \cdot \left(\frac{2\sigma'(x_*)^2}{\sigma(x_*)^2} - \frac{\sigma''(x_*)}{\sigma(x_*)}\right) + 2 \cdot \frac{\mu'(x_*)\sigma'(x_*)}{\sigma(x_*)^2} - \frac{\mu''(x_*)}{\sigma(x_*)}\right) \\ &\qquad - \frac{f_X'(x_*)^2}{f_X(x_*)^2} + \frac{f_X''(x_*)}{f_X(x_*)} \\ &\sim -p_y(x_*)^2 \cdot \left(\frac{3\sigma'(x_*)^2}{\sigma(x_*)^2} - \frac{\sigma''(x_*)}{\sigma(x_*)}\right) - p_y(x_*) \left(\frac{4\mu'(x_*)\sigma'(x_*)}{\sigma(x_*)^2} - \frac{\mu''(x_*)}{\sigma(x_*)}\right) \\ &\qquad - \frac{\mu'(x_*)^2}{\sigma(x_*)^2} - \frac{f_X'(x_*)^2}{f_X(x_*)^2} + \frac{f_X''(x_*)}{f_X(x_*)}. \end{aligned}$$

For $x = x_0$, we have

$$\begin{aligned}
\mu'(x_0) &= \mu_1\mu_2 x_0^{\mu_2-1}, \\
\mu''(x_0) &= -\mu_1\mu_2(1-\mu_2)x_0^{\mu_2-2}, \\
\sigma(x_0) &\sim \sqrt{\sigma_0+\sigma_1}, \\
\sigma'(x_0) &\sim -\sigma_1\sigma_2/(2\sqrt{\sigma_0+\sigma_1}), \\
\sigma''(x_0) &\sim \sigma_1^2\sigma_2^2/(4(\sigma_0+\sigma_1)^{3/2}), \text{ and} \\
p_y(x_0) &\sim \log y/\sqrt{\sigma_0+\sigma_1}.
\end{aligned}$$

So,

$$\begin{aligned}
h_y''(x_0) \sim &-\frac{\log^2 y}{\sigma_0+\sigma_1}\cdot\left(\frac{3\sigma_1^2\sigma_2^2}{4(\sigma_0+\sigma_1)^2}-\frac{\sigma_1^2\sigma_2^2}{4(\sigma_0+\sigma_1)^2}\right) \\
&+\frac{\log y}{\sqrt{\sigma_0+\sigma_1}}\left(\frac{2\mu_1\mu_2 x_0^{\mu_2-1}\cdot\sigma_1\sigma_2}{(\sigma_0+\sigma_1)^{3/2}}-\frac{\mu_1\mu_2(1-\mu_2)x_0^{\mu_2-2}}{\sqrt{\sigma_0+\sigma_1}}\right) \\
&-\frac{\mu_1^2\mu_2^2 x_0^{2\mu_2-2}}{\sigma_0+\sigma_1}+\frac{1}{x_0^2}+\frac{\log x_0-\theta}{x_0^2\alpha^2}-\frac{1}{x_0^2\alpha^2} \\
\sim &-\frac{\sigma_1^2\sigma_2^2}{2(\sigma_0+\sigma_1)^3}\cdot\log^2 y-\frac{\mu_1\mu_2(1-\mu_2)}{\sigma_0+\sigma_1}\cdot\log y\cdot x_0^{\mu_2-2} \\
&-\frac{\mu_1^2\mu_2^2}{\sigma_0+\sigma_1}\cdot x_0^{2\mu_2-2}+\frac{1}{x_0^2}+\frac{\log x_0-\theta}{x_0^2\alpha^2}-\frac{1}{x_0^2\alpha^2}.
\end{aligned}$$

We combine this result with equation (S5), to get

$$h_y''(x_0) \sim -\frac{\mu_1\mu_2(1-\mu_2)}{\sigma_0+\sigma_1}\cdot\log y\cdot x_0^{\mu_2-2} \sim -C\,(\log y)^{2+\frac{1}{1-\mu_2}}$$

with

$$C = \frac{\mu_1\mu_2(1-\mu_2)}{\sigma_0+\sigma_1}\cdot\left(\frac{\sigma_1\sigma_2}{2\mu_1\mu_2(\sigma_0+\sigma_1)}\right)^{1+\frac{1}{1-\mu_2}}.$$

We conclude that $h_y''(x_0) < 0$ and that indeed x_0 corresponds to a local maximum.

For $x = x_{2*}$ with $* = a, b$, we have

$$\begin{aligned}
\mu'(x_{2*}) &= \mu_1\mu_2 x_{2*}^{\mu_2-1}, \\
\mu''(x_{2*}) &= -\mu_1\mu_2(1-\mu_2)x_{2*}^{\mu_2-2}, \\
\sigma(x_{2*}) &\sim \sqrt{\sigma_0}, \\
\sigma'(x_{2*}) &\sim -\sigma_1\sigma_2/(2\sqrt{\sigma_0})\cdot\exp(-\sigma_2 x_{2*}), \\
\sigma''(x_{2*}) &\sim \sigma_1\sigma_2^2/(2\sqrt{\sigma_0})\cdot\exp(-\sigma_2 x_{2*}), \text{ and} \\
p_y(x_{2b}) &\sim \log y/\sqrt{\sigma_0}.
\end{aligned}$$

So,

$$h_y''(x_{2*}) \sim -\frac{\log^2 y}{\sigma_0} \cdot \left(\frac{3\sigma_1^2\sigma_2^2 \cdot \exp(-2\sigma_2 x_{2*})}{4\sigma_0^2} - \frac{\sigma_1\sigma_2^2 \cdot \exp(-\sigma_2 x_{2*})}{2\sigma_0} \right) + \frac{\log y}{\sqrt{\sigma_0}} \left(\frac{2\mu_1\mu_2 x_{2*}^{\mu_2-1} \cdot \sigma_1\sigma_2 \exp(-\sigma_2 x_{2*})}{\sigma_0^{3/2}} - \frac{\mu_1\mu_2(1-\mu_2)x_{2*}^{\mu_2-2}}{\sqrt{\sigma_0}} \right) - \frac{\mu_1^2\mu_2^2 x_{2*}^{2\mu_2-2}}{\sigma_0} - \frac{k-1}{x_{2*}^2} - \frac{k(k-1)x_{2*}^{k-2}}{\lambda^k}.$$

For $x_{2*} = x_{2a}$, we simplify

$$h_y''(x_{2a}) \sim \frac{\sigma_1\sigma_2^2}{2\sigma_0^2} \cdot \log^2 y \cdot \exp(-\sigma_2 x_{2a})$$

which confirms that x_{2a} corresponds to a local minimum. Finally, for $x_{2*} = x_{2b}$, we simplify

$$\begin{aligned} h_y''(x_{2b}) &\sim -\frac{\mu_1\mu_2(1-\mu_2)x_{2b}^{\mu_2-2}}{\sigma_0} \cdot \log y - \frac{k(k-1)x_{2b}^{k-2}}{\lambda^k} \\ &\sim -\left(\frac{\mu_1\mu_2(1-\mu_2)\left(\frac{\lambda^k\mu_1\mu_2}{k\sigma_0}\right)^{\frac{\mu_2-2}{k-\mu_2}}}{\sigma_0} + \frac{k(k-1)\left(\frac{\lambda^k\mu_1\mu_2}{k\sigma_0}\right)^{\frac{k-2}{k-\mu_2}}}{\lambda^k} \right) (\log y)^{\frac{k-2}{k-\mu_2}} \\ &\sim -\left(\frac{\mu_1\mu_2(1-\mu_2)}{\sigma_0} \frac{k\sigma_0}{\lambda^k\mu_1\mu_2} + \frac{k(k-1)}{\lambda^k} \right) \cdot \left(\frac{\lambda^k\mu_1\mu_2}{k\sigma_0}\right)^{\frac{k-2}{k-\mu_2}} (\log y)^{\frac{k-2}{k-\mu_2}} \\ &\sim -\frac{k}{\lambda^k}(k-\mu_2) \cdot \left(\frac{\lambda^k\mu_1\mu_2}{k\sigma_0}\right)^{\frac{k-2}{k-\mu_2}} (\log y)^{\frac{k-2}{k-\mu_2}}. \end{aligned}$$

Finally, it is clear to see that $h_y''(x_{2b}) < 0$ which confirms that x_{2b} is a local maximum.

S2.5 Calculating the survival function of *Y*

We will apply Proposition 2 to g_y from equation (S2), where we find that $k_0 = 2$ and $x_y^* = x_0(y)$. This gives us a lower bound for $\overline{F}_Y(y)$ as $y \to \infty$. We start with evaluating $g_y(x_0)$ and after, we check the smoothness assumption of the proposition for $k_0 = 2$. Finally, we derive an upper bound that is of the same order as the lower bound. Hence, we can combine the lower and upper bound to get an estimate for the rate of convergence to 0 of $\overline{F}_Y(y)$.

Before, we evaluate $g_y(x_0)$ and $h_y''(x_0)$, we first simplify $p_y(x_0)$ and $p_y(x_0)$ as $y \to \infty$. We have

$$p_y(x_0) = \frac{\log y}{\sqrt{\sigma_0+\sigma_1}} - \frac{\mu_0}{\sqrt{\sigma_0+\sigma_1}} + O\left((\log y)^{-\frac{\mu_2}{1-\mu_2}}\right)$$

and

$$\frac{1}{p_y(x_0)} = \frac{\sqrt{\sigma_0+\sigma_1}}{\log y} + O\left((\log y)^{-2-\frac{\mu_2}{1-\mu_2}}\right) = \frac{\sqrt{\sigma_0+\sigma_1}}{\log y} + O\left((\log y)^{-2}\right).$$

So,

$$\begin{aligned} g_y(x_0) &= \varphi\left(\frac{\log y-\mu(x_0)}{\sigma(x_0)}\right)\cdot\left(\frac{\sigma(x_0)}{\log y-\mu(x_0)} + O\left(\left(\frac{\sigma(x_0)}{\log y-\mu(x_0)}\right)^3\right)\right) f_X(x_0) \\ &= \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{\log y-\mu(x_0)}{\sigma(x_0)}\right)^2\right\} \\ &\quad\cdot\left(\frac{\sigma(x_0)}{\log y-\mu(x_0)} + O\left(\left(\frac{\sigma(x_0)}{\log y-\mu(x_0)}\right)^3\right)\right)\frac{\exp\left\{-\frac{(\log x_0-\theta)^2}{2\alpha^2}\right\}}{\sqrt{2\pi}x_0\alpha} \\ &= \frac{\sqrt{\sigma_0+\sigma_1}}{2\pi\alpha}\exp\left\{-\frac{1}{2}\left(\frac{\log y}{\sqrt{\sigma_0+\sigma_1}} - \frac{\mu_0}{\sqrt{\sigma_0+\sigma_1}} + O\left((\log y)^{-\frac{\mu_2}{1-\mu_2}}\right)\right)^2\right\} \\ &\quad\cdot\frac{\frac{1}{\log y}+O\left((\log y)^{-2}\right)}{\left(\frac{\sigma_1\sigma_2}{2\mu_1\mu_2(\sigma_0+\sigma_1)}\right)^{-\frac{1}{1-\mu_2}}\cdot(\log y)^{-\frac{1}{1-\mu_2}} + O\left(\log^{-2}(y)\right)} \\ &\quad\cdot\exp\left\{-\frac{\left(\log\left[\left(\frac{\sigma_1\sigma_2}{2\mu_1\mu_2(\sigma_0+\sigma_1)}\right)^{-\frac{1}{1-\mu_2}}\cdot(\log y)^{-\frac{1}{1-\mu_2}} + O\left(\log^{-2}(y)\right)\right]-\theta\right)^2}{2\alpha^2}\right\} \\ &= \exp\left\{-\frac{1}{2(\sigma_0+\sigma_1)}\left(\log^2 y - 2\mu_0\log y + O\left((\log y)^{\frac{1-2\mu_2}{1-\mu_2}}\right)\right)\right\}. \end{aligned}$$

Next, we check the assumptions of Proposition 2. First, we clearly have $h_y'(x_0)(-h_y''(x_0))^{-1/2} = 0$. Secondly, we have one clearly dominating term in the second derivative of h_y near x_0, so it is enough to show that

$$\begin{aligned} &\lim_{y\to\infty}\frac{h_y''\left(x_0+\frac{x}{\sqrt{-h_y''(x_0)}}\right)}{h_y''(x_0)} \\ &= \lim_{x\to 0}\frac{p_y\left(x_0+\frac{x}{\sqrt{-h_y''(x_0)}}\right)}{p_y(x_0)}\cdot\frac{\mu''\left(x_0+\frac{x}{\sqrt{-h_y''(x_0)}}\right)}{\mu''(x_0)}\cdot\frac{\sigma(x_0)}{\sigma\left(x_0+\frac{x}{\sqrt{-h_y''(x_0)}}\right)} \end{aligned}$$

is equal to 1 for any fixed x. Since $(-h_y''(x_0))^{-1/2} \ll x_0$ as $y\to\infty$ and since p_y and σ are differentiable at $0 = \lim_{y\to\infty} x_0$, it is clear that the first and third term of the equation above tend to 1. Since $\mu''(0)$ does not exist, we would need to work out the term involving the second derivative of μ more carefully. We get

$$\frac{\mu''\left(x_0 + \frac{x}{\sqrt{-h_y''(x_0)}}\right)}{\mu''(x_0)} = \left(1 + \frac{x}{x_0\sqrt{-h_y''(x_0)}}\right)^{\mu_2 - 2}. \tag{S10}$$

We note that $x_0\sqrt{-h_y''(x_0)}$ is asymptotically equal to a constant times $(\log y)^{(1-2\mu_2)/(2-2\mu_2)}$. Since $\mu_2 < 0.5$, the second term within the brackets in equation (S10) tends to 0 when $y \to \infty$. This yields that the right hand side of equation (S10) converges to 1 as $y \to \infty$. This is enough to show the smoothness assumption of the proposition. We get that for any fixed $\tilde{x} > 0$, there exists a constant $C_1(\tilde{x})$ such that

$$\begin{aligned}\int_0^\infty g_y(x)\,\mathrm{d}x &\geq \int_{x_0 - \frac{\tilde{x}}{\sqrt{-h_y''(x_0)}}}^{x_0 + \frac{\tilde{x}}{\sqrt{-h_y''(x_0)}}} g_y(x)\,\mathrm{d}x \\ &\geq C_1(\tilde{x}) g_y(x_0) \cdot \frac{1}{\sqrt{-h_y''(x_0)}} \\ (\text{as } y \to \infty) \quad &= \exp\left\{-\frac{1}{2(\sigma_0 + \sigma_1)}\left[\log^2 y - 2\mu_0 \log y + O\left((\log y)^{\frac{1-2\mu_2}{1-\mu_2}}\right)\right]\right\}.\end{aligned} \tag{S11}$$

Next, we evaluate $g_y(x_{2b})$ but first we work out

$$p_y(x_{2b}) = \frac{\log y}{\sqrt{\sigma_0}} + O\left((\log y)^{\frac{\mu_2}{k-\mu_2}}\right)$$

and

$$\frac{1}{p_y(x_0)} = \frac{\sqrt{\sigma_0}}{\log y} + O\left((\log y)^{-2+\frac{\mu_2}{k-\mu_2}}\right).$$

So,

$$\begin{aligned}
g_y(x_{2b}) &= \varphi\left(\frac{\log y - \mu(x_{2b})}{\sigma(x_{2b})}\right)\left(\frac{\sigma(x_{2b})}{\log y - \mu(x_{2b})} + O\left(\frac{\sigma(x_{2b})^3}{(\log y - \mu(x_{2b}))^3}\right)\right) f_X(x_{2b}) \\
&= \varphi\left(\frac{\log y}{\sqrt{\sigma_0}} + O\left((\log y)^{\frac{\mu_2}{k-\mu_2}}\right)\right) \cdot \frac{\sqrt{\sigma_0}}{\log y} \cdot \left(1 + O\left((\log y)^{-1+\frac{\mu_2}{k-\mu_2}}\right)\right) \\
&\quad \cdot \frac{k}{\lambda^k}\left(\frac{\lambda^k \mu_1 \mu_2}{k\sigma_0}\right)^{\frac{k-1}{k-\mu_2}} (\log y)^{\frac{k-1}{k-\mu_2}} \left(1 + O\left((\log y)^{-\frac{k-2\mu_2}{k-\mu_2}}\right)\right) \\
&\quad \cdot \exp\left\{ - \frac{\left(\frac{\sigma_1 \lambda^k \mu_1 \mu_2}{k\sigma_0^2}\right)^{\frac{k}{k-\mu_2}} (\log y)^{\frac{k}{k-\mu_2}}}{\lambda^k} \left(1 + O((\log y)^{-\frac{k-2\mu_2}{k-\mu_2}}\right) \right\} \\
&= \exp\left\{ -\frac{1}{2\sigma_0}\log^2(y) + O\left((\log y)^{\frac{k}{k-\mu_2}}\right)\right\}.
\end{aligned}$$

In particular, we find that $g_y(x_0) > g_y(x_{2b})$ for y large enough. We have now all tools available to find an upperbound that gives the result directly,

$$g_y(x) \leq \tilde{g}_y(x) := \begin{cases} \max\{g_y(x) : \ x \in [0, x_2]\} & \text{for } 0 \leq x \leq x_2, \\ f_X(x) & \text{for } x > x_2. \end{cases}$$

Since $g_y(x_{2b}) \leq g_y(x_0)$ for y large enough, we have derived that the maximum over the interval $[0, x_2]$ is attained at x_0. We here note that we do not need to show that x_3 and x_4 cannot exist as per definition, as they would clearly need to be larger than x_2 if they exist. So, as $y \to \infty$

$$\begin{aligned}
&\int_0^\infty g_y(x)\,\mathrm{d}x \leq \int_0^{x_{2b}} g_y(x_0)\,\mathrm{d}x + \int_{x_{2b}}^\infty f_X(x)\,\mathrm{d}x = x_{2b} g_y(x_0) + \overline{F}_X(x_{2b}) \\
&= \left(\frac{\lambda^k \mu_1 \mu_2}{k\sigma_0}\right)^{\frac{1}{k-\mu_2}} \left(1 + O\left((\log y)^{-\frac{k-2\mu_2}{k-\mu_2}}\right)\right) (\log y)^{\frac{1}{k-\mu_2}} \qquad \text{(S12)} \\
&\qquad \cdot \exp\left\{-\frac{1}{2(\sigma_0+\sigma_1)}\left[\log^2 y - 2\mu_0 \log y + O\left((\log y)^{\frac{1-2\mu_2}{1-\mu_2}}\right)\right]\right\} \\
&\quad + \exp\left\{-\lambda^{-k}\left(\frac{\lambda^k \mu_1 \mu_2}{k\sigma_0}\right)^{\frac{k}{k-\mu_2}} (\log y)^{\frac{k}{k-\mu_2}} \left[1 + O\left((\log y)^{-\frac{k-2\mu_2}{k-\mu_2}}\right)\right]\right\} \\
&= \exp\left\{-\frac{1}{2(\sigma_0+\sigma_1)}\left[\log^2 y - 2\mu_0 \log y + O\left((\log y)^{\frac{1-2\mu_2}{1-\mu_2}}\right)\right]\right\}. \qquad \text{(S13)}
\end{aligned}$$

Now, combining equation (S11) and equation (S13), yields as $y \to \infty$

$$\mathbb{P}(Y > y) = \int_0^\infty g_y(x)\,\mathrm{d}x = \exp\left\{-\frac{1}{2(\sigma_0+\sigma_1)}\left[\log^2 y - 2\mu_0 \log y + O\left((\log y)^{\frac{1-2\mu_2}{1-\mu_2}}\right)\right]\right\}.$$

S2.6 Calculating η

We use the previous work to transform Y to Y_E on standard exponential margins. Thus

$$\begin{aligned}Y_E &= F_E^{-1}(F_Y(Y)) = -\log(1 - F_Y(Y))\\ &= -\log\left(\exp\left\{-\frac{1}{2(\sigma_0+\sigma_1)}\left[\log^2 Y - 2\mu_0 \log Y + O\left((\log Y)^{\frac{1-2\mu_2}{1-\mu_2}}\right)\right]\right\}\right)\\ &= \frac{1}{2(\sigma_0+\sigma_1)}\left(\log^2 Y - 2\mu_0\log Y + O\left((\log Y)^{\frac{1-2\mu_2}{1-\mu_2}}\right)\right).\end{aligned}$$

So, the function T that transforms $\log Y$ to Y_E is given by

$$T(y) = \frac{y^2}{2(\sigma_0+\sigma_1)} - \frac{\mu_0 y}{\sigma_0+\sigma_1} + O\left(y^{\frac{1-2\mu_2}{1-\mu_2}}\right),$$

as $y \to \infty$. In calculating the extremal dependence measures, we need to solve $T(y) = u$ for large y. We get

$$T^{-1}(u) = \sqrt{2(\sigma_0+\sigma_1)u} + O(1)$$

as $u \to \infty$. We write down a formula for $\chi = \lim_{u\to\infty} \mathbb{P}(Y_E > u \mid (X/\lambda)^k > u)$ as $u \to \infty$

$$\begin{aligned}\mathbb{P}(Y_E > u \mid (X/\lambda)^k > u) &= e^u \int_{\lambda u^{1/k}}^{\infty} \mathbb{P}(T(\log Y) > u \mid X = x) f_X(x)\,\mathrm{d}x\\ &= e^u \int_{\lambda u^{1/k}}^{\infty} \mathbb{P}(\log Y > T^{-1}(u) \mid X = x) f_X(x)\,\mathrm{d}x\\ &= e^u \int_{\lambda u^{1/k}}^{\infty} \overline{\Phi}\left(\frac{T^{-1}(u)-\mu(x)}{\sigma(x)} \mid X = x\right)\cdot \frac{kx^{k-1}}{\lambda^k}\exp\left\{-\left(\frac{x}{\lambda}\right)^k\right\}\mathrm{d}x.\end{aligned}$$

In particular, we have for $I = [\lambda u^{1/k}, \lambda(2+\sigma_1/\sigma_0)^{1/k}]$

$$\mathbb{P}(Y_E > u \mid (X/\lambda)^k > u) \tag{S14}$$
$$> e^u \int_I \overline{\Phi}\left(\frac{T^{-1}(u)-\mu(x)}{\sigma(x)} \mid X = x\right)\cdot \frac{kx^{k-1}}{\lambda^k}\exp\left\{-\left(\frac{x}{\lambda}\right)^k\right\}\mathrm{d}x. \tag{S15}$$

For ease of presentation, we define $p_u(x) = [T^{-1}(u) - \mu(x)]/\sigma(x)$. Similar to the previous section, we define g_u as the integrand and $h_u := \log g_u$ as the log of the integrand, both are specified only on the integration domain I. For x in the integration domain, we have

$$h_u(x) := \log\left(\overline{\Phi}(p_u(x)) f_X(x)\right).$$

We apply Proposition 2 to bound integral (S14) from below. In particular, we first need to find the mode of h_u over the integration domain. Let x_u be a sequence such that for each u, x_u lies in the integration domain. So, then we can write $x = C_u u^{1/k} + o(u^{1/k})$ for some bounded set of constants $C_u \in [\lambda, \lambda(2 + \sigma_1/\sigma_0)]$. We have

$$\begin{aligned} h_u'(x) &= -\frac{\varphi(p_u(x))}{\overline{\Phi}(p_u(x))} \cdot p_u'(x) - \frac{k-1}{x} - \frac{kx^{k-1}}{\lambda^k} \\ &= \frac{\varphi(p_u(x))}{\overline{\Phi}(p_u(x))} \cdot \left(p_u(x) \cdot \frac{\sigma'(x)}{\sigma(x)} + \frac{\mu'(x)}{\sigma(x)} \right) - \frac{k-1}{x} - \frac{kx^{k-1}}{\lambda^k}. \end{aligned}$$

Since, $p_u(x) \sim \sqrt{2(1+\sigma_1/\sigma_0)u} \to \infty$ as $u \to \infty$, we simplify

$$\begin{aligned} h_u'(x) \sim \sqrt{2\left(1 + \frac{\sigma_1}{\sigma_0}\right) u} \cdot \Bigg(\sqrt{2\left(1 + \frac{\sigma_1}{\sigma_0}\right) u} \cdot \frac{-\sigma_1 \sigma_2 e^{-\sigma_2 \left(C_u u^{1/k} + o(u^{1/k})\right)}}{2\sigma_0} \\ + \frac{\mu_1 \mu_2 \left(C_u u^{1/k} + o(u^{1/k})\right)^{\mu_2 - 1}}{\sqrt{\sigma_0}} \Bigg) \\ - \frac{k-1}{C_u u^{1/k} + o(u^{1/k})} - \frac{k\left(C_u u^{1/k} + o(u^{1/k})\right)^{k-1}}{\lambda^k}. \\ \sim -\frac{kC_u^{k-1} u^{1-1/k}}{\lambda^k}. \end{aligned}$$

In particular, we derive that $h_u'(x) < 0$ as $u \to \infty$. So, the maximum of h_u over the integration domain must be attained at the boundary and hence is given by $x_0 = \lambda u^{1/k}$. In particular, we get $h_u'(x_0) \sim -ku^{1-1/k}/\lambda$. We now will show that we can apply Proposition 2 with $k_0 = 1$. We have, as $u \to \infty$,

$$\begin{aligned} h_u(\lambda u^{1/k}) &= -\frac{1}{2} \log(2\pi) - \frac{1}{2} p_u(\lambda u^{1/k})^2 - \log p_u(\lambda u^{1/k}) + \log f_X(\lambda u^{1/k}) \\ &= -\frac{1}{2} \log(2\pi) - \frac{1}{2} \left(\frac{T^{-1}(u) - \mu(\lambda u^{1/k})}{\sigma(\lambda u^{1/k})} \right)^2 \\ &\qquad - \log \left(\frac{T^{-1}(u) - \mu(\lambda u^{1/k})}{\sigma(\lambda u^{1/k})} \right) + \log \left(\frac{ku^{(k-1)/k}}{\lambda} \right) - u \\ &= -\left(2 + \frac{\sigma_1}{\sigma_0}\right) u + O\left(u^{1/2 + \mu_2/k}\right). \end{aligned}$$

Next, we check the smoothness assumption of Proposition 2 with $k_0 = 1$. Let $\delta > 0$ and $0 \le x \le \delta$. It is now enough to show that the limit of u to infinity of the following expression tends to 1. We have

$$\lim_{u\to\infty} \frac{h_u'\left(\lambda u^{1/k} + \frac{x}{-h_u'(\lambda u^{1/k})}\right)}{h_u'(\lambda u^{1/k})} = \lim_{u\to\infty} \frac{\left(\lambda u^{1/k} + \frac{\lambda x}{k u^{1-1/k}}\right)^{k-1}}{u^{(k-1)/k}} = \lim_{u\to\infty} \left(\lambda + \frac{\lambda x}{k}\right)^{k-1} = 1.$$

This is enough to show the smoothness assumption of Proposition 2 with $k_0 = 1$. We conclude that for each $\tilde{x}$, there exists a constant $C_1(\tilde{x})$ such that

$$\begin{aligned}\int_{\lambda u^{1/k}}^{\infty} g_u(x)\,\mathrm{d}x &\geq \int_{\lambda u^{1/k}}^{\lambda u^{1/k} + \frac{\tilde{x}}{-h_u'(x_0)}} g_u(x)\,\mathrm{d}x \geq C_1(\tilde{x}) g_u(\lambda u^{1/k}) \cdot \frac{1}{-h_u'(\lambda u^{1/k})} \\ &\overset{(\text{as } u\to\infty)}{=} e^{-\left(2+\frac{\sigma_1}{\sigma_0}\right)u + O\left(u^{\frac{1}{2}+\frac{\mu_2}{k}}\right)}. \end{aligned} \tag{S16}$$

To get an upper bound, we use the following crude upper bound $\tilde{g}_u$ for g_u,

$$g_u(x) \leq \tilde{g}_u(x) := \begin{cases} g_u(\lambda u^{1/k}) & \text{for } \lambda u^{1/k} \leq x \leq \lambda \left(2 + \frac{\sigma_1}{\sigma_0}\right)^{1/k} u^{1/k}, \\ f_X(x) & \text{for } x > \lambda \left(2 + \frac{\sigma_1}{\sigma_0}\right)^{1/k} u^{1/k}. \end{cases}$$

We get as $u \to \infty$,

$$\begin{aligned}\int_{\lambda u^{1/k}}^{\infty} g_u(x)\,\mathrm{d}x \leq& \left(\lambda \left(2 + \frac{\sigma_1}{\sigma_0}\right)^{1/k} u^{1/k} - \lambda u^{1/k}\right) g_u(\lambda u^{1/k}) \qquad \text{(S17)} \\ &+ \overline{F}_X\left(\lambda \left(2 + \frac{\sigma_1}{\sigma_0}\right)^{1/k} u^{1/k}\right) \\ =& \exp\left(-\left(2 + \frac{\sigma_1}{\sigma_0}\right) u + O\left(u^{1/2+\mu_2/k}\right)\right) + \exp\left(-\left(2 + \frac{\sigma_1}{\sigma_0}\right) u\right) \\ =& \exp\left(-\left(2 + \frac{\sigma_1}{\sigma_0}\right) u + O\left(u^{1/2+\mu_2/k}\right)\right). \qquad \text{(S18)}\end{aligned}$$

Combining equations (S16) and (S18), we get

$$\mathbb{P}(Y_E > u \mid (X/\lambda)^k > u) = \int_{\lambda u^{1/k}}^{\infty} g_u(x)\,\mathrm{d}x = \exp\left(-\left(2 + \frac{\sigma_1}{\sigma_0}\right) u + O\left(u^{1/2+\mu_2/k}\right)\right)$$

as $u \to \infty$. From this expression, it is straightforward to see that $\xi = 0$ and

$$\eta^{-1} = 2 + \frac{\sigma_1}{\sigma_0}.$$

S3 Details on Calculations for the Exact HT model

S3.1 Introduction

Assume model (10) for random vector (X, Y) with $\overline{H}$ as in equation (11). We recall that (X, Y) is a random vector such that X and Y both have standard Laplace margins. Moreover,there exist $0 \leq \alpha \leq 1$, $\beta < 1$ and $u > 0$ such that for $x > u$

$$\mathbb{P}(Y > y \mid X = x) = \overline{H}\left(\frac{y - \alpha x}{x^{\beta}}\right),$$

holds for all $y \in \mathbb{R}$ with

$$\overline{H}\,(z) = \exp(-\gamma z^{\delta})\mathbb{1}\{z > 0\} + \mathbb{1}\{z \leq 0\}$$

for $\gamma > 0$ and $\delta \geq (1-\beta)^{-1}$. In this section, we work out the value for η when $0 < \alpha < 1,\ \beta > 0$ and $\delta = (1-\beta)^{-1}$. The other cases are significantly easier to work out and the results of these cases are stated in the main paper.

S3.2 Calculating η

We write

$$\begin{aligned}
\mathbb{P}(Y > u, X > u) &= \int_u^{\infty} \overline{H}\left(\frac{u - \alpha x}{x^{\beta}}\right) f_X(x)\,\mathrm{d}x \\
&= \frac{1}{2}\int_u^{u/\alpha} \exp\left(-\gamma\left(\frac{u-\alpha x}{x^{\beta}}\right)^{\delta} - x\right)\mathrm{d}x + \frac{1}{2}\int_{u/\alpha}^{\infty} \exp(-x)\,\mathrm{d}x \\
&= \frac{1}{2}\int_u^{u/\alpha} \exp\left(-\gamma\left(\frac{u-\alpha x}{x^{\beta}}\right)^{\delta} - x\right)\mathrm{d}x + \frac{1}{2}\exp\left(-\frac{u}{\alpha}\right).
\end{aligned}$$

In general, we cannot evaluate the first integral in closed form for finite u. However, we can bound it from below using Proposition 2. A bound from above can again be found directly. We define the integration domain $I = [u, u/\alpha]$,

$$g_u(x) := \exp\left(-\gamma\left(\frac{u-\alpha x}{x^{\beta}}\right)^{\delta} - x\right)$$

for $x \in I$ and $h_u := \log g_u$ on I. We now need to determine whether or not the mode $x_0 := x_0(u)$ of the integrand g_u over the integration domain I lies on the boundary of I or in the interior of I. We assume that x_0 lies in the interior of

I, then we have

$$\begin{aligned} 0 = h_u'(x_0) &= \gamma\delta \left(\frac{u-\alpha x_0}{x_0^\beta}\right)^{\delta-1} \cdot \left(\frac{\alpha}{x_0^\beta} + \frac{(u-\alpha x_0)\beta}{x_0^{\beta+1}}\right) - 1 \\ &= \gamma\delta\beta(u-\alpha x_0)^\delta x_0^{-\beta\delta-1} + \gamma\delta\alpha(u-\alpha x_0)^{\delta-1}x_0^{-\beta\delta} - 1 \\ &= \gamma\delta\beta(u-\alpha x_0)^\delta x_0^{-\delta} + \gamma\delta\alpha(u-\alpha x_0)^{\delta-1}x_0^{-\delta+1} - 1 \end{aligned}$$

and we derive that

$$\beta(u-\alpha x_0)^\delta + \alpha(u-\alpha x_0)^{\delta-1}x_0 = \frac{1}{\gamma\delta}x_0^\delta. \tag{S19}$$

Since, we work under the premise that $x_0 \in (u, u/\alpha)$, we are only interested in finding solutions that satisfy $x_0 = \tilde{c}u + o(u)$ as $u \to \infty$ for some $\tilde{c} \in [1, 1/\alpha]$, otherwise the mode of h_u is found at the boundary of the integration domain at u. We try $x_0 = cu$ with $c \in (0, \infty)$ in equation (S19), and we derive that this is an exact solution if c solves

$$0 = \gamma\delta\left(\beta(1-\alpha c)^\delta + \alpha c(1-\alpha c)^{\delta-1}\right) - c^\delta = \gamma(1-\alpha c)^{\delta-1}\left(\delta - 1 + \alpha c\right) - c^\delta. \tag{S20}$$

Since the right hand side is a continuous function of c for $c \in [0, 1/\alpha]$, we show by the intermediate value theorem that $c \in (0, 1/\alpha)$ by inserting $c = 0$ and $c = 1/\alpha$ and comparing signs of the right hand side of equation (S20). Indeed, for $c = 0$, we have that

$$\gamma(1-\alpha c)^{\delta-1}\left(\delta - 1 + \alpha c\right) - c^\delta = \gamma(\delta - 1) > 0$$

and for $c = 1/\alpha$, we have that

$$\gamma(1-\alpha c)^{\delta-1}\left(\delta - 1 + \alpha c\right) - c^\delta = -\alpha^{-\delta} < 0.$$

We recall that we are only interested in the value for c if $c \in (1, 1/\alpha)$. Hence, let $c = 1$ in the right hand side of equation (S20) to give

$$\begin{aligned} \gamma\delta\left(\beta(1-\alpha c)^\delta + \alpha c(1-\alpha c)^{\delta-1}\right) - c^\delta &= \gamma(1-\alpha)^{\delta-1}\left(\left(\delta-1\right)\left(1-\alpha\right) + \delta\alpha\right) - 1 \\ &= \gamma(1-\alpha)^{\delta-1}(\delta - 1 + \alpha) - 1 \end{aligned}$$

which is negative if and only if $\gamma(1-\alpha)^{\delta-1}(\delta-1+\alpha) < 1$. We conclude that $c \in (0, 1)$ if and only if $\gamma(1-\alpha)^{\delta-1}(\delta-1+\alpha) < 1$ and $c \in [1, 1/\alpha)$ if and only if $\gamma(1-\alpha)^{\delta-1}(\delta-1+\alpha) \geq 1$. We term these cases as Case (2a) and Case (2b), respectively. In Case (2b), x_0 lies in the interior of the integration domain I for large enough u, and in Case (2a), the mode over the integration domain I is found at u on the boundary.

We work out $g_u(x_0)$ for both Case (2a) and (2b),

$$\begin{aligned} g_u(x_0) &= \exp\left\{-\gamma\left(\frac{u-\alpha x_0}{x_0^\beta}\right)^\delta - x_0\right\} \\ &= \exp\left\{-\gamma\left(\frac{u-\alpha(cu+o(u))}{(cu+o(u))^\beta}\right)^\delta - cu + o(u)\right\} \\ &= \exp\left\{-\gamma u^{\delta-\beta\delta}\frac{(1-\alpha c+o(1))^\delta}{c^{\beta\delta}+o(1))} - cu + o(u)\right\} \\ &= \exp\left\{-\left(\frac{\gamma(1-\alpha c)^\delta}{c^{\beta\delta}}+c\right)u + o(u)\right\}. \end{aligned}$$

Next, we work out $h'_u(x_0)$ in Case (2a)

$$\begin{aligned} h'_u(x_0) &= \gamma\delta\beta(u-\alpha u)^\delta u^{-\beta\delta-1} + \gamma\delta\alpha(u-\alpha u)^{\delta-1}u^{-\beta\delta} - 1 \\ &= \gamma(1-\alpha)^{\delta-1}(\delta-1+\alpha) - 1. \end{aligned}$$

By definition of Case (2a), we have that $h'_u(x_0) < 0$. Let $C > 0$ and $|x| \leq C$, then as $u \to \infty$

$$\begin{aligned} h'_u\left(x_0 - \frac{x}{h'_u(x_0)}\right) &= \gamma\delta\beta\left(u - \alpha\left(x_0 + \frac{x}{-h'_u(x_0)}\right)\right)^\delta \left(x_0 + \frac{x}{-h'_u(x_0)}\right)^{-\beta\delta-1} \\ &\quad + \gamma\delta\alpha\left(u - \alpha\left(x_0 + \frac{x}{-h'_u(x_0)}\right)\right)^{\delta-1} \left(x_0 + \frac{x}{-h'_u(x_0)}\right)^{-\beta\delta} - 1 \\ &= \gamma\delta\beta\left(u - \alpha u - \frac{x\alpha}{1-\gamma(1-\alpha)^{\delta-1}(\delta-1+\alpha)}\right)^\delta \\ &\quad \cdot \left(u + \frac{x}{1-\gamma(1-\alpha)^{\delta-1}(\delta-1+\alpha)}\right)^{-\beta\delta-1} \\ &\quad + \gamma\delta\alpha\left(u - \alpha u - \frac{x\alpha}{1-\gamma(1-\alpha)^{\delta-1}(\delta-1+\alpha)}\right)^{\delta-1} \\ &\quad \cdot \left(u + \frac{x}{1-\gamma(1-\alpha)^{\delta-1}(\delta-1+\alpha)}\right)^{-\beta\delta} - 1 \\ &= \gamma\delta\beta\left(u^\delta(1-\alpha)^\delta + O\left(u^{\delta-1}\right)\right)\left(u^{-\beta\delta-1} + O\left(u^{-\beta\delta-2}\right)\right) \\ &\quad + \gamma\delta\alpha\left(u^{\delta-1}(1-\alpha)^{\delta-1} + O\left(u^{\delta-2}\right)\right)\left(u^{-\beta\delta} + O\left(u^{-\beta\delta-1}\right)\right) - 1 \\ &= h'_u(x_0) + O\left(u^{\delta-2-\beta\delta}\right). \end{aligned}$$

So,

$$\lim_{u\to\infty} \frac{h'_u\left(x_0 + \frac{x}{-h'_u(x_0)}\right)}{h'_u(x_0)} = 1,$$

which is enough to show the smoothness assumption of Proposition 2 with $k_0 = 1$. We get that for any fixed $\tilde{x} > 0$ there exist a $C_1(\tilde{x}) > 0$ such that

$$\begin{aligned}
\mathbb{P}(X > u,\ Y > u) &= \int_u^{u/\alpha} g_u(x)\,\mathrm{d}x + \frac{1}{2}\exp\left\{-\frac{u}{\alpha}\right\} \\
&\geq \frac{1}{2}\int_{x_0-\frac{\tilde{x}}{-h_u'(x_0)}}^{x_0+\frac{\tilde{x}}{-h_u'(x_0)}} g_u(x)\,\mathrm{d}x + \frac{1}{2}\exp\left\{-\frac{u}{\alpha}\right\} \\
&\geq \frac{1}{2}C_1(\tilde{x})g_u(x_0)\cdot\frac{1}{-h_u'(x_0)} + \frac{1}{2}\exp\left\{-\frac{u}{\alpha}\right\} \\
&\geq \frac{1}{2}C_1(\tilde{x})\exp\left\{-\left(\gamma(1-\alpha)^{\delta}+1\right)u + o(u)\right\} \\
&\qquad \cdot\frac{1}{1-\gamma(1-\alpha)^{\delta-1}(\delta-1+\alpha)} + \frac{1}{2}\exp\left\{-\frac{u}{\alpha}\right\} \\
&= \exp\left\{-\left(\gamma(1-\alpha)^{\delta}+1\right)u + o(u)\right\}.
\end{aligned}$$

In the last step we used that $\gamma(1-\alpha)^{\delta}+1 < 1/\alpha$ holds, which can be directly derived from the assumptions corresponding to Case (2b). Similarly to before, we can find an upper bound rather straightforwardly using the following upperbound for $g_u(x)$

$$g_u(x) \leq \tilde{g}_u(x) := \begin{cases} g_u(x_0) & \text{for } u \leq x \leq u/\alpha, \\ f_X(x) & \text{for } x > u/\alpha. \end{cases}$$

So,

$$\begin{aligned}
\mathbb{P}(X > u,\ Y > u) &= \int_u^{u/\alpha} g_u(x)\,\mathrm{d}x + \frac{1}{2}\exp\left\{-\frac{u}{\alpha}\right\} \\
&\leq u\left(\frac{1}{\alpha}-1\right)g_u(x_0) + \frac{1}{2}\exp\left\{-\frac{u}{\alpha}\right\} \\
&= u\left(\frac{1}{\alpha}-1\right)\exp\left\{-\left(\gamma(1-\alpha)^{\delta}+1\right)u + o(u)\right\} + \frac{1}{2}\exp\left\{-\frac{u}{\alpha}\right\} \\
&= \exp\left\{-\left(\gamma(1-\alpha)^{\delta}+1\right)u + o(u)\right\}.
\end{aligned}$$

We conclude that

$$\mathbb{P}(X > u,\ Y > u) = \exp\left\{-\left(\gamma(1-\alpha)^{\delta}+1\right)u + o(u)\right\},$$

$\chi = 0$ and

$$\eta = \left(\gamma(1-\alpha)^{\delta}+1\right)^{-1}.$$

For Case (2a), we work out $h_u''(x_0)$ as $u \to \infty$

$$\begin{aligned}
h_u''(x_0) &= -\alpha^2\gamma\delta(\delta-1)(u-\alpha x_0)^{\delta-2}x_0^{-\beta\delta} - 2\alpha\beta\gamma\delta^2(u-\alpha x_0)^{\delta-1}x_0^{-\beta\delta-1} \\
&\qquad - \beta\delta(\beta\delta+1)\gamma(u-\alpha x_0)^{\delta}x_0^{-\beta\delta-2} \\
&= -\alpha^2\gamma\delta(\delta-1)(u-\alpha(cu+o(u)))^{\delta-2}(cu+o(u))^{-\beta\delta} \\
&\qquad - 2\alpha\beta\gamma\delta^2(u-\alpha(cu+o(u)))^{\delta-1}(cu+o(u))^{-\beta\delta-1} \\
&\qquad - \beta\delta(\beta\delta+1)\gamma(u-\alpha(cu+o(u)))^{\delta}(cu+o(u))^{-\beta\delta-2} \\
&= -\Big[\alpha^2\gamma\delta(\delta-1)(1-\alpha c)^{\delta-2}c^{-\beta\delta} + 2\alpha\beta\gamma\delta^2(1-\alpha c)^{\delta-1}c^{-\beta\delta-1} \\
&\qquad + \beta\delta(\beta\delta+1)\gamma(1-\alpha c)^{\delta}c^{-\beta\delta-2}\Big]u^{\delta-2-\beta\delta} + o\left(u^{\delta-2-\beta\delta}\right) \\
&= -\beta\delta^2\gamma c^{-\beta\delta-2}(1-\alpha c)^{\delta-2}\left(\alpha^2c^2 + 2\alpha(1-\alpha c)c + (1-\alpha c)^2\right)u^{-1} + o\left(u^{-1}\right) \\
&= -\beta\delta^2\gamma c^{-\beta\delta-2}(1-\alpha c)^{\delta-2}u^{-1} + o\left(u^{-1}\right) \\
&= -\delta(\delta-1)\gamma c^{-\delta-1}(1-\alpha c)^{\delta-2}u^{-1} + o\left(u^{-1}\right).
\end{aligned}$$

Now, let $C > 0$ and $|x| \le C$, then we have $x_0 + x(-h_u''(x_0))^{-1/2} = cu + o(u)$. So,

$$h_u''\left(x_0 + \frac{x}{\sqrt{-h_u''(x_0)}}\right) = h_u''\left(cu + o(u)\right).$$

So,

$$\lim_{u\to\infty} \frac{h_u''\left(x_0 + \frac{x}{\sqrt{-h_u''(x_0)}}\right)}{h_u''(x_0)} = \lim_{u\to\infty} \frac{-\delta(\delta-1)\gamma c^{-\delta-1}(1-\alpha c)^{\delta-2}u^{-1}(1+o(1))}{-\delta(\delta-1)\gamma c^{-\delta-1}(1-\alpha c)^{\delta-2}u^{-1}(1+o(1))} = 1,$$

which is enough to show the smoothness assumption of Proposition 2 with $k_0 = 1$. We get that for any fixed $\tilde{x} > 0$ there exist a $C_1(\tilde{x}) > 0$ such that as $u \to \infty$

$$\begin{aligned}
\mathbb{P}(X > u, Y > u) &= \int_u^{u/\alpha} g_u(x)\,\mathrm{d}x + \frac{1}{2}\exp\left\{-\frac{u}{\alpha}\right\} \\
&\ge \frac{1}{2}\int_{x_0 - \frac{\tilde{x}}{-h_u'(x_0)}}^{x_0 + \frac{\tilde{x}}{-h_u'(x_0)}} g_u(x)\,\mathrm{d}x \\
&\ge \frac{1}{2}C_1(\tilde{x})g_u(x_0)\cdot\frac{1}{-h_u'(x_0)} \\
&= \exp\left\{-\left(\frac{\gamma(1-\alpha c)^{\delta}}{c^{\delta-1}} + c\right)u + o(u)\right\}.
\end{aligned}$$

Similarly to before, we can find an upper bound rather straightforwardly,

$$\begin{aligned}\mathbb{P}(X > u,\ Y > u) &= \int_u^{u/\alpha} g_u(x)\,\mathrm{d}x + \frac{1}{2}\exp\left\{-\frac{u}{\alpha}\right\}\\ &\leq u\left(\frac{1}{\alpha} - 1\right) g_u(x_0) + \frac{1}{2}\exp\left\{-\frac{u}{\alpha}\right\}\\ &= u\left(\frac{1}{\alpha} - 1\right)\exp\left\{-\left(\frac{\gamma(1-\alpha c)^{\delta}}{c^{\beta\delta}} + c\right)u + o(u)\right\} + \frac{1}{2}\exp\left\{-\frac{u}{\alpha}\right\}\\ &= \exp\left\{-\left(\frac{\gamma(1-\alpha c)^{\delta}}{c^{\delta-1}} + c\right)u + o(u)\right\}.\end{aligned}$$

So,

$$\mathbb{P}(X > u,\ Y > u) = \exp\left\{-\left(\frac{\gamma(1-\alpha c)^{\delta}}{c^{\beta\delta}} + c\right)u + o(u)\right\},$$

and we conclude that $\chi = 0$ and

$$\eta = \left(\frac{\gamma(1-\alpha c)^{\delta}}{c^{\delta-1}} + c\right)^{-1}.$$

References

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